Chapter 2

First order PDE

2.1 How and Why First order PDE appear?

2.1.1 Physical origins

Conservation laws form one of the two fundamental parts of any mathematical model of Continuum Mechanics. These models are PDEs. Discussion is beyond the scope of this course.

2.1.2 Mathematical origins

1. **Two-parameter family of surfaces:**[1] Let $f : \mathbb{R}^2 \times A \times B \longrightarrow \mathbb{R}$ be a smooth function. Then
$$z = f(x, y, a, b), \tag{2.1}$$
roughly speaking, represents a two-parameter family of surfaces in $\mathbb{R}^3$. Differentiating (2.1) with respect to x and y yields the relations
$$z_x = f_x(x, y, a, b), \tag{2.2a}$$
$$z_y = f_y(x, y, a, b). \tag{2.2b}$$
Eliminating a and b from (2.1)-(2.2), we get a relation of the form
$$F(x, y, z, z_x, z_y) = 0.$$
This is a PDE for the unknown function of two independent variables.

   Exercise 2.1 *Let $f(x, y, a, b) = (x-a)^2+(y-b)^2$. Get a PDE by eliminating the parameters a and b. (Answer: $u_x^2 + u_y^2 = 4u$.)*

2. **Unknown function of known functions:**

   (a) **Unknown function of a single known function:** Let $u = f(g)$ where f is an unknown function and g is a known function of two independent variables x and y. Differentiating $u = f(g)$ w.r.t. x and y yields the equations $u_x = f'(g)g_x$ and $u_y = f'(g)g_y$ respectively. Eliminating the arbitrary function f from these two equations, we obtain
$$g_y u_x - g_x u_y = 0,$$
which is a first order PDE for u.

[1] Let $y = f(x, c_1, c_2, \cdots, c_n)$ denote an n-parameter family of plane curves. By eliminating $c_1, \cdots, c_n$ we get an n-th order ODE of the form $F(x, y, y', \cdots, y^{(n)}) = 0$. If we consider a family of space curves, then we get systems of ODE after eliminating parameters. Of course, all this we get only if we are able to eliminate the parameters!

(b) **Unknown function of two known functions:** Let

$$u = f(x - ay) + g(x + ay). \tag{2.3}$$

Denoting $v(x, y) = x - ay$ and $w(x, y) = x + ay$, the above equation becomes

$$u = f(v) + g(w). \tag{2.4}$$

where f, g are unknown functions and v, w are known functions.

Differentiating (2.4) with respect to x and y yields the relations

$$p = u_x = f'(x - ay) + g'(x + ay), \tag{2.5a}$$
$$q = u_y = -af'(x - ay) + ag'(x + ay). \tag{2.5b}$$

Eliminating f and g from (2.5a)-(2.5b) (after differentiating them w.r.t. y and x respectively), we get a relation of the form

$$q_y = a^2 p_x$$

In terms of u the above first order PDE is the well-known Wave equation

$$u_{yy} = a^2 u_{xx}.$$

2.2 Quasi-linear PDE

Consider the quasi-linear PDE given by

$$a(x, y, u)\, u_x + b(x, y, u)\, u_y = c(x, y, u), \tag{2.6}$$

where a, b, c are continuously differentiable functions on a domain $\Omega \subseteq \mathbb{R}^3$. Let Ω_0 be the projection of Ω to the XY-plane.

Definition 2.2 (Integral Surface) *Let $D \subseteq \Omega_0$ and $u : D \longrightarrow \mathbb{R}$ be a solution of the equation* (2.6)*. The surface S represented by $z = u(x, y)$ is called an* **Integral Surface**.

Remark 2.3

1. *Any point on an integral surface S has the form $(x, y, u(x, y))$ for some $(x, y) \in D$. Since S is an integral surface, such an (x, y) is unique.*

2. *Note that any integral surface S is of the form $z = u(x, y)$ for some solution u (defined on its domain) of the equation* (2.6)*. The projection of such an S to the XY-plane will be the domain of u.*

3. *For the surface $z = u(x, y)$, the normal at any point $(x_0, y_0, u(x_0, y_0))$ on S is given by $(u_x(x_0, y_0), u_y(x_0, y_0), -1)$. We can write the PDE* (2.6) *in the form*

   $$(u_x(x_0, y_0), u_y(x_0, y_0), -1) \bullet (a(x_0, y_0, u(x_0, y_0)), b(x_0, y_0, u(x_0, y_0)), c(x_0, y_0, u(x_0, y_0))) = 0.$$

   Thus the vector $(a(x_0, y_0, u(x_0, y_0)), b(x_0, y_0, u(x_0, y_0)), c(x_0, y_0, u(x_0, y_0)))$ belongs to the tangent space to S at the point $(x_0, y_0, u(x_0, y_0))$. By definition of tangent space, there exists a curve $\gamma : (-\delta, \delta) \to \mathbb{R}^3$ such that (i) γ lies on S, (ii) $\gamma(0) = (x_0, y_0, u(x_0, y_0))$, and (iii) $\gamma'(0) = (a(x_0, y_0, u(x_0, y_0)), b(x_0, y_0, u(x_0, y_0)), c(x_0, y_0, u(x_0, y_0)))$. This motivates the definition of a Characteristic curve that we are going to define shortly.

