Chapter 5

Laplace Equation

The following equation is called **Laplace equation** in two independent variables x, y:

$$u_{xx} + u_{yy} = 0. \tag{5.1}$$

The non-homogeneous problem

$$u_{xx} + u_{yy} = F, \tag{5.2}$$

where F is a function of the independent variables x, y only is called the **Poisson equation**.

Definition 5.1 (Solution) (i) *A twice continuously differentiable function $\phi : D \longrightarrow \mathbb{R}$ is said to be a solution of the Laplace equation* (5.1) *if*

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = 0 \quad \forall (x, y) \in D. \tag{5.3}$$

(ii) *Let F be a given continuous function on D. A twice continuously differentiable function $\phi : D \longrightarrow \mathbb{R}$ is said to be a solution of the Poisson equation* (5.2) *if*

$$\phi_{xx}(x, y) + \phi_{yy}(x, y) = F(x, y) \quad \forall (x, y) \in D. \tag{5.4}$$

Boundary Value Problems associated to Laplace Equation

For domains other than the whole XY-plane, there is always a boundary and we can impose restrictions on the unknown function u or its derivatives or a mix of both on the boundary. These kinds of problems are called boundary value problems.

There are at least three boundary value problems associated Laplace and Poisson equations. For simplicity, we describe them only for Laplace equation and the corresponding notions can be easily extended to Poisson equation.

(1) **Dirichlet Problem** Let D be a domain in $\mathbb{R}^2$ with piecewise smooth boundary and f be a continuous function on the closure of D. To find a solution u of (5.1) in the domain D such that

- (i) u is continuous on closure of D and
- (ii) $u|_{\partial D} = f$

If D is the unit disk, then corresponding Dirichlet problem is called **interior Dirichlet problem**. If D is the complement of the closed unit disk, then corresponding Dirichlet problem is called **exterior Dirichlet problem**.

(2) **Second Boundary Value Problem a.k.a. Neumann Problem** To find a solution u of (5.1) in the domain D such that

- (i) u, u_x, u_y are continuous on closure of D and

(ii) at every point of ∂D, the directional derivative of u in the direction of normal (denoted by $\frac{\partial u}{\partial n}$ or $\partial_{\mathbf{n}} u$) satisfies $\partial_{\mathbf{n}} u = f$.

(3) **Third Boundary Value Problem a.k.a. Robin Problem** To find a solution u of (5.1) in the domain D such that

(i) u, u_x, u_y are continuous on closure of D and

(ii) at every point of ∂D, $u + \alpha\frac{\partial u}{\partial n} = f$, where α is a given constant.

Remark 5.2

(i) *There are other kinds of boundary value problems possible. For example, on a part of the boundary ∂D one of the three boundary value problems described above is imposed and on the remaining part another one of the above three. We do not consider such boundary value problems here.*

(ii) *Cauchy-Kowalewski theorem guarantees that a solution of an analytic Cauchy problem for an elliptic equation exists and is unique (locally), but is not always well-posed.* ∎

Hadamard's Example

We proved that Cauchy problem for the Wave equation is well-posed for $x \in \mathbb{R}$ and $t \in [0, T]$ for every $T > 0$. Let us consider a similar problem for the Laplace equation now. More precisely, we consider the following Cauchy problem for Laplace equation in the upper half-plane:

$$\Delta u = 0 \quad \text{on } \mathbb{R} \times (0, \infty), \tag{5.5a}$$
$$u(x, 0) = f(x) \quad \text{for all } x \in \mathbb{R}, \tag{5.5b}$$
$$u_y(x, 0) = g(x) \quad \text{for all } x \in \mathbb{R}. \tag{5.5c}$$

Note that we proved the well-posedness of Cauchy problem for Wave equation, the initial conditions were exactly same as (5.5b)-(5.5c), except that the Laplace equation (5.5b) was replaced with Wave equation.

