MA 417 Ordinary Differential Equations

Problem Set #2

September 3, 2007

1. Prove the following sharp result concerning saturated solutions.

   (Classification of saturated solutions) Let $\mathbf{f} : \mathbb{I} \times \Omega \to \mathbb{R}^n$ be continuous on $\mathbb{I} \times \Omega$ and assume that it maps bounded subsets in $\mathbb{I} \times \Omega$ into bounded subsets in $\mathbb{R}^n$. Let $\mathbf{u} : [x_0, d) \to \mathbb{R}^n$ be a saturated right solution of IVP. Then one of the following alternatives holds.

   $(1)'$ The function $\mathbf{u}$ is unbounded on the interval $[x_0, d)$. If $d < \infty$ there exists $\lim_{x \to d_-} \|\mathbf{u}(x)\| = \infty$.

   (2) The function $\mathbf{u}$ is bounded on the interval $[x_0, d)$, and $\mathbf{u}$ is global *i.e.,* $d = \sup \mathbb{I}$.

   $(3)'$ The function $\mathbf{u}$ is bounded on the interval $[x_0, d)$, and $\mathbf{u}$ is not global *i.e.,* $d < \sup \mathbb{I}$ and limit of $\mathbf{u}$ as $x \to d_-$ exists and lies on the boundary of Ω.

2. Let $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $\mathbf{u} : [x_0, d) \to \mathbb{R}^n$ be a saturated right solution of IVP. Use the previous exercise and conclude that one of the following two alternatives holds.

   (1) The function $\mathbf{u}$ is global *i.e.,* $d = \infty$.

   (2) The function $\mathbf{u}$ is not global *i.e.,* $d < \infty$ and $\lim_{x \to d_-} \|\mathbf{u}(x)\| = \infty$. This phenomenon is often referred to as $\mathbf{u}$ **blows up in finite time**.

3. The IVP

$$y'' - y = -\cos(e^x) - e^x \sin(e^x)$$
$$y(0) = \sin 1, \quad y'(0) = \cos 1 - \sin 1.$$

   via d'Alembert transformations becomes

$$z_1' = z_2, \quad z_2' = z_1 - \cos(e^x) - e^x \sin(e^x)$$
$$z_1(0) = \sin 1, \quad z_2(0) = \cos 1 - \sin 1.$$

Note $y(x) = e^{-x}\sin(e^x)$ is a solution of scalar equation and $\lim_{x\to\infty} y(x) = 0$. On the other hand, $\mathbf{z}(x) = (e^{-x}\sin(e^x),\ \cos(e^x) - e^{-x}\sin(e^x))$ is the solution of corresponding system, and it has no limit as $x \to \infty$.

From the experience with this example, do we have to develop a theory for extension of solutions, classifying saturated solutions for second order equations in normal form separately? Or, is the theory developed for first order systems enough?

4. Solve the IVP $y' = \sqrt{|y|}$, $y(0) = -1$. For solving the IVP, you may use the property of solutions of ODE $y' = \sqrt{|y|}$, namely if y is a solution of ODE, then z defined by $z(x) = -y(-x)$ is also a solution. Mention the interval on which your solution to IVP exists. Comment on the uniqueness of solutions to the above IVP.

5. Find the general solution of the second order ODE
$$y'' + 2y' + y = e^{-x}$$

6. Let y_1 be a solution of the second order linear equation
$$y'' + a(x)y' + b(x)y = 0,$$
where a and b are continuous functions on an interval $\mathbb{I}$. Find a second solution y_2 by ansatz $y_2(x) = y_1(x)\int^x u(s)ds$. This procedure is called *method of reduction of order*.

7. Solve the IVP
$$\begin{aligned} y'' + 2y' + y &= 0 \\ y(0) = 0, \quad y'(0) &= 0. \end{aligned}$$
Can you generalize for a general second order linear equation in normal form?