Chapter 2

First order PDE

2.1 ▪ How and why first order PDE appear?

2.1.1 ▪ Physical origins

Conservation laws form one of the two fundamental parts of any mathematical model of Continuum Mechanics. These models are PDEs. Discussion is beyond the scope of this course.

2.1.2 ▪ Mathematical origins

1. **Two-parameter family of surfaces:**[2] Let $f : \mathbb{R}^2 \times A \times B \to \mathbb{R}$ be a smooth function. Then

$$z = f(x, y, a, b), \tag{2.1}$$

roughly speaking, represents a two-parameter family of surfaces in $\mathbb{R}^3$. Differentiating (2.1) with respect to x and y yields the relations

$$z_x = f_x(x, y, a, b), \tag{2.2a}$$

$$z_y = f_y(x, y, a, b). \tag{2.2b}$$

Eliminating a and b from (2.1)-(2.2), we get a relation of the form

$$F(x, y, z, z_x, z_y) = 0.$$

This is a PDE for the unknown function of two independent variables.

2. **Unknown function of known functions:**

   (a) **Unknown function of a single known function:** Let $u = f(g)$ where f is an unknown function and g is a known function of two independent variables x and y. Differentiating $u = f(g)$ w.r.t. x and y yields the equations $u_x =$

[2] Let $y = f(x, c_1, c_2, \cdots, c_n)$ denote an n-parameter family of plane curves. By eliminating $c_1, \cdots, c_n$ we get an n-th order ODE of the form $F(x, y, y', \cdots, y^{(n)}) = 0$. If we consider a family of space curves, then we get systems of ODE after eliminating parameters. Of course, all this we get only if we are able to eliminate the parameters!

$f'(g)g_x$ and $u_y = f'(g)g_y$ respectively. Eliminating the arbitrary function f from these two equations, we obtain

$$g_y u_x - g_x u_y = 0,$$

which is a first order PDE for u.

(b) **Unknown function of two known functions:** Let

$$u = f(x - ay) + g(x + ay). \tag{2.3}$$

Denoting $v(x,y) = x - ay$ and $w(x,y) = x + ay$, the above equation becomes

$$u = f(v) + g(w). \tag{2.4}$$

where f, g are unknown functions and v, w are known functions.

Differentiating (2.4) with respect to x and y yields the relations

$$p = u_x = f'(x - ay) + g'(x + ay), \tag{2.5a}$$
$$q = u_y = -af'(x - ay) + ag'(x + ay). \tag{2.5b}$$

Eliminating f and g from (2.5a)-(2.5b) (after differentiating them w.r.t. y and x respectively), we get a relation of the form

$$q_y = a^2 p_x$$

In terms of u the above first order PDE is the well-known Wave equation

$$u_{yy} = a^2 u_{xx}.$$

2.2 ▪ Cauchy problem for quasilinear PDE

Consider the single quasilinear first order PDE given by

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u), \tag{2.6}$$

where a, b, c are continuously differentiable functions on a domain $\Omega \subseteq \mathbb{R}^3$. Let Ω_0 denote the projection of Ω to the xy-plane.

Definition 2.1 (Integral Surface). *Let $D \subseteq \Omega_0$ and $u : D \to \mathbb{R}$ be a solution of the equation (2.6). The surface S represented by $z = u(x,y)$ is called an integral surface corresponding to a given solution u.*

Let us introduce the notion of a Cauchy problem for the quasilinear PDE (2.6).

Cauchy Problem

Cauchy problem for the quasilinear PDE (2.6) consists of finding an integral surface $z = u(x,y)$ of the quasilinear PDE (2.6), containing a given space curve Γ whose parametric equations are

$$x = f(s),\ y = g(s),\ z = h(s),\ s \in I, \tag{2.7}$$

where f, g, h are assumed to be continuously differentiable on the interval I and $h(s) = u(f(s), g(s))$ for $s \in I$. The curve Γ is called Data curve or Datum curve.

Initial value problem

Initial value problem for the quasilinear PDE (2.6) is a special Cauchy problem for (2.6), wherein the initial curve Γ lies in the zx-plane and the y variable has an interpretation of the time-variable. That is, Γ has the following parametric form:

$$x = f(s),\, y = 0,\, z = h(s),\ s \in I, \tag{2.8}$$

In this section we prove the existence of solutions to the Cauchy problem for the equation (2.6) under some sufficient conditions on the datum curve Γ, and on Cauchy data.

2.2.1 ▪ Integral surface and Characteristic curves

In this subsection, we clarify some of the misleading usages of the word integral surface, most importantly in the context of results like Theorem 2.7.

