Chapter 3

Classification of Second order PDEs

This chapter deals with a finer classification of second order quasilinear PDEs, as compared to Chapter 1. In Section 3.1, we present a formal procedure to solve the Cauchy problem for a quasilinear PDE which forms the central idea in the proof of Cauchy-Kowalewski theorem. The formal procedure works under a condition, the failure of which motivates the concept of a characteristic curve. Notion of a characteristic curve is introduced in Section 3.2, and it is shown in Section 3.3 that characteristic curves are carriers of discontinuities.

In Section 3.2 we classify all second order quasilinear PDEs in two independent variables, which are given by

$$a(x,y,u,u_x,u_y)u_{xx}+2b(x,y,u,u_x,u_y)u_{xy}+c(x,y,u,u_x,u_y)u_{yy}+d(x,y,z,u_x,u_y)=0, \tag{3.1}$$

where a,b,c,d are functions, into three classes: *hyperbolic, parabolic, elliptic*. In Section 3.4 we derive canonical forms for each of the classes for linear PDEs which are of the form

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u+g(x,y)=0.$$

While the classification is similar to that of conic sections in plane geometry, the canonical forms are similar to their standard forms. We generalize the ideas of classification to second order equations in more than two independent variables in Section 3.5.

3.1 ▪ Cauchy problem for a quasilinear PDE and its solution

Cauchy problem for a quasilinear PDE (3.1) is to find a twice continuously differentiable function u that solves (3.1) and the surface $z=u(x,y)$ contains a given space curve Γ prescribed parametrically by

$$\Gamma: x=f(s),\, y=g(s),\, z=h(s)\; s\in I, \tag{3.2}$$

where f,g,h are continuously differentiable functions, and having a prescribed normal derivative along the projection of Γ to xy-plane which is denoted by Γ_0.

We would like to construct an integral surface $z=u(x,y)$ in a neighbourhood of a point $P(x_0,y_0,z_0)=(f(s_0),g(s_0),h(s_0))$ on Γ. Equivalently, we would like to determine a solution of (3.1) in a neighbourhood of the point $P_0(x_0,y_0)$ which satisfies $u(P_0)=z_0$.

An attempt to solve Cauchy problem is made using a classical strategy, which consists of determining derivatives of u of all orders at P_0, and then propose a formal power series expansion for u around the point P_0 as a solution to the Cauchy problem. This is the essential idea behind the proof of Cauchy-Kowalewski theorem for second order quasilinear PDEs in two independent variables, and we will not discuss its proof.

In what follows, we try to compute first, second, and third order partial derivatives of u at P_0, and derive conditions under which it is possible.

Computation of first order derivatives

Step 1:

Let $[p(s), q(s), -1]$ be the direction numbers of the normal to the surface $z = u(x, y)$ at points of Γ, where p and q are continuously differentiable functions. Note that we do not have the knowledge of $p(s)$ and $q(s)$. Then the function u must satisfy the relations

$$u(f(s), g(s)) = h(s) \tag{3.3a}$$
$$u_x(f(s), g(s)) = p(s) \tag{3.3b}$$
$$u_y(f(s), g(s)) = q(s) \tag{3.3c}$$

since $\Big(u_x(f(s), g(s)), u_y(f(s), g(s)), -1\Big)$ is the normal direction on the surface at points of Γ. Let the normal derivative prescribed on Γ_0 be given by the function $\chi(s)$. Then we get the relation

$$\chi(s) = \frac{-p(s)g'(s) + q(s)f'(s)}{\sqrt{(f')^2 + (g')^2}} \tag{3.4}$$

assuming that $\sqrt{(f')^2 + (g')^2} \neq 0$, since the unit normal on Γ_0 is given by

$$\frac{(-g'(s), f'(s), 0)}{\sqrt{(f')^2 + (g')^2}}.$$

Step 2: If the curve Γ lies on a surface $z = u(x, y)$, then $h(s) = u(f(s), g(s))$. Differentiating this relation w.r.t. s, we get the following 'strip condition' which has to be satisfied along Γ (compatibility condition). For all $s \in I$,

$$h'(s) = \frac{d}{ds}(u(f(s), g(s))) = u_x(f(s), g(s))f'(s) + u_y(f(s), g(s))g'(s)$$

In view of (3.3), the strip condition reduces to

$$h'(s) = p(s)f'(s) + q(s)g'(s) \tag{3.5}$$

Step 3: The linear system of equations (3.4) - (3.5) has a unique solution for $p(s)$ and $q(s)$, since $\sqrt{(f')^2 + (g')^2} \neq 0$. Thus both the first order derivatives have been determined along points of Γ_0.