Definition 2.4 (Characteristic Vector Field) *The vector field* $(a(x,y,z),\, b(x,y,z),\, c(x,y,z))$ *is called the* **Characteristic Vector Field** *of the equation* (2.6). *The direction of the vector* $(a(x,y,z),\, b(x,y,z),\, c(x,y,z))$ *is called the* **Characteristic Direction** *at* $(x,y,z) \in \Omega$.

Definition 2.5 (Characteristic Curve) *A curve in* $\mathbb{R}^3$ *which is tangential to the characteristic direction at each of its points is called a* **Characteristic Curve** *(i.e., the tangent to the curve is parallel to the characteristic direction at every point on the curve).*

Definition 2.6 (Base Characteristics) *The projections of* **Characteristic Curves** *to the* XY*-plane are called* **Base characteristics**.

Remark 2.7

1. *The characteristic curves of the equation* (2.6) *are solutions of the following autonomous system of ODEs:*

$$\begin{aligned} \frac{dx}{dt} &= a(x,y,z) \\ \frac{dy}{dt} &= b(x,y,z) \\ \frac{dz}{dt} &= c(x,y,z) \end{aligned}$$

   Note that the above system is **autonomous** *and we are interested not on the solutions but only the trace of the solutions in the* XYZ*-space as* t *varies in the interval on which solutions to the above system exist. Thus the parameter* t *is somewhat artificial (see [18]) and replacing it with any other parameter along a characteristic curve will amount to replacing* a, b, c *by proportional quantities.*

2. *Since* a, b, c *are assumed to be continuously differentiable on* Ω*, through any point of* Ω *there exists a unique characteristic curve of the equation* (2.6) *(Prove this!). Hence distinct characteristic curves do not intersect (why?), however their projections to the* XY*-plane might intersect(Does it not contradict the existence and uniqueness theorem? Explain).*

3. *In the case of semi-linear PDE, neither distinct characteristic curves nor their projections on* XY*-plane intersect. However, if at least one of the functions* a, b *is such that existence and uniqueness theorem for solutions to IVPs cannot be applied, then it may happen that characteristic curves and their projections to* XY*-plane may intersect. For example, see Exercise 2.23.*

4. *The characteristic curves form a two-parameter family. However the solutions to the characteristic system of ODEs form a three-parameter family.* ■

Exercise 2.8 *Justify the above remarks.*

Theorem 2.9 *Let* $S:\ z = u(x,y)$ *be a surface in* $\mathbb{R}^3$*. Then the following statements are equivalent.*

1. *The surface* S *is an integral surface of equation* (2.6).

2. *The surface* S *is a union of characteristic curves of the equation* (2.6).

PROOF :
Proof of (1) $\Longrightarrow$ (2): Let $S:\ z = u(x,y)$ be an integral surface of equation (2.6). That is, there is a domain $D \subseteq \mathbb{R}^2$ and a function $u:\ D \longrightarrow \mathbb{R}$ such that

$$a(x,y,u(x,y))\, u_x(x,y) + b(x,y,u(x,y))\, u_y(x,y) = c(x,y,u(x,y)) \text{ for all } (x,y) \in D. \tag{2.7}$$

The statement (2) of the theorem is equivalent to

$$S = \bigcup_{\gamma \text{ is a characteristic curve}} \gamma.$$

Thus, to prove that S is a union of characteristic curves, it is sufficient to prove that the characteristic curve γ_p lies entirely[1] on S for every $p \in S$ (why?). Let $p = (x_0, y_0, z_0)$ be an arbitrary point on the surface S. Through p, there exists a unique characteristic curve γ_p and we want to prove that γ_p lies entirely[2] on S. Suppose that γ_p is given by

$$x = x(t),\ y = y(t),\ z = z(t),\ t \in I \quad \text{and} \quad P = (x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0)) \text{ for some } t_0 \in I.$$