Hadamard proved that Cauchy problem (5.5) is not well-posed on $\mathbb{R} \times [0, T]$ for any $T > 0$, by proving that stability estimate does not hold. For the moment, let us accept that solution to Cauchy problem exists and is unique[1]. If the problem were to be well-posed, the following stability estimate is expected to be satisfied: Given $\epsilon > 0$, there exists a $\delta > 0$ such that for every f_1, f_2, g_1, g_2 satisfying

$$|f_1(x) - f_2(x)| + |g_1(x) - g_2(x)| < \delta \;\; \forall x \in \mathbb{R}, \tag{5.6}$$

the corresponding solutions u_1, u_2 of the Cauchy problem satisfy

$$|u_1(x, y) - u_2(x, y)| < \epsilon \;\; \forall x \in \mathbb{R},\, y \geq 0. \tag{5.7}$$

What should we do to prove that solutions to Cauchy problem do not have stability?[2] We produce a sequence $(f_n), (g_n)$ which are "close" to the zero function but the corresponding solutions are "far" from the function $u(x, y) \equiv 0$ which is the solution with zero Cauchy data. Consider f_n, g_n given by

$$f_n(x) \equiv 0, \quad g_n(x) = \frac{\sin nx}{n}, \tag{5.8}$$

[1]This assumption says that if we somehow find a solution of the Cauchy problem then that is the only solution.
[2]Formulate the negation and convince yourself that what we are dong here is indeed equivalent.

note that they are "very close" to the function zero. The solution to Cauchy problem with the Cauchy data (5.8) is given by

$$u_n(x,y) = \frac{1}{n^2}\sin(nx)\,\sinh(ny). \tag{5.9}$$

For n large, the initial conditions are very close to zero and hence can be thought of as a perturbation of the zero initial state. However the corresponding solution of Cauchy problem $u_n(x,y)$, due to the presence of hyperbolic sine function, is unbounded even on $[\mathbb{R}\times[0,T]$ for any $T>0$.

Summary': This example shows the difference between the Cauchy problems for Wave and Laplace equations posed on the upper half-plane. Even though the nature of the Cauchy data imposed is the same, changing the equation from Wave to Laplace changes the stability property drastically.

5.1 Green's identities

Green's Identities form an important tool in the analysis of Laplace equation. They all are derived from divergence theorem, equivalently from Integration by parts formula. Let us recall the divergence theorem now.

Theorem 5.3 (Divergence theorem) *Let $D\subseteq\mathbb{R}^2$ be a bounded piecewise smooth domain. Let $\Psi: D\to\mathbb{R}^2$ be a function. Let $\Psi=(\psi_1,\psi_2)$ where $\psi_i\in C^1(\overline{D})\cap C(\overline{D})$ for $i=1,2$. Then*

$$\int_D \nabla\bullet\Psi(x,y)\,dx\,dy = \int_{\partial D}\Psi(x(s),y(s))\bullet\mathbf{n}(x(s),y(s))\,ds, \tag{5.10}$$

where $\mathbf{n}(x(s),y(s))$ is the unit outward normal at the point $(x(s),y(s))\in\partial D$. ■

Green's Identity-I

Let $u: D\to\mathbb{R}$ be such that $u\in C^2(\overline{D})\cap C^1(\overline{D})$. Now apply (5.10) with $\Psi=\nabla u$ to get Green's identity-I:

$$\int_D \Delta u(x,y)\,dx\,dy = \int_{\partial D}\nabla u(x(s),y(s))\bullet\mathbf{n}(x(s),y(s))\,ds \tag{5.11}$$

With the notation $\partial_{\mathbf{n}}u := \nabla u\bullet\mathbf{n}$, Green's identity-I takes the form

$$\int_D \Delta u(x,y)\,dx\,dy = \int_{\partial D}\partial_{\mathbf{n}}u\,ds \tag{5.12}$$

The quantity $\partial_{\mathbf{n}}u$ is called the normal derivative of u; as it represents the directional derivative of u in the dirction of the outward unit normal $\mathbf{n}$.

Green's Identity-II

Let $u: D\to\mathbb{R}$ be such that $u\in C^2(\overline{D})\cap C^1(\overline{D})$. Now apply (5.10) with $\Psi=v\nabla u-u\nabla v$ to get Green's identity-II:

$$\int_D \big(v\Delta u-u\Delta v\big)(x,y)\,dx\,dy = \int_{\partial D}\big(v\,\partial_{\mathbf{n}}u-u\,\partial_{\mathbf{n}}v\big)\,ds \tag{5.13}$$

Green's Identity-III

Let $u: D\to\mathbb{R}$ be such that $u\in C^2(\overline{D})\cap C^1(\overline{D})$. Now apply (5.10) with $\Psi=v\nabla u$ to get Green's identity-III:

$$\int_D \nabla u\bullet\nabla v\,dx\,dy = \int_{\partial D} v\,\partial_{\mathbf{n}}u\,ds - \int_D v\Delta u\,dx\,dy. \tag{5.14}$$