Computation of second order derivatives

On differentiating the equations (3.3b) and (3.3c), we get

$$p'(s) = u_{xx}(f(s), g(s))f'(s) + u_{xy}(f(s), g(s))g'(s) + 0.u_{yy}(f(s), g(s)) \tag{3.6}$$
$$q'(s) = 0.u_{xx}(f(s), g(s))f'(s) + u_{xy}(f(s), g(s))f'(s) + u_{yy}(f(s), g(s))g'(s). \tag{3.7}$$

If $z = u(x,y)$ solves the PDE (3.1), then we also have, for all $s \in I$

$$a(f(s),g(s),h(s),p(s),q(s))u_{xx}(f(s),g(s)) + 2b(f(s),g(s),h(s),p(s),q(s))u_{xy}(f(s),g(s)) \\ + c(f(s),g(s),h(s),p(s),q(s))u_{yy}(f(s),g(s)) = -d(f(s),g(s),h(s),p(s),q(s)). \tag{3.8}$$

If

$$\Delta = \begin{vmatrix} f'(s) & g'(s) & 0 \\ 0 & f'(s) & g'(s) \\ a(f(s),g(s),h(s),p(s),q(s)) & 2b(f(s),g(s),h(s),p(s),q(s)) & c(f(s),g(s),h(s),p(s),q(s)) \end{vmatrix} \neq 0, \tag{3.9}$$

then u_{xx}, u_{xy}, u_{yy} can be solved for uniquely on Γ_0, from the equations (3.6)-(3.8). Thus all second order partial derivatives of u have been determined along points of Γ_0 under the condition $\Delta \neq 0$.

Computation of third and higher order derivatives

Moreover, if $\Delta \neq 0$, we can solve for all the higher order derivatives $u_{xxx}, u_{xxy}, u_{xyy}, \cdots$ uniquely on Γ_0. For, differentiating the PDE (3.1) w.r.t. x yields

$$\frac{\partial}{\partial x} d(x,y,u,u_x,u_y) = a u_{xxx} + 2b u_{xxy} + c u_{xyy} + \left(\frac{\partial a}{\partial x} u_{xx} + 2\frac{\partial b}{\partial x} u_{xy} + \frac{\partial c}{\partial x} u_{yy} \right) \tag{3.10}$$

Note that all the terms appearing within parantheses on the RHS of the equation (3.10) are known functions along Γ_0. This is because u_{xx}, u_{xy}, u_{yy} have already been determined uniquely on Γ under the assumption that $\Delta \neq 0$. Similarly the LHS of the equation (3.10) is also a known function along Γ_0 since

$$\begin{aligned} \frac{\partial}{\partial x} d(x,y,z,u_x,u_y)\Big|_{(f(s),g(s),h(s),p(s),q(s))} &= d_x(f(s),g(s),h(s),p(s),q(s)) \\ &+ d_z(f(s),g(s),h(s),p(s),q(s))u_x(f(s),g(s)) \\ &+ d_p(f(s),g(s),h(s),p(s),q(s))u_{xx}(f(s),g(s)) \\ &+ d_q(f(s),g(s),h(s),p(s),q(s))u_{xy}(f(s),g(s)). \end{aligned}$$

Thus, along Γ_0 we have

$$a u_{xxx} + 2b u_{xxy} + c u_{xyy} = \text{ a known function of } s \tag{3.11}$$

Note that the following system of equations holds:

$$\frac{d}{ds}(u_{xx}(f(s),g(s))) = u_{xxx}(f(s),g(s))f'(s) + u_{xxy}(f(s),g(s))g'(s) \tag{3.12a}$$

$$\frac{d}{ds}\Big(u_{xy}(f(s),g(s))\Big) = u_{xxy}(f(s),g(s))f'(s) + u_{xyy}(f(s),g(s))g'(s) \tag{3.12b}$$

$$\frac{d}{ds}\Big(u_{yy}(f(s),g(s))\Big) = u_{xyy}(f(s),g(s))f'(s) + u_{yyy}(f(s),g(s))g'(s). \tag{3.12c}$$

The system of linear equations given by (3.11) and (3.12) has a unique solution since Δ was assumed to be non-zero, and $\sqrt{(f')^2 + (g')^2} \neq 0$, a proof of this is left to the reader as an exercise.

Thus, if $\Delta \neq 0$ along Γ, all the successive partial derivatives of a solution u of the PDE (3.1) satisfying the Cauchy data can be solved for uniquely, and hence a formal power series expansion for u in the neighbourhood of a point (x_0, y_0) on Γ_0 can be written provided a, b, c, d are infinitely differentiable functions.

If we assume a, b, c, d are analytic and a solution u of the given Cauchy problem is also analytic at P_0, then the formal power series expansion for u around P_0 yields a solution of the Cauchy problem and there is no other analytic solution of this Cauchy problem. This is the essence of Cauchy-Kowalewski theorem for second order quasilinear PDEs in two independent variables.

3.2 ▪ Classification of quasilinear PDEs

Recall from Section 3.1 that if $\Delta \neq 0$ for all $s \in I$, all the partial derivatives of a solution (if exist) may be determined along Γ_0, from the Cauchy data. However if $\Delta = 0$ for all $s \in I$, then we cannot implement the procedure (described in Section 3.1) to solve the Cauchy problem. This motivates the following definition.