Without loss of generality assume that $(x(t), y(t)) \in D$ for all $t \in I$; if not we replace I by an interval I' for which this holds. To prove that γ_p lies entirely on S, we will prove

$$z(t) = u\left(x(t), y(t)\right)^{3} \text{ for all } t \in I.$$

Thus we are led to consider the following function which is defined on I:

$$V(t) = z(t) - u\left(x(t), y(t)\right) \text{ for all } t \in I.$$

We need to show that V is the zero function. Note that $V(t_0) = 0$ as $P \in S$. Let us compute the derivative of V.

$$\begin{aligned} V'(t) &= z'(t) - u_x\left(x(t), y(t)\right)\frac{dx}{dt} - u_y\left(x(t), y(t)\right)\frac{dy}{dt} \\ &= c\left(x(t), y(t), z(t)\right) - u_x\left(x(t), y(t)\right) a\left(x(t), y(t), z(t)\right) - u_y\left(x(t), y(t)\right) b\left(x(t), y(t), z(t)\right) \\ &= c\left(x(t), y(t), V(t) + u\left(x(t), y(t)\right)\right) - u_x\left(x(t), y(t)\right) a\left(x(t), y(t), V(t) + u\left(x(t), y(t)\right)\right) \\ &\qquad - u_y\left(x(t), y(t)\right) b\left(x(t), y(t), V(t) + u\left(x(t), y(t)\right)\right). \end{aligned}$$

Thus the function $V : I \longrightarrow \mathbb{R}$ is a solution (why?) of the ODE

$$U' = f(t, U), \tag{2.8}$$

where

$$\begin{aligned} f(t, U) &= c\left(x(t), y(t), U + u\left(x(t), y(t)\right)\right) - u_x\left(x(t), y(t)\right) a\left(x(t), y(t), U + u\left(x(t), y(t)\right)\right) \\ &\qquad - u_y\left(x(t), y(t)\right) b\left(x(t), y(t), U + u\left(x(t), y(t)\right)\right) \end{aligned}$$

The RHS of (2.8) is a locally Lipschitz function w.r.t. U since a, b, c, u are continuously differentiable functions on D if u is assumed to be continuously differentiable. Further note that $U(t) \equiv 0$ is a solution of the ODE (2.8) (why?). Also $U(t_0) = 0$ in view of (2.7) as $z = u(x, y)$ is an integral surface. Hence by uniqueness of solutions of Initial value problems to ODEs, we conclude that $V \equiv 0$.

Proof of (2) $\Longrightarrow$ (1): Let the surface $S :\ z = u(x, y)$ be a union of characteristic curves of the equation (2.6). We want to show that S is an integral surface. In other words, we want to show that u solves the equation (2.6). Let $p_0 = (x_0, y_0, u(x_0, y_0))$ be any point on the surface S. We want to show

$$(u_x(x_0, y_0), u_y(x_0, y_0), -1) \bullet (a(x_0, y_0, u(x_0, y_0)), b(x_0, y_0, u(x_0, y_0)), c(x_0, y_0, u(x_0, y_0))) = 0. \tag{2.9}$$

Since S is a union of characteristics, there is a characteristic passing through p_0 that lies entirely on S. Since $(u_x(x_0, y_0), u_y(x_0, y_0), -1)$ is the normal direction to S at p_0 and $(a(x_0, y_0, u(x_0, y_0)), b(x_0, y_0, u(x_0, y_0)), c(x_0, y_0, u(x_0, y_0)))$ is the direction of tangent to γ at p_0, we get (2.9). This finishes the proof. ■

[1] Many books claim this statement. This is FALSE. Note that u is given to us. We cannot assume anything about the domain on which u is defined.

[2] In fact, that part of γ_p lies on S for which the corresponding base characterisitcs reside in D, the domain on which u is defined.

[3] We dont have the right to write this equation if $(x(t), y(t))$ does not belong to D for all $t \in I$.

Remark 2.10 (On the domain D) *In the quasi-linear equation* (2.6)*, recall that a, b, c are defined on $\Omega \subseteq \mathbb{R}^3$ and were assumed to be continuously differentiable. Let Ω_0 be the projection of Ω into the XY-plane. That is,*

$$\Omega_0 = \left\{(x, y) \in \mathbb{R}^2 \,:\, (x, y, z) \in \Omega \text{ for some } z \in \mathbb{R}\right\}.$$

Then $z = u(x, y)$ is an intgral surface on some domain $D \subseteq \Omega_0$. Note that characteristic curve lives in Ω, by virtue of being a soluton of the system of ODE where the vector field is defined on Ω. Projection of characteristic curves to XY-plane live inside Ω_0. ■