5.1.1 Consequences of Green's identities

Lemma 5.4 *Let $F \in C(\overline{D})$. If u solves $\Delta u = F$ on a domain D, then*

$$\int_D F(x,y)\,dx\,dy = \int_{\partial D} \partial_\nu u\,d\sigma. \tag{5.15}$$

Proof: Integrating both sides of the equation $\Delta u = F$ on D yields

$$\int_D F(x,y)\,dx\,dy = \int_D \Delta u(x,y)\,dx\,dy \tag{5.16}$$

Applying Green's identity-I (5.12), the integral on the right hand side of the equation (5.16) becomes

$$\int_D \Delta u(x,y)\,dx\,dy = \int_{\partial D} \partial_{\mathbf{n}} u\,ds. \tag{5.17}$$

This finishes the proof. ∎

In particular, when u solves $\Delta u = 0$, we have

$$\int_{\partial D} \partial_{\mathbf{n}} u\,d\sigma = 0. \tag{5.18}$$

Theorem 5.5 (Uniqueness theorem) *Let D be a smooth domain. Then*

(a) *The Dirichlet problem has at most one solution.*

(b) *If u solves the Neumann problem, then any other solution is of the form $v = u + c$ for some real number c.*

(c) *If $\alpha \geq 0$, then the Robin problem has at most one solution.*

Proof: **Proof of (c):** Note that (a) is a special case of (c), hence it is enough to prove (c). Let u_1, u_2 be solutions of the Robin problem. Define $w := u_1 - u_2$. Observe that w is a harmonic function(why?), and w satisfies the boundary condtion

$$w + \alpha \partial_{\mathbf{n}} w = 0. \tag{5.19}$$

Using the Green's identity-III (5.14) with $u = v = w$, we get

$$\int_D |\nabla w|^2\,dx\,dy = -\alpha \int_{\partial D} \left(\partial_{\mathbf{n}} w\right)^2\,ds. \tag{5.20}$$

Since the left hand side of (5.20) is non-negative while the right hand side is non-positive, we conclude that both sides must be zero. This implies that $\nabla w = 0$ in D and also $\partial_{\mathbf{n}} w = 0$ on ∂D. Since $\nabla w = 0$ in D, w must be a constant function. Since $\partial_{\mathbf{n}} w = 0$, in view of the equality (5.19), we conclude that $w = 0$ on the boundary ∂D. Since $w \in C(\overline{D})$, it follows that $w \equiv 0$ in D. This finishes the proof of (c).

Proof of (b): Let u_1, u_2 be solutions of the Neumann problem. Define $w := u_1 - u_2$. Observe that w is a harmonic function(why?), and w satisfies the boundary condtion

$$\partial_{\mathbf{n}} w = 0. \tag{5.21}$$

Using the Green's identity-III (5.14) with $u = v = w$, we get

$$\int_D |\nabla w|^2\,dx\,dy = 0. \tag{5.22}$$

This implies that w is a constant function. Thus $u_1 = u_2 + c$ for some constant $c \in \mathbb{R}$. ∎

5.2 Weak Maximum Principle

Let $D \subseteq \mathbb{R}^2$ be a bounded domain. Let $u : D \longrightarrow \mathbb{R}$ be a continuous function such that u can be extended to the closure of D as a continuous function. Such class of functions is denoted by $C(\overline{D})$. When D is bounded every function in $C(\overline{D})$ attains both its maximum and minimum values, somewhere in $\overline{D}$. Weak Maximum principle says that if u happens to be a harmonic function, then the maximum and minimum values of u are definitely attained on the boundary of the domain D whether or not they are attained in D.

Theorem 5.6 (The Weak Maxmimum Principle) *Let $D \subseteq \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D. Then the maximum value of u in $\overline{D}$ is achieved on the boundary ∂D.*

Proof: **Step 1:** Recall from differential calculus of two variables that at a point of interior maximum, $\frac{\partial^2 u}{\partial x^2} \leq 0$ and $\frac{\partial^2 u}{\partial y^2} \leq 0$. As a consequence, $\Delta u \leq 0$ at an interior maximum point. Thus if v is a function such that $\Delta v > 0$ in D, then the maximum value of v on $\overline{D}$ cannot be attained in D. Hence v attains its maximum value only on the boundary ∂D. The idea to prove Weak maximum principle is to find such a function v starting from the given harmonic function u.