Definition 3.1 (characteristic curve).

(i) *If $\Delta = 0$ for all $s \in I$, then Γ_0 is said to be characteristic w.r.t. the partial differential equation (3.1) and the Cauchy data (3.2).*

(ii) *If $\Delta \neq 0$ for all $s \in I$, then Γ_0 is said to be non-characteristic for equation (3.1) and the Cauchy data (3.2).*

Remark 3.2 (characteristic curves for semilinear equations). If the PDE (3.1) is semilinear, whether the curve Γ_0 is characteristic or not depends only on the equation, and is independent of the Cauchy data. ∎

The curve Γ_0 which is given parametrically by $(f(s), g(s))$ $(s \in I)$ is a characteristic curve if the following equation is satisfied along Γ_0:

$$a(f(s), g(s), h(s))\left(g'\right)^2 - 2b(f(s), g(s), h(s))f'g' + c(f(s), g(s), h(s))\left(f'\right)^2 = 0. \tag{3.13}$$

Eliminating the parameter s, we get

$$\frac{dy}{dx} = \frac{g'}{f'} \qquad \text{along} \quad \Gamma_0 \tag{3.14}$$

Thus the characteristic equation (3.13) takes the form

$$a(x, y, u(x, y))\left(\frac{dy}{dx}\right)^2 - 2b(x, y, u(x, y))\frac{dy}{dx} + c(x, y, u(x, y)) = 0, \tag{3.15}$$

where u is a given solution of the quasilinear PDE (3.1).

We may solve for $\frac{dy}{dx}$ from the equation (3.15), and obtain

$$\frac{dy}{dx} = \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{3.16}$$

In general the differential equation (3.15) is called the characteristic equation for the quasilinear equation (3.1) and the given Cauchy data. The solutions of the characteristic equation are called characteristic curves.

If the equation (3.1) is linear (or is quasilinear and an integral surface $z = u(x,y)$ is given), the characteristic equation (3.15) becomes an ordinary differential equation, and the characteristic curves can be obtained by solving (3.16).

If $b^2 - ac > 0$, the equation (3.15) gives rise to two different families of real characteristics. If $b^2 - ac = 0$, there is only one family of real characteristic curves. If $b^2 - ac < 0$, there are no real characteristic curves.

Definition 3.3. *Let u be a solution of the quasilinear PDE*

$$a(x,y,u,u_x,u_y)u_{xx} + 2b(x,y,u,u_x,u_y)u_{xy} + c(x,y,u,u_x,u_y)u_{yy} + d(x,y,z,u_x,u_y) = 0.$$

Define a function δ by

$$\delta(x,y) = (b^2 - ac)(x,y,u(x,y),u_x(x,y),u_y(x,y))$$

With respect to the integral surface $z = u(x,y)$, we say that the given quasilinear equation is

(i) *of hyperbolic type at the point (x,y) if $\delta(x,y)$ is positve.*
(ii) *of parabolic type at the point (x,y) if $\delta(x,y)$ is zero.*
(iii) *of elliptic type at the point (x,y) if $\delta(x,y)$ is negaitve.*

Remark 3.4. (i) The type of a quasilinear PDE of second order depends on the terms involving second order derivatives only.
(ii) In view of the law of trichotomy which holds for real numbers, at every point (x,y) the quasilinear PDE (with a given integral surface) must be of one of the three types.
(iii) The type of a quasilinear PDE might vary from point to point, depending on the coefficients.
(iv) If a,b,c are constant functions, then the equation is of the same type at every point (x,y).
(v) If the quasilinear equation is semilinear, then the type of the equation depends only on the point (x,y) in the plane. ∎

Example 3.5. (a) The wave equation $u_{tt} - u_{xx} = 0$ is of hyperbolic type at every point (x,t), since $\delta(x,t) = 4$.
(b) The heat equation $u_t - u_{xx} = 0$ is of parabolic type at every point (x,t), since $\delta(x,t) = 0$.
(c) The Laplace equation $u_{xx} + u_{yy} = 0$ is of elliptic type at every point (x,y), since $\delta(x,y) = -4$.
∎

Example 3.6 (Tricomi equation). The linear PDE

$$u_{yy} - y u_{xx} = 0$$

is known as Tricomi equation. For this equation, note that $a = -y, b = 0, c = 1$. Thus $b^2 - ac = y$, and hence the Tricomi equation is of hyperbolic type in the upper half plane, is of parabolic type on x-axis, and is of elliptic type in the lower half plane. ∎

3.3 ▪ Characteristics as carriers of discontinuities

Characteristic curves were introduced as curves of indeterminacy of higher order derivatives of a solution to Cauchy problems. It also turns out that the curves of discontinuities of a 'weak solution' to a second order quasilinear PDE also turn out to be characteristic curves. This is the content of the next result.