Remark 2.2 (On the notion of integral surface).

(i) An integral surface corresponding to a given solution u of the equation (2.6) is simply referred to as an integral surface. This is responsible for many a confusion, since the underlying solution of the equation (2.6) giving rise to a given integral surface is forgotten.

(ii) An integral surface corresponding to a given solution u of the equation (2.6) is simply the graph of the function $u : D \to \mathbb{R}$, and thus the map $(x,y) \longmapsto (x,y,u(x,y))$ is 1-1.

(iii) Any point on an integral surface S has the form $(x,y,u(x,y))$ for some $(x,y)\in D$.

(iv) Note that any integral surface S is of the form $z = u(x,y)$ for some solution u (defined on its domain) of the equation (2.6). The projection of such an S to the xy-plane will be the domain on which the solution u of the equation (2.6) is defined.

(v) For the integral surface $z = u(x,y)$, the normal at any point $(x_0,y_0,u(x_0,y_0))$ on S is given by $\Big(u_x(x_0,y_0), u_y(x_0,y_0), -1\Big)$. We can write the PDE (2.6) in the form

$$\Big(u_x(x_0,y_0), u_y(x_0,y_0), -1\Big).\big(a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0))\big) = 0.$$

Thus the vector $(a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0)))$ belongs to the tangent space to the integral surface S at the point $(x_0,y_0,u(x_0,y_0))$.
By definition of tangent space, there exists a $\delta > 0$ and a curve $\gamma : (-\delta,\delta) \to \mathbb{R}^3$ such that

(a) the curve γ lies on S. That is $\gamma(-\delta,\delta) \subseteq S$.

(b) $\gamma(0) = (x_0,y_0,u(x_0,y_0))$.

(c) $\gamma'(0) = (a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0)))$.

This motivates the definition of a Characteristic curve that we are going to define shortly. ∎

Definition 2.3 (Characteristic vector field). *The vector field $(a(x,y,z),\, b(x,y,z),\, c(x,y,z))$ is called the Characteristic vector field of the equation (2.6). The Characteristic direction at $(x,y,z)\in\Omega$ is given by the direction of the vector $(a(x,y,z),\, b(x,y,z),\, c(x,y,z))$.*

Definition 2.4 (Characteristic curve). *A curve in $\mathbb{R}^3$ which is tangential to the characteristic direction at each of its points is called a Characteristic curve. That is, at each of the points on the curve, the tangent to the curve is parallel to the characteristic direction.*

Definition 2.5 (Base characteristics). *The projections of Characteristic curves to the xy-plane are called Base characteristics.*

Remark 2.6 (On characteristic curves).

(i) Let γ be a curve in $\mathbb{R}^3$, that γ is the image of a function $\gamma : J \to \mathbb{R}^3$ defined on some interval J given by $\gamma(t) = (x(t), y(t), z(t))$. If γ is a characteristic curve corresponding to the quasilinear PDE (2.6), then the functions $x(t), y(t), z(t)$ are solutions of the following autonomous system of ODEs:

$$\frac{dx}{dt} = a(x,y,z) \tag{2.9a}$$

$$\frac{dy}{dt} = b(x,y,z) \tag{2.9b}$$

$$\frac{dz}{dt} = c(x,y,z). \tag{2.9c}$$

Whenver γ is a characteristic curve, it continues to be so even after a re-parametrization $\tilde{\gamma}$. In such a case, denoting $\tilde{\gamma}(s) = (\tilde{x}(s), \tilde{y}(s), \tilde{z}(s))$, the RHS of the system of characteristic ODEs (2.9) satisfied by $\tilde{x}, \tilde{y}, \tilde{z}$ would at most be replaced by quantities proportional to the characteristic direction $(a(\tilde{x},\tilde{y},\tilde{z}), b(\tilde{x},\tilde{y},\tilde{z}), c(\tilde{x},\tilde{y},\tilde{z}))$. Thus the parameter t is somewhat artificial (see [22]).

(ii) Since a, b, c are assumed to be continuously differentiable on Ω, the RHS of the system (2.9) is a locally Lipschitz vector field. Thus Cauchy-Picard-Lipschitz theorem assures the existence of a characteristic curve of the equation (2.6) through any point of Ω, which is locally unique. Hence two distinct characteristic curves do not intersect unless one of them lies completely on the other. However base characteristics corresponding to two distinct characteristic curves might intersect, and the reader is instructed to show that this observation does not contradict existence-uniqueness theorems concerning initial value problems for ODEs.