Example 2.11 *If $u : D \longrightarrow \mathbb{R}$ be a solution to the Quasi-linear PDE* (2.6)*, then so is $v : D_1 \longrightarrow \mathbb{R}$ where v is defined by $v(x, y) = u(x, y)$. The integral surfaces $z = u(x, y)$ and $z = v(x, y)$ are different since u and v are different as functions. But both the integral surfaces coincide on D_1. Thus interesection of two integral surfaces could be another integral surface.*

The following corollary follows immediately from Theorem 2.9

Corollary 2.12 *Let S_1 and S_2 be two integral surfaces such that $p \in S_1 \cap S_2$. Then some part of the characteristic passing through p lies on both S_1 and S_2*[1]*.*

Corollary 2.13 *If two integral surfaces intersect without touching and the intersection is a curve γ, then γ is a characteristic curve.*

Proof :

To prove that γ is a characteristic curve, we have to prove that at any point P on the curve γ, the tangent has the characteristic direction. At P the tangent to the curve γ lies in the tangent planes to S_1 as well as S_2 and also the characteristic direction $(a(P), b(P), c(P))$. Since the tangent planes do not coincide (why?), the only direction common to both S_1 and S_2 is $(a(P), b(P), c(P))$. Hence tangent to the curve γ at P is proportional to the characterisitc direction at P. Since P is an arbitrary point on γ, it follows that γ is a characteristic curve. ■

Exercise 2.14 *Suppose γ is a curve that lies on two integral surfaces. Can we use the above proof to conclude that γ is a characteristic curve? If yes, give a proof. If not, explain where the above proof fails.*

2.2.1 Cauchy Problem for Quasi-linear PDE

Cauchy Problem

To find an integral surface $z = u(x, y)$ of the quasi-linear PDE (2.6), containing a given space curve Γ whose parametric equations are

$$x = f(s),\ y = g(s),\ z = h(s),\ \ s \in I, \tag{2.10}$$

where f, g, h are assumed to be continuously differentiable on the interval I and $h(s) = u(f(s), g(s))$ for $s \in I$.

Initial Value Problem

Initial value problem for the quasi-linear PDE (2.6) is a special Cauchy problem for (2.6), wherein the initial curve Γ lies in the ZX-plane and the y variable has an interpretation of the time-variable. That is, Γ has the following parametric form:

$$x = f(s),\ y = 0,\ z = h(s),\ \ s \in I, \tag{2.11}$$

[1] This is stated more confusingly and also WRONGLY as "If two integral surfaces intersect at a point p, then they intersect along the entire characteristic curve through p." **Exercise:** Which part of this statement is confusing and which part is wrong?

2.2.2 Cauchy Problem: What to expect?: Three examples

We consider three Cauchy problems for linear PDEs where the PDEs can be solved explicitly. These three examples illustrate that all three possibilities concerning a mathematical problem can occur, namely,

(i) Cauchy problem has a unique solution. (Example 2.15)

(ii) Cauchy problem has an infinite number of solutions. (Example 2.16)

(iii) Cauchy problem has no solution. (Example 2.17)

Consider the following equation

$$u_x = cu + d(x, y), \tag{2.12}$$

where $c \in \mathbb{R}$ and d is a continuously differentiable function. The equation (2.12) can be thought of as an ODE where y appears as a parameter. Its explicit solution is given by

$$u(x, y) = e^{cx} \left(\int_0^x e^{-c\xi} d(\xi, y)\, d\xi + u(0, y) \right) \tag{2.13}$$

Example 2.15 (Cauchy Problem 1: Existence of a Unique solution) *The Cauchy data is prescribed on the Y-axis:*

$$u_x = cu + d(x, y),\ u(0, y) = y. \tag{2.14}$$

The unique solution is given by

$$u(x, y) = e^{cx} \left(\int_0^x e^{-c\xi} d(\xi, y)\, d\xi + y \right). \quad \blacksquare$$

Example 2.16 (Cauchy Problem 2: Non-uniqueness of solutions) *Cauchy data is prescribed on the X-axis:*

$$u_x = cu,\ u(x, 0) = e^{cx}. \tag{2.15}$$

This Cauchy problem has infinitely many solutions:

$$u(x, y) = e^{cx}\, T(y),$$

T is any function of a single variable such that $T(0) = 2$. $\blacksquare$

Example 2.17 (Cauchy Problem 3: Non-Existence of solutions) *Cauchy data is prescribed on the X-axis:*

$$u_x = cu,\ u(x, 0) = \sin x. \tag{2.16}$$

This Cauchy problem has no solution. For, if it has a solution then, in view of the formula (2.13)*, the solution satisfies*

$$\sin x = u(x, 0) = e^{cx} u(0, 0), \text{ for all } x \in \mathbb{R}.$$

The above equation cannot hold and hence the Cauchy problem has no solution. $\blacksquare$

Summary

Observe that in these examples, when the Cauchy data was prescribed on X axis we encountered the non-existence or multiplicity of solutions to the Cauchy problems, while prescribing Cauchy data on Y-axis gave us a unique solution. The following question also arises due to these examples: What was so nice about Y-axis w.r.t. the given PDE and what was bad about X-axis in the same context. Thus, if we want a well-posed Cauchy problem, these three examples tell us that we cannot prescribe Cauchy data on arbitrary curves in the XY-plane.