Theorem 3.7. *Let $\Omega \subseteq \mathbb{R}^2$ be a region, and γ be a curve in $\mathbb{R}^2$ that divides Ω into two parts such that $\Omega \setminus \gamma$ is composed of two disjoint regions Ω_1 and Ω_2. Given $v^1(x,y) \in C(\overline{\Omega_1})$ and $v^2(x,y) \in C(\overline{\Omega_2})$, define*

$$v(x,y) = \begin{cases} v^1(x,y) & \text{if } (x,y) \in \Omega_1, \\ v^2(x,y) & \text{if } (x,y) \in \Omega_2. \end{cases} \tag{3.17}$$

Let the jump in the values of v across γ be denoted by $[v]$. That is,

$$[v](x,y) = v^2(x,y) - v^1(x,y) \quad \text{for} \quad (x,y) \in \gamma. \tag{3.18}$$

Let u, u_x, u_y be continuous on Ω, and u_{xx}, u_{xy}, u_{yy} have jump discontinuities on γ. That is, $u^i_{xx}, u^i_{xy}, u^i_{yy}$ belong to $C(\overline{D_i})$ for $i = 1,2$. Let γ be given by $\gamma : x = \varphi(y)$, where φ is a continuously differentiable function. Then

(i) *the jumps in the quantities u_{xx}, u_{xy}, u_{yy} across γ are not independent of one another. Define $\lambda(y) := [u_{xx}](\varphi(y), y)$, then*

$$[u_{xx}](\varphi(y), y) = \lambda(y), \tag{3.19a}$$
$$[u_{xy}](\varphi(y), y) = -\lambda(y)\varphi'(y), \tag{3.19b}$$
$$[u_{yy}](\varphi(y), y) = \lambda(y)\big(\varphi'(y)\big)^2 \tag{3.19c}$$

at each point $(\varphi(y), y)$ on γ.

(ii) *Further assume that u^1 and u^2 solve the quasilinear equation in the regions Ω_1 and Ω_2 respectively, then we get that discontinuities in the second order derivatives of u propagate along a characteristic curve.*

(iii) *Further if u^1 and u^2 solve the linear equation in the regions Ω_1 and Ω_2 respectively then λ saitisfies the following ODE*

$$0 = 2(b - c\varphi')\lambda' + \Big(a_x - 2b_x\varphi' + c_x\left(\varphi'\right)^2 + d - e\varphi' - c\varphi''\Big)\lambda. \tag{3.20}$$

Proof. **Proof of (i):** Since u, u_x, u_y are assumed to be continuous on Ω, these functions are continuous across γ. Let the superscripts 1 and 2 on a function defined on Ω denote its restrictions to the regions Ω_1 and Ω_2 respectively. Thus we have along γ

$$0 = [u] = u^2(\varphi(y), y) - u^1(\varphi(y), y) \tag{3.21}$$
$$0 = [u_x] = u^2_x(\varphi(y), y) - u^1_x(\varphi(y), y) \tag{3.22}$$
$$0 = [u_y] = u^2_y(\varphi(y), y) - u^1_y(\varphi(y), y) \tag{3.23}$$

Along γ, let $\lambda(y) = [u_{xx}] = u^2_{xx}(\varphi(y), y) - u^1_{xx}(\varphi(y), y)$. Note that λ is a function of y along γ. Differentiating both sides of the equations (3.22) and (3.23) w.r.t. y, we get

$$0 = u^2_{xx}(\varphi(y), y)\varphi'(y) - u^1_{xx}(\varphi(y), y)\varphi'(y) + u^2_{xy}(\varphi(y), y) - u^1_{xy}(\varphi(y), y)$$
$$0 = u^2_{xy}(\varphi(y), y)\varphi'(y) - u^1_{xx}(\varphi(y), y)\varphi'(y) + u^2_{yy}(\varphi(y), y) - u^1_{yy}(\varphi(y), y)$$

which can be written as (in terms of jumps)

$$0 = [u_{xx}](\varphi(y), y)\varphi'(y) + [u_{xy}](\varphi(y), y), \tag{3.24a}$$
$$0 = [u_{xy}](\varphi(y), y)\varphi'(y) + [u_{yy}](\varphi(y), y). \tag{3.24b}$$

From the last two equations (3.24), the relations (3.19) follow.