(iii) In the case of semilinear PDE, neither distinct characteristic curves nor their projections on xy-plane intersect. However, if at least one of the functions a, b is such that existence-uniqueness theorems concerning initial value problems for ODEs cannot be applied, then it may happen that characteristic curves and the corresponding base characteristics might intersect. For example, see Exercise 2.3.

(iv) The characteristics form a two-parameter family. However the solutions to the characteristic system of ODEs form a three-parameter family. Justification of these is left to the reader. ∎

The following result helps in constructing an integral surface via characteristic curves.

Theorem 2.7. *Let $D \subseteq \Omega_0$, and $S : z = u(x,y)$ be a surface in $\mathbb{R}^3$ where $u : D \to \mathbb{R}$ is a continuously differentiable function. Then the following two statements are equivalent.*

(1). *The surface S is an integral surface of equation (2.6).*

(2). *The surface S is a union of characteristic curves of the equation (2.6).*

Proof.
Proof of (1) $\Longrightarrow$ (2): Let $S : z = u(x,y)$ be an integral surface corresponding to the quasilinear PDE (2.6). That is, $u : D \to \mathbb{R}$ satisfies the equation

$$a(x,y,u(x,y))\,u_x(x,y) + b(x,y,u(x,y))\,u_y(x,y) = c(x,y,u(x,y)) \text{ for all } (x,y) \in D. \tag{2.10}$$

The statement (2) of the theorem is equivalent to

$$S = \bigcup_{\gamma_P \text{ is a characteristic curve passing through } P\in S} \gamma_P.$$

Thus, to prove that S is a union of characteristic curves, it is sufficient to prove that the characteristic curve γ_P lies entirely[1] on S for every $P \in S$. Let $P(x_0, y_0, z_0)$ be an arbitrary point on the surface S. Through P, there exists a unique characteristic curve γ_P and we want to prove that γ_P lies entirely[2] on S. Suppose that γ_P is given by

$$x = x(t),\ y = y(t),\ z = z(t),\ t \in I \quad \text{and} \quad P(x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0)) \text{ for some } t_0 \in I.$$

Without loss of generality assume that $(x(t), y(t)) \in D$ for all $t \in I$; if not we replace I by an interval I' for which this holds. To prove that γ_P lies on S, we will prove

$$z(t) = u\,(x(t), y(t))^3 \text{ for all } t \in I.$$

Thus we are led to consider the following function which is defined on I:

$$V(t) = z(t) - u\,(x(t), y(t)) \text{ for all } t \in I.$$

We need to show that V is the zero function. Note that $V(t_0) = 0$ as $P \in S$. Let us compute the derivative of V.

$$\begin{aligned}
V'(t) &= z'(t) - u_x\,(x(t),y(t))\,\frac{dx}{dt} - u_y\,(x(t),y(t))\,\frac{dy}{dt} \\
&= c\,(x(t),y(t),z(t)) - u_x\,(x(t),y(t))\,a\,(x(t),y(t),z(t)) - u_y\,(x(t),y(t))\,b\,(x(t),y(t),z(t)) \\
&= c\,(x(t),y(t),V(t)+u\,(x(t),y(t))) - u_x\,(x(t),y(t))\,a\,(x(t),y(t),V(t)+u\,(x(t),y(t))) \\
&\qquad - u_y\,(x(t),y(t))\,b\,(x(t),y(t),V(t)+u\,(x(t),y(t))).
\end{aligned}$$

Thus the function $V : I \to \mathbb{R}$ is a solution of the ODE

$$U' = f(t,U), \tag{2.11}$$

where

$$\begin{aligned}
f(t,U) &= c\,(x(t),y(t),U+u\,(x(t),y(t))) - u_x\,(x(t),y(t))\,a\,(x(t),y(t),U+u\,(x(t),y(t))) \\
&\qquad - u_y\,(x(t),y(t))\,b\,(x(t),y(t),U+u\,(x(t),y(t))).
\end{aligned}$$

Since the functions a, b, c are continuously differentiable, and the function u defining the surface $z = u(x,y)$ is also assumed to be continuously differentiable, the function $f(t,U)$ is a locally Lipschitz function w.r.t. U. Further note that $U(t) \equiv 0$ is a solution

[1] Many books claim this statement. This is FALSE. Note that u is given to us. We cannot assume anything about the domain on which u is defined.

[2] In fact, that part of γ_P lies on S for which the corresponding base characteristics reside in D, the domain on which u is defined.

of the ODE (2.11). As $P \in S$, in view of (2.10), we get $U(t_0) = 0$. Hence by uniqueness of solutions to initial value problems for ODEs, we conclude that $U \equiv 0$. Since V is a solution of the ODE (2.11) satisfying the initial condition $V(t_0) = 0$, we conclude that $V \equiv 0$. That is, γ_P lies on S.