We will discuss these examples in Remark 2.21.

2.2.3 Method of Characteristics

To determine (prove the existence of) a solution of the equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{2.17}$$

satisfying the condition

$$u\left(f(s), g(s)\right) = h(s), \quad s \in I \tag{2.18}$$

where I is an interval in $\mathbb{R}$. *i.e.,* the integral surface passes through the curve

$$\Gamma: \; x = f(s),\; y = g(s),\; z = h(s), \quad s \in I. \tag{2.19}$$

Assume f, g, h are continuously differentiable in a neighbourhood of $s = s_0 \in I$.
In view of Theorem 2.9, in order to find a solution of the Cauchy problem for (2.17) we try to find the integral surface and for that it is a natural idea to find characteristics passing through points on Γ near $P_0 \in \Gamma$.

$$\text{Let } P_0 = (x_0, y_0, z_0) = \left(f(s_0), g(s_0), h(s_0)\right). \tag{2.20}$$

The characteristic differential equations of (2.17) are the system of ODE

$$\frac{dx}{dt} = a(x,y,z) \tag{2.21a}$$

$$\frac{dy}{dt} = b(x,y,z) \tag{2.21b}$$

$$\frac{dz}{dt} = c(x,y,z). \tag{2.21c}$$

Solve the above system (2.21) with the initial conditions at $t = 0$,

$$x = f(s),\; y = g(s),\; z = h(s) \quad \text{for every } s \text{ near } s_0. \tag{2.22}$$

Since a, b, c are continuously differentiable functions, there exists a unique solution of the Initial Value Problem (2.21)-(2.22). Denote this solution by

$$x = X(s,t),\; y = Y(s,t),\; z = Z(s,t). \tag{2.23}$$

Question: What is the domain of the vector-valued function $(s,t) \longmapsto (X(s,t), Y(s,t), Z(s,t))$?[2]
Answer: Note that for each fixed $s \in I$, $(X(s,t), Y(s,t), Z(s,t))$ are solutions to characteristic system of ODEs on a t interval which *a priori* depends on the s, say J_s. So, the domain is $I \times J_s$. If we want to consider a range of s, then we must intersect $I \times J_s$ to get a domain independent of s. that is,

$$\bigcap_{s \in I} I \times J_s$$

We do not know what kind of set this is. Can we assure that there is some open set inside this set? We need such a thing in order to even think about applying Inverse function theorem. Luckily we can choose an interval J independent of s provided we are willing to restrict ourselves to a fixed neighbourhood of $s = s_0$.[3],[4] Note the functions $x = X(s,t)$, $y = Y(s,t)$, $z = Z(s,t)$ satisfy

$$X(s_0, 0) = x_0 = f(s_0),\; Y(s_0, 0) = y_0 = g(s_0),\; Z(s_0, 0) = z_0 = h(s_0) \text{ and} \tag{2.24}$$

[2] Answering this question is very important and quite non-trivial to say that this domain has an open set in it. Many books do not address this question. They directly apply Inverse function theorem without caring for the details other than a certain non-vanishing of a certain Jacobian!

[3] This is a key-idea in proving continuous dependence of solutions to initial value problems for Ordinary differential equations. Refer to Wolfgang's ODE book, or my lecture notes on ODE.

[4] Recall that Cauchy-Lipschitz-Picard theorem gives a lower estimate about the set J_s, of course for each fixed s and that involves the largest possible rectangle centered at the initial condition that we can put inside certain domain, Lipschitz constant etc. However if the characteristic vector field is such that one can have a global solution for the characteristic system, then we can take $J_s = \mathbb{R}$ and we overcome the problem of choosing a J which is independent of s.

$$\frac{dX}{dt} = a\left(X(s,t), Y(s,t), Z(s,t)\right) \tag{2.25a}$$
$$\frac{dY}{dt} = b\left(X(s,t), Y(s,t), Z(s,t)\right) \tag{2.25b}$$
$$\frac{dZ}{dt} = c\left(X(s,t), Y(s,t), Z(s,t)\right) \tag{2.25c}$$