Proof of (ii):

We now assume that u^1 and u^2 are solutions of a second order quasilinear PDE

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d(x, y, u, u_x, u_y) = 0$$

in the regions Ω_1 and Ω_2 respectively. That is,

$$0 = a u^i_{xx} + 2b u^i_{xy} + c u^i_{yy} + d(x, y, u^i, u^i_x, u^i_y) \qquad i = 1, 2. \tag{3.25}$$

On taking difference of the two equations in (3.25), we get

$$0 = a[u_{xx}] + 2b[u_{xy}] + c[u_{yy}] \quad \text{on} \quad \gamma.$$

Using the relations (3.19) in the last equation, we get

$$\begin{aligned} 0 &= a(\varphi(y), y)\lambda(y) - 2b(\varphi(y), y)\lambda(y)\varphi + c(\varphi(y), y)\lambda(y)\left(\varphi'\right)^2 \\ &= \lambda(y)\left(a(\varphi(y), y) - 2b(\varphi(y), y)\frac{dx}{dy} + c(\varphi(y), y)\left(\frac{dx}{dy}\right)^2\right) \end{aligned}$$

Assuming that $\lambda(y) \neq 0$ along γ, we get

$$a(\varphi(y), y) - 2b(\varphi(y), y)\frac{dx}{dy} + c(\varphi(y), y)\left(\frac{dx}{dy}\right)^2 = 0.$$

This implies that $x = \varphi(y)$ is a solution of characteristic differential equation, and thus γ is a characteristic curve.

Proof of (iii):

Since $\lambda(y) = [u_{xx}](\varphi(y), y)$, we have

$$\lambda(y) = u^2_{xx}(\varphi(y), y) - u^1_{xx}(\varphi(y), y) \tag{3.26}$$

Differentiating on both sides of the equation (3.26) w.r.t. y gives

$$\begin{aligned} \frac{d\lambda}{dy}(y) &= u^2_{xxx}(\varphi(y), y)\varphi'(y) - u^1_{xxx}(\varphi(y), y)\varphi'(y) + u^2_{xxy}(\varphi(y), y) - u^1_{xxy}(\varphi(y), y) \\ &= [u_{xxx}](\varphi(y), y)\varphi'(y) + [u_{xxy}](\varphi(y), y) \end{aligned} \tag{3.27}$$

Similarly, we get

$$\frac{d[u_{xy}]}{dy}(y) = [u_{xxy}](\varphi(y), y)\varphi'(y) + [u_{xyy}](\varphi(y), y) \tag{3.28}$$

In view of the relation $[u_{xy}] = -\lambda\varphi'$, the equation (3.27) becomes

$$\frac{d}{dy}\left(-\lambda\varphi'\right) = [u_{xxy}](\varphi(y),y)\varphi'(y) + [u_{xyy}](\varphi(y),y). \tag{3.29}$$

If u^i $(i = 1,2)$ solve the linear equation in the domain D_i, then we also have

$$a u^i_{xx} + 2b u^i_{xy} + c u^i_{yy} + d u^i_x + e u^i_y + f u^i + g = 0, \qquad i = 1,2. \tag{3.30}$$

First differentiate each of the above two equations given by (3.30) w.r.t. x, and then subtract one from another to yield

$$a[u_{xxx}] + 2b[u_{xxy}] + c[u_{xyy}] + d[u_{xx}] + e[u_{xy}] + a_x[u_{xx}] + 2b_x[u_{xy}] + c_x[u_{yy}] = 0$$

Using the relations (3.19) in the last equation, we get

$$a[u_{xxx}] + 2b[u_{xxy}] + c[u_{xyy}] + d\lambda - e\lambda\varphi' + a_x\lambda - 2b_x\lambda\varphi' + c_x\lambda\left(\varphi'\right)^2 = 0. \tag{3.31}$$

Eliminating the jumps in the third order derivatives of u across γ from the equations (3.27)-(3.29), (3.31), and using the relation $a - 2b\varphi' + c\,(\varphi')^2 = 0$, we get the following first order ordinary differential equation satisfied by λ

$$0 = 2(b - c\varphi')\lambda' + \left(a_x - 2b_x\varphi' + c_x\left(\varphi'\right)^2 + d - e\varphi' - c\varphi''\right)\lambda \tag{3.32}$$

□

Remark 3.8. We may interpret $\lambda(y) = [u_{xx}](\varphi(y),y)$ as the intensity of jump in u_{xx} across γ. If $\lambda(y_0) = 0$ for some y_0, and if the initial value problem for the ODE (3.20) has a unique solution (which is the case if $b - c\varphi' \neq 0$ along γ), then $\lambda(y) \equiv 0$. That is, if u_{xx} is continuous at some point of γ, then u_{xx} is continuous at all points of γ, and as a consequence of the relations (3.19), all second order derivatives of u are continuous across γ.

If the variable y is given the interpretation of time, then the equation $x = \varphi(y)$ gives the location of the jump in u_{xx} for varioius times. The speed of propagation of discontinuities is given by $\varphi'(y)$, which is equal to $\frac{dx}{dy}$, and satsifies the characteristic differential equation

$$a(\varphi(y),y) - 2b(\varphi(y),y)\frac{dx}{dy} + c(\varphi(y),y)\left(\frac{dx}{dy}\right)^2 = 0.$$

■

3.4 ▪ Canonical forms for linear PDEs

In this section we will obtain canonical forms for the second order linear PDE given by

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u + g(x,y) = 0. \tag{3.33}$$

Canonical forms for hyperbolic, parabolic, and elliptic types of equations are modeled after wave, heat, and laplace equations respectively. In other words, a change of coordinates

from (x,y) to (ξ,η) will be introduced so that when the equation (3.33) is written in the new coordinate system, terms with second order derivatives look like those in wave, heat, or laplace equations depending on the type of the equation in the region.