Proof of (2) $\Longrightarrow$ (1): Let the surface $S : z = u(x,y)$ be a union of characteristic curves of the equation (2.6). We want to show that S is an integral surface. In other words, we want to show that u solves the equation (2.6). Let $P\,(x_0, y_0, u(x_0,y_0))$ be any point on the surface S. We want to show

$$\Big(u_x(x_0,y_0), u_y(x_0,y_0), -1\Big).\big(a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0))\big) = 0. \tag{2.12}$$

Since S is a union of characteristic curves, there is a characteristic curve γ_P passing through P that lies on S. Since normal to S at P is in the direction of $\Big(u_x(x_0,y_0), u_y(x_0,y_0), -1\Big)$, and $(a(x_0,y_0,u(x_0,y_0)), b(x_0,y_0,u(x_0,y_0)), c(x_0,y_0,u(x_0,y_0)))$ is the direction of tangent to γ_P at P, we get (2.12). This finishes the proof.

□

Remark 2.8 (On the domain D). In the quasilinear equation (2.6), recall that a, b, c are defined on $\Omega \subseteq \mathbb{R}^3$ and were assumed to be continuously differentiable. Let Ω_0 be the projection of Ω into the xy-plane. That is,

$$\Omega_0 = \left\{(x,y) \in \mathbb{R}^2 : (x,y,z) \in \Omega \text{ for some } z \in \mathbb{R}\right\}.$$

Then $z = u(x,y)$ is an intgral surface on some domain $D \subseteq \Omega_0$. Note that characteristic curve lives in Ω, by virtue of being a soluton of the system of ODE where the vector field is defined on Ω. Projection of characteristic curves to xy-plane live inside Ω_0. ∎

Example 2.9. If $u : D \to \mathbb{R}$ is a solution to the Quasilinear PDE (2.6), then so is $v : D_1 \to \mathbb{R}$ where $D_1 \subseteq D$ and v is defined by $v(x,y) = u(x,y)$. The integral surfaces $z = u(x,y)$ and $z = v(x,y)$ are different since u and v are different as functions. But both the integral surfaces coincide on D_1. Thus interesection of two integral surfaces could be another integral surface. ∎

The following corollary follows immediately from Theorem 2.7

Corollary 2.10. *Let S_1 and S_2 be two integral surfaces such that $P \in S_1 \cap S_2$. Then some part of the characteristic passing through P lies on both S_1 and S_2*[1].

Corollary 2.11. *If two integral surfaces intersect without touching*[2] *and the intersection is a curve γ, then γ is a characteristic curve.*

Proof. To prove that γ is a characteristic curve, we have to prove that at any point P on the curve γ, the tangent has the characteristic direction. At P the tangent to the curve γ lies in the tangent planes to S_1 as well as S_2 and also the characteristic direction $(a(P), b(P), c(P))$. Since the tangent planes do not coincide (why?), the only direction common to both S_1

[1]This is stated more confusingly and also WRONGLY as "If two integral surfaces intersect at a point P, then they intersect along the entire characteristic curve through P." Discovering which part of this statement is confusing and which part is wrong is left to the reader.

[2]means that the two surfaces do not have a common tangent plane at the point(s) of intersection

and S_2 is $(a(P), b(P), c(P))$. Hence tangent to the curve γ at P is proportional to the characteristic direction at P. Since P is an arbitrary point on γ, it follows that γ is a characteristic curve. □

2.2.2 ▪ Three examples of Cauchy problems for a linear PDE

In this section, we present three examples of Cauchy problems for a special class of quasilinear PDE, namely linear PDE. These Cauchy problems can be solved explicitly, and they illustrate that all three possibilities concerning a mathematical problem can occur, namely,

(i) Cauchy problem has a unique solution. (Example 2.12)
(ii) Cauchy problem has an infinite number of solutions. (Example 2.13)
(iii) Cauchy problem has no solution. (Example 2.14)

Consider the following equation

$$u_x = cu + d(x,y), \tag{2.13}$$

where $c \in \mathbb{R}$ and d is a continuously differentiable function. The equation (2.13) can be thought of as an ODE where y appears as a parameter. Its explicit solution is given by

$$u(x,y) = e^{cx}\left(\int_0^x e^{-c\xi} d(\xi,y)\,d\xi + u(0,y)\right). \tag{2.14}$$

Example 2.12 (Cauchy Problem 1: Existence of a Unique solution). The Cauchy data is prescribed on the Y-axis:

$$u_x = cu + d(x,y),\ u(0,y) = y. \tag{2.15}$$

The unique solution is given by

$$u(x,y) = e^{cx}\left(\int_0^x e^{-c\xi} d(\xi,y)\,d\xi + y\right). \tag{2.16}$$

■

Example 2.13 (Cauchy Problem 2: Non-uniqueness of solutions). Cauchy data is prescribed on the X-axis:

$$u_x = cu,\ u(x,0) = e^{cx}. \tag{2.17}$$

This Cauchy problem has infinitely many solutions:

$$u(x,y) = e^{cx}\,T(y), \tag{2.18}$$

T is any function of a single variable such that $T(0) = 1$.