Step 2: Define the function v_ϵ by

$$v_\epsilon(x, y) := u(x, y) + \epsilon(x^2 + y^2). \tag{5.23}$$

Then $v_\epsilon \in C^2(D) \cap C(\overline{D})$. Note that $\Delta v_\epsilon > 0$ in D and thus v_ϵ attains its maximum only on the boundary ∂D. Denoting

$$M := \mathrm{Max}_{\partial D} u, \ L := \mathrm{Max}_{\partial D}(x^2 + y^2), \tag{5.24}$$

we have

$$v_\epsilon(x, y) \leq M + \epsilon L, \ \forall (x, y) \in D. \tag{5.25}$$

Since $u(x, y) \leq v_\epsilon(x, y)$ for all $(x, y) \in D$, we have

$$u(x, y) \leq M + \epsilon L, \quad \forall (x, y) \in D. \tag{5.26}$$

Note that last inequality holds for every $\epsilon > 0$. Thus taking limits as $\epsilon \to 0$, we get

$$u(x, y) \leq M, \quad \forall (x, y) \in D. \tag{5.27}$$

This finishes the proof. ■

Corollary 5.7 (The weak minimum principle) *Let $D \subseteq \mathbb{R}^2$ be a bounded domain. Let $u \in C^2(D) \cap C(\overline{D})$ be a harmonic function in D. Then the minimum value of u in $\overline{D}$ is achieved on the boundary ∂D.*

Proof: Define by $v(x, y) = -u(x, y)$ on D. Then v is a harmonic function. Now apply the weak maximum principle to the harmonic function v and conclude. ■

Remark 5.8 *We have to understand clearly what the Weak maximum principle says and what it does not talk about. The weak maximum principle says that on bounded domains, any harmonic function takes its maximum value on the boundary surely. The weak maximum principle is silent on whether the harmonic function will take or will not take the maximumum value in the domain.*

For example, consider D to be the unit disk centered at the origin. Then the constant function $u \equiv 1$ is definitely a harmonic function in D. It attains its maximum in D also, apart from attaining on ∂D.

There is another maximum principle, called **Strong maximum priniciple***, which says that on bounded domains non-constant harmonic functions will never attain maxima in D and maxima are attained only on the boundary ∂D.* ■

5.2.1 Consequences of Weak Maximum Principle

We already discussed uniqueness of solutions to all the three boundary value problems for Laplace equation posed on bounded domains using Green's identities. For Wave and Heat equations, we discussed uniqueness via Energy method and same analysis can be doen for Laplace equation as well.

We are now going to discuss uniqueness questions concerning the boundary value problems using another tool, namely, the weak maximum principle for harmonic functions on a bounded domain.

Theorem 5.9 (Uniqueness of solutions to Dirichlet problem) *Let $D \subseteq \mathbb{R}$ be a bounded domain and consider the Dirichlet problem on D:*

$$\begin{aligned} \Delta u &= f \text{ on } D, \\ u &= g \text{ on } \partial D, \end{aligned}$$

where g is a continuous function on ∂D. Then the Dirichlet problem has at most one solution in the class $C^2(D) \cap C(\overline{D})$.

Proof: Assume that u and v belonging to the class $C^2(D) \cap C(\overline{D})$ solve the Dirichlet problem. Define $w := u - v$. Then w belongs to the class $C^2(D) \cap C(\overline{D})$ and solves the homogeneous Dirichlet problem:

$$\begin{aligned} \Delta w &= 0 \text{ on } D, \\ w &= 0 \text{ on } \partial D. \end{aligned}$$

By weak maximum principle $w \leq 0$ and by weak minimum principle $w \geq 0$. Thus $w \equiv 0$. This finishes the proof of the theorem. ∎

Remark 5.10 *In the above theorem, it is essential that D is a bounded domain. For unbounded domains the theorem may not hold good. For example, consider the following Dirichlet problem on the upper half plane:*

$$\begin{aligned} \Delta u(x,y) &= 0 \text{ for } x \in \mathbb{R},\, 0 < y < \infty, \\ u(x,0) &= 1 \text{ for all } x \in \mathbb{R}. \end{aligned}$$

This problem has at least two solutions, namely $u_1(x,y) = xy$, and, $u_2(x,y) = 0$. ∎