We have the following result which asserts that the type of the equation in a region is independent of the coordinate system w.r.t. which the equaion is written.

Lemma 3.9. *Consider a change of coordinates from (x,y) to (ξ,η), where $(\xi,\eta)=(\xi(x,y),\eta(x,y))$. Then the equation (3.33) transforms in the new coordinate system to the equation*

$$A(\xi,\eta)w_{\xi\xi}+2B(\xi,\eta)w_{\xi\eta}+C(\xi,\eta)w_{\eta\eta}+2D(\xi,\eta)w_{\xi}+2E(\xi,\eta)w_{\eta}+F(\xi,\eta)w+G(\xi,\eta)=0, \tag{3.34}$$

where

$$A(\xi,\eta)=a\,(\xi_x(x,y))^2+2b\xi_x(x,y)\xi_y(x,y)+c\left(\xi_y(x,y)\right)^2 \tag{3.35a}$$

$$B(\xi,\eta)=a\xi_x(x,y)\eta_x(x,y)+b\left(\xi_x(x,y)\eta_y(x,y)+\xi_y(x,y)\eta_x(x,y)\right)+c\xi_y(x,y)\eta_y(x,y) \tag{3.35b}$$

$$C(\xi,\eta)=a\left(\eta_y(x,y)\right)^2+2b\eta_x(x,y)\eta_y(x,y)+c\left(\eta_y(x,y)\right)^2 \tag{3.35c}$$

and the type of the equation does not change. In fact, the following equality holds:

$$B^2-AC=\left(\xi_x(x,y)\eta_y(x,y)-\xi_y(x,y)\eta_x(x,y)\right)^2(b^2-ac) \tag{3.36}$$

Proof. Define

$$w(\xi,\eta):=u\,(x(\xi,\eta),y(\xi,\eta)). \tag{3.37}$$

We then have the following identities:

$$w(\xi,\eta)=u\,(x(\xi,\eta),y(\xi,\eta)),\ \text{and}\ u\,(x,y)=w\,(\xi(x,y),\eta(x,y)). \tag{3.38}$$

In order to find an equation in (ξ,η) variables, we will compute partial derivatives of u of first and second orders in terms of new coordinates, and then substitute in the given equation (3.33). This computation is done by choosing an appropriate identity from (3.38). Applying one of the most fundamental and important results in differential calculus, namely, chain rule, to $u\,(x,y)=w\,(\xi(x,y),\eta(x,y))$ we get

$$\begin{aligned}
u_x(x,y)&=w_\xi\,(\xi(x,y),\eta(x,y))\,\xi_x(x,y)+w_\eta\,(\xi(x,y),\eta(x,y))\,\eta_x(x,y)\\
u_y(x,y)&=w_\xi\,(\xi(x,y),\eta(x,y))\,\xi_y(x,y)+w_\eta\,(\xi(x,y),\eta(x,y))\,\eta_y(x,t)
\end{aligned}$$

Differentiating the above set of equations once more, we get

$$\begin{aligned}
u_{xx}(x,y)&=w_{\xi\xi}\,(\xi(x,y),\eta(x,y))(\xi_x(x,y))^2+2w_{\xi\eta}\,(\xi(x,y),\eta(x,y))\,\xi_x(x,y)\eta_x(x,y)+\\
&\quad w_{\eta\eta}\,(\xi(x,y),\eta(x,y))(\eta_x(x,y))^2,\\
u_{xy}(x,y)&=w_{\xi\xi}\,(\xi(x,y),\eta(x,y))\,\xi_x(x,y)\xi_y(x,y)+w_{\xi\eta}\,(\xi(x,y),\eta(x,y))\,\xi_x(x,y)\eta_y(x,y)+\\
&\quad w_{\xi\eta}\,(\xi(x,y),\eta(x,y))\,\xi_y(x,y)\eta_x(x,y)+w_{\eta\eta}\,(\xi(x,y),\eta(x,y))\,\eta_x(x,y)\eta_y(x,y),\\
u_{yy}(x,y)&=w_{\xi\xi}\,(\xi(x,y),\eta(x,y))\left(\xi_y(x,y)\right)^2+2w_{\xi\eta}\,(\xi(x,y),\eta(x,y))\,\xi_y(x,y)\eta_y(x,y)+\\
&\quad w_{\eta\eta}\,(\xi(x,y),\eta(x,y))\left(\eta_y(x,y)\right)^2.
\end{aligned}$$

Substituting the values of $u_{xx}(x,y)$ and $u_{yy}(x,y)$ into (3.33), we get the equation for w as (3.34) Note that we need not compute D,E,F,G explicitly as they do not play any role in determining type of the equation. It is left to the reader to check that the equation (3.36) holds.