■

Example 2.14 (Cauchy Problem 3: Non-Existence of solutions). Cauchy data is prescribed on the X-axis:

$$u_x = cu,\ u(x,0) = \sin x. \tag{2.19}$$

This Cauchy problem has no solution. For, if it has a solution then, in view of the formula (2.14), the solution satisfies

$$\sin x = u(x,0) = e^{cx} u(0,0), \text{ for all } x \in \mathbb{R}. \tag{2.20}$$

The above equation cannot hold and hence the Cauchy problem has no solution.
■

Remark 2.15 (On the three examples). In this remark, we make some observations on the three examples considered above.

(i) When the Cauchy data was prescribed on x-axis we encountered the non-existence or multiplicity of solutions to the Cauchy problems, while prescribing Cauchy data on y-axis gave us a unique solution.
(ii) Thus the following question also arises: What was so nice about y-axis w.r.t. the given PDE and what was bad about x-axis at the same time? The reader is instructed to think about these questions, and answer them at least after going through Remark 2.22.
(iii) The three examples considered above suggest that Cauchy data cannot be prescribed on arbitrary curves in xy-plane if we expect the Cauchy problem to be well-posed.
■

2.2.3 ▪ Solution via method of characteristics

In this subsection, we determine (prove the existence of) a solution of the equation

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{2.21}$$

satisfying the condition

$$u\,(f(s), g(s)) = h(s), \quad s \in I \tag{2.22}$$

where I is an interval in $\mathbb{R}$. *i.e.,* the integral surface which passes through the curve

$$\Gamma:\ x = f(s),\, y = g(s),\, z = h(s), \quad s \in I. \tag{2.23}$$

Assume f, g, h are continuously differentiable in a neighbourhood of $s = s_0 \in I$.

In view of Theorem 2.7, in order to find a solution of the Cauchy problem for (2.21) we try to find the corresponding integral surface, and for that it is a natural idea to find characteristics passing through points on Γ near each $P_0 \in \Gamma$.

$$\text{Let } P_0(x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0)). \tag{2.24}$$

The characteristic differential equations of (2.21) is given by the system of ODEs

$$\frac{dx}{dt} = a(x,y,z) \tag{2.25a}$$

$$\frac{dy}{dt} = b(x,y,z) \tag{2.25b}$$

$$\frac{dz}{dt} = c(x,y,z). \tag{2.25c}$$

For each $s_0 \in I$, solve the above system (2.25) with the initial conditions at $t = 0$,

$$x(0) = f(s_0),\ y(0) = g(s_0),\ z(0) = h(s_0). \tag{2.26}$$

Since a, b, c are continuously differentiable functions, there exists a unique solution of the initial value problem (2.25)-(2.26) for $t \in J_{s_0}$ where $J_{s_0} \in \mathbb{R}$ is an interval containing 0. Denote this solution by

$$x = X(s_0, t),\ y = Y(s_0, t),\ z = Z(s_0, t). \tag{2.27}$$

Repeating the above procedure for each $s \in I$, we get solution of the system of characteristic ODEs (2.25) defined for $t \in J_s$ where $J_s \in \mathbb{R}$ is an interval containing 0, given by

$$x = X(s, t),\ y = Y(s, t),\ z = Z(s, t) \tag{2.28}$$

and satisfying the initial conditions

$$X(s, 0) = f(s),\ Y(s, 0) = g(s),\ Z(s, 0) = h(s). \tag{2.29}$$

The equations (2.28) represent the parametric equations of an integral surface, provided we can solve for (s, t) in terms of x, y from the two equations $x = X(s, t)$, $y = Y(s, t)$ and then substitute for s, t in terms of x, y in $z = Z(s, t)$. This is achieved by applying Inverse function theorem for the vector-valued function

$$(s, t) \longmapsto (X(s, t), Y(s, t)). \tag{2.30}$$

We need to check if the vector-valued function given in (2.30) satisfies the hypotheses of Implicit function theorem. First of all, the function must be continuously differentiable (total differentiability). In order to answer this question, we should know the domain on which the function in (2.30) is defined, and then comes the question of differentiability[3].