The equations (2.23) represent the parametric equations of an integral surface, provided we can solve for (s,t) in terms of x, y from the two equations $x = X(s,t)$, $y = Y(s,t)$ and then substitute for s, t in terms of x, y in $z = Z(s,t)$. This is achieved by applying Inverse function theorem for the vector-valued function

$$(s,t) \in I \times J \longmapsto \left(X(s,t), Y(s,t)\right).$$

Several things needs to be checked for applying Inverse function theorem. The first thing is the map given above must be continuously differentiable (total differentiability). It is enough if we know that partial derivatives exist and are continuous on $I \times J$. Can you prove this?[5] Once we prove this, we can go ahead and check the other conditions in the hypothesis of Inverse function theorem. Applying Inverse function theorem yields a neighbourhood of $(s_0, 0)$ and a neighbourhood of $(f(s_0), g(s_0))$ such that inversion is possible[6] and we get

$$s = S(x,y),\ t = T(x,y). \tag{2.26}$$

Define

$$u(x,y) = Z\left(S(x,y), T(x,y)\right). \tag{2.27}$$

Then u solves the PDE (2.17)[7] Then the integral surface defined by u is given by

$$S:\ z = u(x,y) = Z\left(S(x,y), T(x,y)\right). \tag{2.28}$$

The equation (2.28) represents a continuously differentiable solution of the Cauchy problem in a neighbourhood of (x_0, y_0, z_0).
Thus a sufficient condition for (2.28) to be a continuously differentiable integral surface is that

$$x = X(s,t),\ y = Y(s,t),\ x_0 = X(s_0,0) = f(s_0),\ y_0 = Y(s_0,0) = g(s_0), X(s,0) = f(s), Y(s,0) = g(s) \tag{2.29}$$

$$J = \frac{\partial(X,Y)}{\partial(s,t)}\Big|_{(s,t)=(s_0,0)} = \begin{vmatrix} X_s(s_0,0) & X_t(s_0,0) \\ Y_s(s_0,0) & Y_t(s_0,0) \end{vmatrix} \neq 0. \tag{2.30}$$

That is,

$$\begin{vmatrix} f'(s_0) & a\left(f(s_0), g(s_0), h(s_0)\right) \\ g'(s_0) & b\left(f(s_0), g(s_0), h(s_0)\right) \end{vmatrix} \neq 0. \tag{2.31}$$

Remark 2.18 *The surface S given by the equation* (2.28) *has parametric equations*

$$x = X(s,t),\ y = Y(s,t),\ z = Z(s,t). \tag{2.32}$$

The curves $s =$ constant *are the characteristic curves of equation* (2.17)*. The tangential direction at any point of the curve $s = c$ is given by (X_t, Y_t, Z_t) at $s = c$. This direction is also tangential to the surface S and hence the normal to S is perperndicular to the characteristic direction at any point. This implies S is an integral surface.*

[5] This is an important exercise.

[6] **Exercise:** State Inverse function theorem. Apply it in the present situation and write down the conclusion CORRECTLY.

[7] **Exercise:** Prove that u is indeed a solution. Hint: Compute the derivatives and substitute in the equation.

Uniqueness: If S_1 is any other integral surface containing Γ, then S_1 must contain the characteristic curves of equation (2.28) through Γ (why?). The integral surface S containing the characteristic curves of equation (2.28) passing through pionts of Γ in a neighbourhood of s_0 *i.e.,* of the point $P_0 = (x_0, y_0, z_0)$. Hence

$$S_1 \equiv S \quad \text{in a neighbourhood of } \; P_0. \tag{2.33}$$

Remark 2.19 *The curve Γ is called Data curve or Datum curve.*

Remark 2.20 *$J = 0$ implies Γ is a characteristic curve. If $J = 0$ at some point s_0 on an integral surface $z = u(x, y)$ passing through Γ (if exists), then*

$$\frac{f'}{a} = \frac{g'}{b} = \frac{h'}{c} \quad \text{at } s = s_0 \tag{2.34}$$