Denoting the determinant of the Jacobian matrix corresponding to the change of coordinates transformation by J, we note that $\left(\xi_x(x,y)\eta_y(x,y)-\xi_y(x,y)\eta_x(x,y)\right)^2 = J^2$ which is positive as $J \neq 0$. Thus the type of the equation does not change under a change of coordinates follows from the equation (3.36). □

3.4.1 ▪ Hyperbolic equations

Theorem 3.10. *Let the equation (3.33) be hyperbolic in a region Ω of the xy-plane. Let $(x_0,y_0)\in\Omega$. Then there exists a change of coordinates $(x,y)\mapsto(\xi,\eta)$ in an open set containing the point (x_0,y_0) such that the equation (3.33) is transformed in the (ξ,η) variables into*

$$w_{\xi\eta} + 2D(\xi,\eta)w_\xi + 2E(\xi,\eta)w_\eta + F(\xi,\eta)w + G(\xi,\eta) = 0. \tag{3.39}$$

Proof. If $a(x_0,y_0) = c(x_0,y_0) = 0$, then we introduce a change of coordinates $(x,y)\mapsto(X,Y)$ where $X(x,y) = x+y$, $Y(x,y) = x-y$. Under this change of coordinates the given equation (3.33) takes a form where the corresponding A and B are non-zero at (X_0,Y_0). Thus we may assume that at least one of the two quantitiess $a(x_0,y_0)$ and $c(x_0,y_0)$ is not zero. Without loss of generality, assume that $a(x_0,y_0)\neq 0$. As a consequence, there exists an open set U containing (x_0,y_0) such that $a(x,y)\neq 0$ for all $(x,y)\in U$.

We know that under a change of coordinates the equation (3.33) transforms to (3.34). Thus for proving the theorem, it is sufficient to find a system of coordinates (ξ,η) so that $A(\xi,\eta) = C(\xi,\eta) = 0$ where A, C are given by (3.35). Thus we need to find ξ,η satisfying

$$a\,(\xi_x(x,y))^2 + 2b\,\xi_x(x,y)\xi_y(x,y) + c\left(\xi_y(x,y)\right)^2 = 0, \tag{3.40a}$$

$$a\left(\eta_y(x,y)\right)^2 + 2b\,\eta_x(x,y)\eta_y(x,y) + c\left(\eta_y(x,y)\right)^2 = 0. \tag{3.40b}$$

Note that both ξ and η must satisfy the same equation, as the equations in (3.40a) and (3.40b) are the same. Thus we need to solve for ξ,η using only one equation. Note that the equation (3.40a) factorizes as

$$\frac{1}{a}\left(a\xi_x + (b-\sqrt{b^2-ac})\xi_y\right)\left(a\xi_x + (b+\sqrt{b^2-ac})\xi_y\right) = 0. \tag{3.41}$$

A function ξ satisfies the equation (3.41) whenever it satisfies either of the first order linear PDEs given by

$$a\xi_x + (b-\sqrt{b^2-ac})\xi_y = 0, \tag{3.42a}$$

$$a\xi_x + (b+\sqrt{b^2-ac})\xi_y = 0. \tag{3.42b}$$

Since we want to find a coordinate change transformation $(\xi,\eta)=(\xi(x,y),\eta(x,y))$, we choose ξ to be a solution of (3.42a), and η to be a solution of (3.42b). Both the equations (3.42) may be solved by method of characteristics. The characteristics corresponding to (3.42a) are governed by the characteristic ODEs given by

$$\frac{dx}{dt}=a,\ \frac{dy}{dt}=b-\sqrt{b^2-ac},\ \frac{d\xi}{dt}=0. \tag{3.43}$$

Thus ξ is constant along the curves $(x(t),y(t))$ which satisfy the ODEs (3.43). On eliminating the parameter t, the base characteristics are governed by the ODE

$$\frac{dy}{dx}=\frac{b(x,y)-\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)}.$$

On differentiating the equation

$$\xi(x(t),y(t))=c$$

w.r.t. t, and using the fact that $(x(t),y(t))$ satisfies the ODEs (3.43), we get

$$\xi_x(x(t),y(t))x'(t)+\xi_y(x(t),y(t))y'(t)=0.$$

From the last equation, we get

$$-\frac{\xi_x}{\xi_y}(x,y)=\frac{dy}{dx}=\frac{b(x,y)-\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)}. \tag{3.44}$$

Similarly the function η will be constant along the characteristic curves determined by the ODE corresponding to (3.42b)

$$\frac{dy}{dx}=\frac{b(x,y)+\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)},$$

and the following relation holds:

$$-\frac{\eta_x}{\eta_y}(x,y)=\frac{dy}{dx}=\frac{b(x,y)-\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)}. \tag{3.45}$$

Since $b^2-ac>0$, we get

$$\begin{vmatrix} \xi_x(x,y) & \xi_y(x,y) \\ \eta_x(x,y) & \eta_y(x,y) \end{vmatrix} \neq 0$$

in view of the relations (3.45) and (3.45). Thus (ξ,η) defines a coordinate transformation near the point (x_0,y_0), by inverse function theorem. □