Luckily we can choose an interval J independent of s (on which IVPs for characteristic equations have solutions) provided we are willing to restrict ourselves to an interval I_0 containing $s = s_0$ instead of the entire interval I.[4],[5] Thus we may assume that the domain of the vector-valued function given in (2.30) is $I_0 \times J$ Thanks to the differentiable dependence of solutions to initial value problems for ODEs, the vector-valued function given in (2.30) is continuously differentiable.

Thus we are interested in the invertibility, near $(s, t) = (s_0, 0)$, of the function

$$(s, t) \in I_0 \times J \longmapsto (X(s, t), Y(s, t)) \in \mathbb{R} \times \mathbb{R}. \tag{2.31}$$

Note that at the point $(s, t) = (s_0, 0)$, the Jacobian of the function in (2.31) is given by

$$J = \left.\frac{\partial(X, Y)}{\partial(s, t)}\right|_{(s,t)=(s_0,0)} = \begin{vmatrix} X_s(s_0, 0) & X_t(s_0, 0) \\ Y_s(s_0, 0) & Y_t(s_0, 0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & a(f(s_0), g(s_0), h(s_0)) \\ g'(s_0) & b(f(s_0), g(s_0), h(s_0)) \end{vmatrix}, \tag{2.32}$$

[3] Most of the books do not address this question. They directly apply Inverse function theorem without caring for the details other than non-vanishing of a certain Jacobian!

[4] This is a key-idea in proving continuous dependence of solutions to initial value problems for Ordinary differential equations.

[5] If the characteristic vector field is such that one can have a global solution for the characteristic system, then we can take $J_s = \mathbb{R}$ and we overcome the problem of choosing a J which is independent of s.

since the functions $x = X(s,t)$, $y = Y(s,t)$, $z = Z(s,t)$ satisfy

$$X(s_0,0) = x_0 = f(s_0),\ Y(s_0,0) = y_0 = g(s_0),\ Z(s_0,0) = z_0 = h(s_0) \text{ and} \tag{2.33}$$

the characteristic system of ODEs

$$\frac{dX}{dt} = a\,(X(s,t), Y(s,t), Z(s,t)) \tag{2.34a}$$
$$\frac{dY}{dt} = b\,(X(s,t), Y(s,t), Z(s,t)) \tag{2.34b}$$
$$\frac{dZ}{dt} = c\,(X(s,t), Y(s,t), Z(s,t)) \tag{2.34c}$$

The above discussion motivates the following definition of transversality.

Definition 2.16 (Transversality condition). *The quasilinear equation (2.39) and the initial curve Γ given by (2.40) are said to satisfy* **transversality condition at a point** $(f(s), g(s), h(s)) \in \Gamma$ *(or equivalently, at $s \in I$) if the base characterstics corresponding to the characteristic curve passing through the point $(f(s), g(s), h(s))$ intersects the projection of the initial curve Γ* **non-tangentially** *(i.e., the tangential directions for these intersecting curves are not parallel). That is,*

$$J|_{(s,t)=(s_0,0)} = \begin{vmatrix} X_s(s_0,0) & X_t(s_0,0) \\ Y_s(s_0,0) & Y_t(s_0,0) \end{vmatrix} = \begin{vmatrix} f'(s_0) & a\,(f(s_0), g(s_0), h(s_0)) \\ g'(s_0) & b\,(f(s_0), g(s_0), h(s_0)) \end{vmatrix} \neq 0. \tag{2.35}$$

If we assume that the Cauchy data and the characteristic vector field satisfies the transversality condition, we can apply Inverse function theorem, and conclude that there exists a neighbourhood of $(s_0, 0)$ and a neighbourhood of $(f(s_0), g(s_0))$ such that the map (2.31) is invertible[6]. That is, we get two functions S, T defined on a neighbourhood of the point (x_0, y_0) such that

$$s = S(x,y),\ t = T(x,y). \tag{2.36}$$

Let us define

$$u(x,y) := Z\,(S(x,y), T(x,y)). \tag{2.37}$$

Checking that u solves the quasilinear PDE (2.21) is left as an exercise to the reader[7]. We will give a geometric proof in Remark 2.18.

The integral surface defined by u is given by

$$S : z = u(x,y) = Z\,(S(x,y), T(x,y)). \tag{2.38}$$

The equation (2.38) represents an integral surface containing the point (x_0, y_0, z_0).

Thus we have proved the local existence part of the following theorem.