and hence Γ has the characteristic direction at s_0. This is because

$$J = 0 \Longleftrightarrow bf' - ag' = 0. \tag{2.35}$$

Now

$$\begin{aligned}
cg' - bh' = \;& c\left(f(s_0), g(s_0), h(s_0)\right) g'(s_0) - b\left(f(s_0), g(s_0), h(s_0)\right) h'(s_0) \\
= \;& \left[a\left(f(s_0), g(s_0), h(s_0)\right) u_x(f(s_0), g(s_0)) + b\left(f(s_0), g(s_0), h(s_0)\right) u_y(f(s_0), g(s_0))\right] g'(s_0) \\
& \quad -b\left(f(s_0), g(s_0), h(s_0)\right)\left[u_x(f(s_0), g(s_0))f'(s_0) + u_y(f(s_0), g(s_0))g'(s_0)\right] \\
= \;& \left[a\left(f(s_0), g(s_0), h(s_0)\right) g'(s_0) - b\left(f(s_0), g(s_0), h(s_0)\right) f'(s_0)\right] u_x(f(s_0), g(s_0)) = 0,
\end{aligned}$$

since $u_x(f(s_0), g(s_0))$ is a real number. Thus we have proved that if $J \equiv 0$ on I, then the curve Γ has characteristic direction at all of its points. Hence Γ is a characteristic. ■

Characteristic Cauchy Problem

If $J = 0$ for all $s \in I$, then by Remark 2.20, Γ has the characteristic direction at every $s \in I$ and Γ is a characteristic curve.

To find an integral surface passing through Γ

Choose any point P_0 on Γ and any curve Γ_1 through P_0 satisfying $J_{\Gamma_1}(P_0) \neq 0$. Then there exists a unique integral surface of equation (2.17) passing through Γ_1 in a neighbourhood of P_0. Then the integral surface containing Γ_1 (exists by our theory as $J_{\Gamma_1}(P_0) \neq 0$) also contains a part of Γ containing P_0, since Γ is a characteristic curve through P_0.

Then there are infinitely many solutions to the Characteristic Cauchy problem corresponding to infinitely many ways of choosing Γ_1. Proof of this is left as an exercise.

Remark 2.21 *If $J \equiv 0$, then two cases arise: either there are infinitely many solutions or no solution. We constructed an infinite number of solutions when the initial curve Γ is a characteristic curve. If Γ is not a characteristic curve, we could not prove the existence of an integral surface containing Γ; but can we prove that no one can prove the existence of a solution*[8]*. By Remark 2.20, we can only conclude that $J_\Gamma \equiv 0$ and Γ is not a characteristic curve are incompatible if $u_x(f(s_0), g(s_0))$ is a real number. So we can conclude that u is not a smooth function at points on Γ if it exists. So we can conclude that there is no smooth integral surface that contains Γ when Γ is not a characteristic but $J \equiv 0$.*

Let us return to the three examples that we considered in Subsection 2.2.2. In Example 2.15, Jacobian is non-zero and hence we have a unique solution. In Example 2.16, $J \equiv 0$ and the initial curve is a characteristic curve and hence it has infinitely many solutions. In Example 2.17, $J \equiv 0$ and the initial curve is not a characteristic curve and hence it has no solutions.

[8] Think about this.

Exercise 2.22 *Consider the first order quasi-linear PDE given by*

$$uu_x + u_y = 1. \tag{2.36}$$

Solve the Cauchy problems associated with (2.36) *and each of the following Cauchy data.*

(1) *(Jacobian is non-vanishing)*

$$x = s,\ y = s,\ z = \frac{s}{2},\ 0 \le s \le 1.$$

(2) *(Jacobian is identically equal to zero, infinite number of solutions to Cauchy problem)*

$$x = \frac{s^2}{2},\ y = s,\ z = s,\ 0 \le s \le 1.$$

Find all solutions and sketch some of them.

(3) *(Jacobian is identically equal to zero, no solutions to Cauchy problem)*

$$x = s^2,\ y = 2s,\ z = s,\ 0 \le s \le 1.$$

Try solving this problem and see what happens.

(4) *(Jacobian is identically equal to zero on a subset)*

$$x = s^2,\ y = 2s,\ z = s,\ 0 \le s \le 3.$$

Try solving this problem and see what happens.

Exercise 2.23 *Solve the equation* $u_x + 3y^{\frac{2}{3}} u_y = 2$ *subject to the condition* $u(x,1) = 1 + x$. *Discuss how the non-smoothness of characteristic vector field affects the solution. (Answer:* $u(x,y) = x + y^{\frac{1}{3}}$*)*

Some more terminology

(1) Characteristic direction (a, b, c) is also called **Monge's direction**.

(2) Characteristic equation is also called **Monge's equation**.

(3) Characteristic curves is also called **Monge's curves**.