Theorem 5.11 *Let $D \subseteq \mathbb{R}$ be a bounded domain. For $i = 1, 2$, let $u_i \in C^2(D) \cap C(\overline{D})$ solve the following Dirichlet problems on D:*

$$\begin{aligned} \Delta u_i &= f \text{ on } D, \\ u &= g_i \text{ on } \partial D, \end{aligned}$$

where g_i is a continuous function on ∂D for each $i = 1, 2$. Then u_1 and u_2 satisfy

$$Max_D |u_1(x,y) - u_2(x,y)| \leq Max_{\partial D} |g_1(x,y) - g_2(x,y)|. \tag{5.28}$$

Proof: Define $w := u_1 - u_2$. Then w solves the following Dirichlet problem

$$\begin{aligned} \Delta w &= 0 \text{ on } D, \\ w &= g_1 - g_2 \text{ on } \partial D. \end{aligned}$$

Applying the weak maximum principle and the weak minimum principle, we get

$$\text{Min}_{\partial D}(g_1 - g_2) \leq w(x,y) \leq \text{Max}_{\partial D}(g_1 - g_2) \quad \forall (x,y) \in D, \tag{5.29}$$

which finishes the proof of the theorem. ∎

5.3 Mean Value Property

Definition 5.12 *Let $D \subseteq \mathbb{R}^2$ be a domain. A continuous function $u : D \longrightarrow \mathbb{R}$ is said to possess the* **Mean Value Property-I on** *D if for every point $P = (x, y) \in D$ and for every $r > 0$ such that the open disk $B_r(P) \subsetneqq D$, the function u satisfies the relation*

$$u(x,y) = \frac{1}{\pi r^2} \iint\limits_{|\xi - x|^2 + |\eta - y|^2 \le r^2} u(\xi, \eta)\, d\xi\, d\eta. \tag{5.30}$$

Definition 5.13 *Let $D \subseteq \mathbb{R}^2$ be a domain. A continuous function $u : D \longrightarrow \mathbb{R}$ is said to possess the* **Mean Value Property-II on** *D if for every point $P = (x, y) \in D$ and for every $r > 0$ such that the open disk $B_r(P) \subsetneqq D$, the function u satisfies the relation*

$$u(x,y) = \frac{1}{2\pi r} \oint_{C_r} u(s)\, ds \tag{5.31}$$

where C_r denotes the circle of radius r that is centered at P.

On expanding the line integral in the equation (5.31), we get the following form of the equation (5.31):

$$u(x,y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta. \tag{5.32}$$

Lemma 5.14 *Let $u : D \longrightarrow \mathbb{R}$ be a continuous function. Then the following statements are equivalent.*

(i) *u has the mean value property-I on D.*

(ii) *u has the mean value property-II on D.*

Proof: We will show (ii) implies (i) and all the steps in this proof can be reversed thereby proving that (i) implies (ii). Let $P = (x, y) \in D$ be an arbitrary point. Further let $R > 0$ be such that the open disk $B_R(P) \subsetneqq D$. Then if u satisfies mean value property-II, then for every $0 < \tau \le R$ we have

$$u(x,y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + \tau\cos\theta, y + \tau\sin\theta)\, d\theta. \tag{5.33}$$

Multiply the equation (5.33) with τ and then integrate w.r.t. τ over the interval $[0, r]$. This yields

$$\int_0^r \tau\, u(x,y)\, d\tau = \frac{1}{2\pi} \int_0^r \left(\int_0^{2\pi} u(x + \tau\cos\theta, y + \tau\sin\theta)\, d\theta \right) \tau\, d\tau. \tag{5.34}$$

The last equation simplifies to

$$\begin{aligned} \frac{r^2}{2} u(x,y) &= \frac{1}{2\pi} \int_0^r \int_0^{2\pi} u(x + \tau\cos\theta, y + \tau\sin\theta)\, \tau\, d\theta\, d\tau \\ &= \frac{1}{2\pi} \iint_{|\xi - x|^2 + |\eta - y|^2 \le r^2} u(\xi, \eta)\, d\xi\, d\eta. \end{aligned}$$

Thus we get

$$u(x,y) = \frac{1}{\pi r^2} \iint\limits_{|\xi - x|^2 + |\eta - y|^2 \le r^2} u(\xi, \eta)\, d\xi\, d\eta, \tag{5.35}$$

which proves that u has mean value property-I as well. ■

Remark 5.15 *In view of Lemma 5.14, we say that a continuous function u has the mean value property on a domain D if either of the mean value properties holds on D and as a consequence both the mean value properties hold on D. The mean value property-I may also be called the Solid Mean Value Property (as the average is taken over the entire disk) and the mean value property-II may also be called Surface/Circle mean value theorem. This nomenclature is not standard.* ■