Theorem 2.17 (Local existence theorem). *Consider the following Cauchy problem for a quasilinear PDE given by*

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u) \tag{2.39}$$

[6]**Exercise:** State Inverse function theorem. Apply it in the present situation and write down the conclusion CORRECTLY.

[7]**Exercise:** Prove that u is indeed a solution. Hint: Compute the derivatives and substitute in the equation.

and the Cauchy data

$$\Gamma: \ x = f(s), y = g(s), z = h(s), \quad s \in I, \tag{2.40}$$

where $I \subseteq \mathbb{R}$ is an interval.

1. *Assume that the functions a, b, c continuously differentiable on an open set in $\mathbb{R}^3$ that contains the initial curve (2.40).*
2. *Let $P_0(x_0, y_0, z_0) = (f(s_0), g(s_0), h(s_0)) \in \Gamma$.*
3. *Assume that the transversality condition holds at s_0.*

Then

(i) *the Cauchy problem (2.39)-(2.40) has a solution in a neighbourhood of the point $(x_0, y_0) \in \Omega_0$.*

(ii) *Further if D_1 and D_2 are neighbourhoods of the point (x_0, y_0), and if $u_1 : D_1 \to \mathbb{R}^3$ and $u_2 : D_2 \to \mathbb{R}^3$ are solutions of the Cauchy problem (2.39)-(2.40), then $u_1 \equiv u_2$ on some neighbourhood of (x_0, y_0). That is, solution to Cauchy problem is locally unique near the datum curve Γ.*

Proof. Local existence was proved in the discussion leading to the statement of the theorem. Let us turn to the proof of uniqueness now. If $S_1 : z = u_1(x, y)$ and $S_2 : z = u_1(x, y)$ are integral surfaces containing Γ (or a part thereof near a point P_0), then some part of the characteristic curves of equation (2.38) that pass through Γ (respectively, a part thereof near a point P_0) by Corollary 2.10. Thus

$$S_1 \equiv S_2 \quad \text{in a neighbourhood of } P_0. \tag{2.41}$$

□

Remark 2.18. The surface S given by the equation (2.38) has parametric equations

$$x = X(s, t), y = Y(s, t), z = Z(s, t). \tag{2.42}$$

The curves $s =$ constant are the characteristic curves of equation (2.21). The tangential direction at any point of the curve $s = c$ is given by (X_t, Y_t, Z_t) at $s = c$. This direction is also tangential to the surface S and hence the normal to S is perperndicular to the characteristic direction at any point. This implies S is an integral surface. ■

What happens if transversality condition fails?

Lemma 2.19. *Consider the Cauchy problem (2.39)-(2.40). Assume that there exists an integral surface $z = u(x, y)$ passing through Γ (or a part theoreof containing $P_0 = (f(s_0), g(s_0), h(s_0))$ where $s_0 \in I$. Further let*

$$J(s_0) = \begin{vmatrix} f'(s_0) & a(f(s_0), g(s_0), h(s_0)) \\ g'(s_0) & b(f(s_0), g(s_0), h(s_0)) \end{vmatrix} = 0. \tag{2.43}$$

Then

(i) Γ *has characteristic direction at* s_0. *That is,*

$$\frac{f'}{a} = \frac{g'}{b} = \frac{h'}{c} \quad \text{at } s = s_0. \tag{2.44}$$

(ii) *If* $J \equiv 0$ *along* Γ, *then* Γ *is a characteristic curve.*

Proof. Note that

$$J(s_0) = 0 \Longleftrightarrow b\left(f(s_0), g(s_0), h(s_0)\right) f'(s_0) - a\left(f(s_0), g(s_0), h(s_0)\right) g'(s_0) = 0. \tag{2.45}$$

Also, we have

$$\begin{aligned} c g' - b h' &= c\left(f(s_0), g(s_0), h(s_0)\right) g'(s_0) - b\left(f(s_0), g(s_0), h(s_0)\right) h'(s_0) \\ &= \Big[a\left(f(s_0), g(s_0), h(s_0)\right) u_x(f(s_0), g(s_0)) + b\left(f(s_0), g(s_0), h(s_0)\right) u_y(f(s_0), g(s_0)) \Big] g'(s_0) \\ &\quad - b\left(f(s_0), g(s_0), h(s_0)\right) \Big[u_x(f(s_0), g(s_0)) f'(s_0) + u_y(f(s_0), g(s_0)) g'(s_0) \Big] \\ &= \Big[a\left(f(s_0), g(s_0), h(s_0)\right) g'(s_0) - b\left(f(s_0), g(s_0), h(s_0)\right) f'(s_0) \Big] u_x(f(s_0), g(s_0)) = 0, \end{aligned}$$