2.2.4 Basic Existence theorem for Cauchy problem

We are considering the Quasi-linear PDE given by

$$a(x,y,u)\, u_x + b(x,y,u)\, u_y = c(x,y,u) \tag{2.37}$$

and the associated Cauchy Problem

$$\Gamma:\ x = f(s),\ y = g(s),\ z = h(s), \quad s \in I. \tag{2.38}$$

Definition 2.24 (Transversality condition) *The quasi-linear equation* (2.37) *and the initial curve* Γ *given by* (2.38) *are said to satisfy* **transversality condition at a point** $(f(s), g(s), h(s)) \in \Gamma$ *(or equivalently, at* $s \in I$*) if the base characterstics corresponding to the characteristic curve passing through the point* $(f(s), g(s), h(s))$ *intersects the projection of the initial curve* Γ **non-tangentially** *(i.e., the tangential directions for these intersecting curves are not parallel). That is,*

$$J|_{t=0} = \begin{vmatrix} X_s(s_0,0) & X_t(s_0,0) \\ Y_s(s_0,0) & Y_t(s_0,0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & a\left(f(s_0), g(s_0), h(s_0)\right) \\ g'(s_0) & b\left(f(s_0), g(s_0), h(s_0)\right) \end{vmatrix} \neq 0. \tag{2.39}$$

Theorem 2.25

(1) *Assume that the functions a, b, c appearing in the quasi-linear PDE* (2.37) *are smooth functions on an open set containing the initial curve* (2.38).

(2) *Let* $P_0 = (f(s_0), g(s_0), h(s_0)) \in= \Gamma$.

Then

(i) *If the transversality condition holds at each point s in the interval $(s_0 - \delta, s_0 + \delta)$, then the Cauchy problem* (2.37)-(2.38) *has a unique solution in a neighbourhood of the point P_0. That is there exists an $\epsilon > 0$ such that the Cauchy problem* (2.37)-(2.38) *has a unique solution in a region defined by $(s, t) \in (s_0 - \delta, s_0 + \delta) \times (-\epsilon, \epsilon)$*[9].

(ii) *If the transversality condition does not hold for s in an interval containing s_0, then the Cauchy problem* (2.37)-(2.38) *has either no solution at all or it has infinitely many solutions.*

Exercise 2.26 *Explain why the procedure given to construct a solution when $J \equiv 0$ fails in the case where there is no solution for the Cauchy problem.*

Exercise 2.27 *Go back to the proof of this theorem. Explain what are the reasons why we expect only a local solution. A solution of the Cauchy problem around a fixed point P on the initial curve Γ. Illustrate with one example each for your reasons.*

2.2.5 Difficulties in having global solutions to Cauchy problem: How to overcome them?

(1) Even when the PDE is linear, the characterisitc equations are nonlinear. From the theory of Ordinary differential equations we know that, in general, we can assert only the local existence of solutions to IVPs for nonlinear ODEs even when the vector fields are smooth. This immediately suggests that we can expect at most a local existence theorem even for a linear first-order PDE, and hence for the quasi-linear first order PDE. Thus we understand that in order to have a 'global existence theorem' for quasi-linear PDE, then the coefficients should satisfy extra hypothesis so that at least the corresponding characterisitc system of ODEs have global solutions.

(2) The parametric representation of an integral surface might be misleading. We have seen many examples where the integral surface is described by smooth functions (a consequence of dependence of solutions to IVPs for ODEs on initial conditions: More precisely dependence of solutions on t is due to the characterisitc vector field *i.e.,* the PDE; and the dependence on s is coming through IVPs for characteristic system of ODEs and also on the description of the initial curve) in (s, t)-variables. But in terms of (x, y)-variables it is not smooth. Inverse function theorem plays its role here. Now either we will be able to apply Inverse function theorem or we can not apply. For a given quasi-linear PDE, we can always find initial curves such that traversality condition does not hold, and hence we can not apply Inverse function theorem. Even when we can apply Inverse function theorem, the conclusions are only local. A good question to think about is "When can we have global conclusions from Inverse function theorem? When is such a situation arise for a quasi-linear PDE?"

(3) A characteristic curve might intersect the initial curve more than once. Since a characteristic curve carries information from the initial curve, when a characteristic curve intersects the initial curve more than once, the information carried from different intersection points should not be in conflict if we expect to have global solutions for the Cauchy problem. Also base characteristics corresponding to distinct characteristics should not intersect for a similar reason if we are hoping for a global solution for the Cauchy problem.

[9] visualise this geometrically

(4) For ODEs when the vector field is defined in the whole space, through every point there passes a solution curve. For quasi-linear first order PDE, characteristic curves may not pass through some parts of $\mathbb{R}^3$ and hence information from the initial curves does not propagate in those areas. This suggests the concepts of domain of influence (of initial curve), domain of dependence (of a solution at a point on the initial curve), finite speed of propagation (of information from the initial curve when one of the independent variables has the interpretation of time).