Theorem 5.16 (Mean Value Principle) *Let $D \subseteq \mathbb{R}^2$ be a domain and u be a harmonic function in D. Then u has the mean value property on D.*

Proof: Let $P_0 = (x_0, y_0) \in D$ and let $R > 0$ be such that the disk, denoted by $B_R(P_0)$, of radius R centered at (x_0, y_0) with the property $B_R(P_0) \subsetneqq D$. We have to prove that, the function u satisfies the relation

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x + R\cos\theta, y + R\sin\theta)\, d\theta., \tag{5.36}$$

Note that the left hand side of (5.36) is a real number, whereas the right hand side of (5.36) depends on r. Thus to prove (5.36), the strategy is to prove that the right hand side is a constant function of the variable r. So, let us consider $V(r)$ defined for $r \in (0, R]$ by

$$V(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta. \tag{5.37}$$

Let us compute $\frac{dV}{dr}$ for $0 < r < R$ and prove that it is zero. As a consequence, we will then have

$$V(r) = \lim_{\rho \to 0} V(\rho) = u(x_0, y_0), \tag{5.38}$$

which completes the proof of the mean value principle. It remain to show that $\frac{dV}{dr} = 0$ on $(0, R)$.

$$\frac{dV}{dr}(r) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} u(x + r\cos\theta, y + r\sin\theta)\, d\theta = \int_{\partial D} \partial_\nu u \, d\sigma = 0. \tag{5.39}$$

In proving the last equality, we have used the equation (5.17) and the fact that u is a harmonic function on D. ∎

Theorem 5.17 *If u is continuous and has the mean value property on a domain D, then it has continuous derivatives of all orders and all of them have the mean value property on D.*

Proof: **Step 1:** It is sufficient to show that the first order partial derivatives of u exist and are continuous. Once it is known that first order partial derivatives of u exist, it follows by computing partial derivatives from the equation

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\theta, y + r\sin\theta)\, d\theta \tag{5.40}$$

which holds as u satisfies mean value property on D to obtain

$$\begin{aligned} u_x(x, y) &= \frac{1}{2\pi} \int_0^{2\pi} u_x(x + r\cos\theta, y + r\sin\theta)\, d\theta, \\ u_y(x, y) &= \frac{1}{2\pi} \int_0^{2\pi} u_y(x + r\cos\theta, y + r\sin\theta)\, d\theta. \end{aligned}$$

By repeating the arguments by replacing u with its partial derivatives, we conclude that u has partial derivatives of all orders and all of them satisfy the mean value relation.

Step 2: Let us now prove that first order partial derivatives of u exist and are continuous. Let us recall the mean value relation (5.30) now.

$$u(x, y) = \frac{1}{\pi r^2} \iint\limits_{|\xi - x|^2 + |\eta - y|^2 \le r^2} u(\xi, \eta)\, d\xi\, d\eta. \tag{5.41}$$

Writing the double integral on the right hand side as iterated integrals, the equation (5.41) takes the form

$$u(x,y) = \frac{1}{\pi r^2} \iint\limits_{|\xi-x|^2+|\eta-y|^2\leq r^2} u(\xi,\eta)\,d\xi\,d\eta \quad = \quad \frac{1}{\pi r^2}\int_{y-r}^{y+r} d\eta \int_{x-\sqrt{r^2-(\eta-y)^2}}^{x+\sqrt{r^2-(\eta-y)^2}} u(\xi,\eta)\,d\xi$$

The last equation tells us that $u_x(x,y)$ exists, in view of fundamental theorem of integral calculus. In fact $u_x(x,y)$ is given by

$$u_x(x,y) = \frac{1}{\pi r^2}\int_{y-r}^{y+r} \left[u(x+\sqrt{r^2-(\eta-y)^2},\eta) - u(x-\sqrt{r^2-(\eta-y)^2},\eta)\right]\,d\eta. \tag{5.42}$$

Since the right hand side of the equation (5.42) is a continuous function of (x,y), we conclude that u_x is a continuous function on D. Similarly one can deduce the existence of u_y at each point and continuity of u_y on D. ∎