This proves (2.44). That is, Γ has characteristic direction at s_0. Clearly Γ will be a characteristic curve if $J \equiv 0$ along Γ. This finishes the proof of lemma. □

Lemma 2.19 tells us that if $J(s_0) \neq 0$ then there exists a local solution to Cauchy problem for the quasilinear PDE. We will now analyze the situation where $J(s) \equiv 0$ for $s \in I$. Once again Lemma 2.19 asserts that Γ must be a characteristic curve if there is a solution to the Cauchy problem. In the next result, we will prove that if Γ is a characteristic curve, and $J(s) \equiv 0$ for $s \in I$, then there exists infinitely many solutions to the Cauchy problem. A Cauchy problem where the initial curve Γ is a characteristic, is called a Characteristic Cauchy problem.

Lemma 2.20.

1. *Consider the Cauchy problem (2.39)-(2.40), where* Γ *is a characteristic curve. Further assume that* $J(s) \equiv 0$ *for* $s \in I$. *Let* P_0 *be a point on* Γ. *Then there exists an infinite number of integral surfaces which contain a part of the initial curve near* P_0.
2. *Consider the Cauchy problem (2.39)-(2.40), where* Γ *has nowhere a characteristic direction. Then Cauchy problem does not admit a solution that is differentiable.*

Proof. **Proof of (1):** Choose any curve Γ_1 through P_0 satisfying $J_{\Gamma_1}(P_0) \neq 0$. Then there exists a unique integral surface of equation (2.21) passing through Γ_1 in a neighbourhood of P_0. Then the integral surface containing Γ_1 (which exists by the local existence theorem as $J_{\Gamma_1}(P_0) \neq 0$) also contains a part of Γ containing P_0, since Γ is a characteristic curve through P_0.

Then there are infinitely many solutions to the Characteristic Cauchy problem corresponding to infinitely many ways of choosing Γ_1, a proof of this is left as an exercise to the reader.

Proof of (2): Lemma 2.19 asserts that Γ must be a characteristic curve if there is a solution to the Cauchy problem when $J(s) \equiv 0$ for all $s \in I$. Thus $J_\Gamma \equiv 0$ and Γ does not have a characteristic direction anywhere are incompatible if the Cauchy problem admits a differentiable solution. Thus we conclude that the Cauchy problem does not admit a solution. This completes the proof of (2). □

Remark 2.21. In the previous lemma, we discussed the case of $J \equiv 0$. It will be interesting to know what happens when J vanishes but does not vanish identically. In view of the last two results, we understand the situation completely whenever J is not equal to zero or stretches where J is identically equal to zero. If Γ is not a characteristic curve, then we expect integral surfaces to have singularities near the points where J vanishes. See Exercise 2.10. ■

Remark 2.22 (On the three examples of Subsection 2.2.2). Let us return to the three examples that we considered in Subsection 2.2.2. We have the following observations regarding them.

(i) In Example 2.12, Jacobian is non-zero and hence we have a unique solution. In other words, the transversality condition is satisfied. Geometrically, this means that the tangential directions for the two intersecting curves, namely, the base characteristics corresponding to the characteristic curve passing through a point P, and the projection of the initial curve Γ are not parallel at any point.

(ii) In Example 2.13, $J \equiv 0$ and the initial curve is a characteristic curve and hence it has infinitely many solutions. In other words, the transversality condition is not satisfied. That is, the tangential directions for the two intersecting curves, namely, the base characteristics corresponding to the characteristic curve passing through a point P, and the projection of the initial curve Γ are parallel at any point P. However in this example, the initial curve is a characteristic. That is, the characteristic direction coincides with the tangential direction for initial curve, and thus by Lemma 2.20 guranteed the existence of an infinite number of solutions.

(iii) In Example 2.14, $J \equiv 0$ and the initial curve is not a characteristic curve and hence it has no solutions. Even though the situation is similar to that of Example 2.13 (namely, transversality condition is not satisfied), in this example we do not have solutions. The reason for non-existence of solutions is that the initial curve is not a characteristic. ■

Some more terminology

(i) Characteristic direction (a, b, c) is also called **Monge's direction**.
(ii) Characteristic equation is also called **Monge's equation**.
(iii) Characteristic curves is also called **Monge's curves**.

2.2.4 ▪ Burgers equation

Let us solve the Cauchy problem for Burgers' equation.

Example 2.23 (Burgers' equation).
Cauchy Problem to be solved is

$$u_y + u u_x = 0,\ u(x,0) = h(x) \text{ for } x \in \mathbb{R}. \tag{2.46}$$

This is a Cauchy problem for a first order quasilinear PDE.

Parametrization of the initial curve Γ