Theorem 5.18 *If $u : D \longrightarrow \mathbb{R}$ is continuous and u has the mean value property on D, then u is harmonic in D.*

Proof: If u has the mean value property, then by Theorem 5.17 we know that all partial derivatives of u exist and all of them are continuous on D. We want to show that $\Delta u = 0$. On the contrary, suppose that there exists a point P in D such that $\Delta u(P) \neq 0$; and without loss of generality assume that $\Delta u(P) > 0$. By continuity of the second order partial derivatives, there exists a disk of radius $\epsilon > 0$ centered at P (denote it by $B_\epsilon(P)$ on which $\Delta u > 0$. We now have

$$\begin{aligned} 0 < \int_{B_\epsilon(P)} \Delta u(x,y)\,dx\,dy &= \int_{C_\epsilon(P)} \partial_\nu u(s)\,ds \\ &= \epsilon \frac{\partial}{\partial \epsilon}\int_0^{2\pi} u(x+\epsilon\cos\theta, y+\epsilon\sin\theta)\,d\theta \\ &= \epsilon\frac{\partial}{\partial\epsilon}\left[2\pi u(x,y)\right] = 0. \end{aligned}$$

This contradiction means that our assumption that Δu at some point is non-zero is wrong. Thus u is a harmonic function. ∎

5.3.1 Consequences of Mean Value Property

Let $D \subseteq \mathbb{R}^2$ be a bounded domain and $u : D \longrightarrow \mathbb{R}$ be a harmonic function. The Weak maximum principle asserted that maximum value of u on the closure of D is attained on the boundary of D, and was silent about attaining the said maximum value in D. Strong maximum principle that we are going to prove says that if such a maximum is also attained inside D, then necessarily the harmonic function u is a constant function.

Theorem 5.19 (Strong Maximum Principle) *Let $D \subseteq \mathbb{R}^2$ be a domain (domain need not be bounded) and $u : D \longrightarrow \mathbb{R}$ be a harmonic function. If u attains its maximum in D, then u is constant.*

Proof: **Step 1:** Let u assume its maximum at $P_0 \in D$ and let this maximum value be denoted by M. We want to prove that u is the constant function that takes the value M everywhere in D. Let P be an arbitrary point in D. We wil show that $u(P) = M$.

Step 2: Join P to P_0 by a smooth curve γ[3]. Since γ is a compact set, it maintains a positive distance from ∂D (in case ∂D is non-empty), let us denote this distance by d_γ. Take a disk of

[3] This is possible since D is open and connected and hence it is path-connected. Prove this.

radius $\frac{d_\gamma}{2}$ with center at P_0 denoted by $B_{\frac{d_\gamma}{2}}(P_0)$. Since u is a harmonic function in D, it has the mean value property in D. Applying the mean value property[4] on this disk, we conclude that u equals M on the entire disk.

Step 3:(Continuation argument) We now take a point P_1 on γ which lies in the disk $B_{\frac{d_\gamma}{2}}(P_0)$ which is at least at a distance $\frac{d_\gamma}{4}$ from P_0. Note that $u(P_1) = M$. Now repeat the arguments of Step 2 with the disk $B_{\frac{d_\gamma}{2}}(P_1)$ and then find a P_2 in a similar way. Continuing this process till we get a $k \in \mathbb{N}$ with the property that $P \in B_{\frac{d_\gamma}{2}}(P_k)$[5]. We will then have $u(P) = M$. This finishes the proof of the theorem. ■

Remark 5.20 *This is a continuation of Remark 5.8. It is important to note that Strong maximum principle says that if a harmonic function attains its maximum in a domain D (bounded domain or otherwise), then it is necessarily a constant function. In particular Strong maximum principle does not talk about where the maximum will be attained; sometimes a harmonic function may not have a maximum value also. This happens when the domain D is not bounded, because in that case though u is continuous on $\overline{D}$ we cannot assert the existence of a maximum value, since $\overline{D}$ is not compact. See the next example in this context.* ■

Example 5.21 *Let D be the domain exterior to the unit disk. The function $u(x, y) = \log(x^2 + y^2)$ is a harmonic function on D. It has neither a maximum value nor a minimum value in D.* ■

Exercise 5.22 *Formulate the strong minimum principle and prove it by following the method of proof of the Strong maximum principle.*

[4]Mean Value Property-I or Mean Value Property-II ?
[5]Why is it possible to find such a k?