Remark 3.11. We may make a further change of variables $x'=\xi+\eta$, $y'=\xi-\eta$. The equation (3.39) then transforms into

$$v_{y'y'}-v_{x'x'}+2D'(x',y')v_{x'}+2E'(x',y')v_{y'}+F'(x',y')v+G'(x',y')=0, \tag{3.46}$$

on verifying that

$$A'(x',y')=1,\ B'(x',y')=0,\ C'(x',y')=-1. \tag{3.47}$$

Both the equations (3.39) and (3.46) are known as canonical forms of an equation that is of hyperbolic type in a region Ω. ∎

Example 3.12. Consider the following linear PDE

$$x^2u_{xx}-2xyu_{xy}-3y^2u_{yy}+u_y=0. \tag{3.48}$$

For this equation $a=x^2$, $b=-xy$, $c=-3y^2$. Thus $b^2-ac=4x^2y^2$. Thus the equation (3.48) is of hyperbolic type at every point (x,y) such that $xy\neq 0$, which means that the equation is of hyperbolic type at all points of xy-plane except for the coordinate axes $x=0$ and $y=0$. At points on the coordinate axes, the equation is of parabolic type.

Let us transform the equation (3.48) into its canonical form in the first quadrant. In order to find the new coordinate system (ξ,η), we need to solve the ODEs

$$\frac{dy}{dx}=\frac{b(x,y)\pm\sqrt{b^2(x,y)-a(x,y)c(x,y)}}{a(x,y)}=\frac{-xy\pm 2|xy|}{x^2}=\frac{-y\pm 2y}{x}.$$

Thus we need to solve the two ODEs

$$\frac{dy}{dx}=\frac{y}{x},\quad \text{and}\quad \frac{dy}{dx}=\frac{-3y}{x},$$

whose solutions are given by $x^{-1}y=$ constant, and $x^3y=$ constant respectively. We introduce the following change of coordinates

$$\xi=\xi(x,y)=x^{-1}y,\quad \text{and}\quad \eta=\eta(x,y)=x^3y.$$

On differentiating the equation

$$u(x,y)=w(\xi(x,y),\eta(x,y))=w(x^{-1}y,x^3y) \tag{3.49}$$

w.r.t. x and y we obtain

$$\begin{aligned}
u_x&=-x^{-2}yw_\xi+3x^2yw_\eta,\\
u_{xx}&=x^{-4}y^2w_{\xi\xi}-6y^2w_{\xi\eta}+9x^4y^2w_{\eta\eta}+2x^{-3}w_\xi+6xyw_\eta,\\
u_{xy}&=-x^{-3}yw_{\xi\xi}+2xyw_{\xi\eta}+3x^5yw_{\eta\eta}-x^{-2}w_\xi+3x^2w_\eta,\\
u_y&=x^{-1}w_\xi+x^3w_\eta,\\
u_{yy}&=x^{-2}w_{\xi\xi}+2x^2w_{\xi\eta}+x^6w_{\eta\eta}.
\end{aligned}$$

On substituting these values in the equation (3.48), we get

$$-16x^2y^2w_{\xi\eta}+5x^{-1}w_\xi+x^3w_\eta=0. \tag{3.50}$$

The last equation can be written in the variables ξ,η completely, on expressing x and y as functions of ξ,η. Indeed we have

$$x=x(\xi,\eta)=\sqrt[4]{\frac{\eta}{\xi}},\quad y=y(\xi,\eta)=\sqrt[4]{\xi^3\eta}.$$

Thus the equation (3.50) becomes

$$-16\xi\eta w_{\xi\eta} + 5\sqrt[4]{\frac{\xi}{\eta}}w_\xi + \sqrt[4]{\left(\frac{\eta}{\xi}\right)^3}w_\eta = 0. \tag{3.51}$$

On simplification, the last equation (3.51) takes the form

$$w_{\xi\eta} - \frac{5}{16}\frac{1}{\sqrt[4]{\xi^3\eta^5}}w_\xi - \frac{1}{16}\frac{1}{\sqrt[4]{\xi^7\eta}}w_\eta = 0,$$

which is a canonical form of the given PDE. ∎

3.4.2 ▪ Parabolic equations

Theorem 3.13. *Let the equation (3.33) be parabolic in a region Ω of the xy-plane. Let $(x_0, y_0) \in \Omega$. Then there exists a change of coordinates $(x,y) \mapsto (\xi,\eta)$ in an open set containing the point (x_0, y_0) such that the equation (3.33) is transformed in the (ξ,η) variables into*

$$w_{\eta\eta} + 2D(\xi,\eta)w_\xi + 2E(\xi,\eta)w_\eta + F(\xi,\eta)w + G(\xi,\eta) = 0. \tag{3.52}$$

Proof. Note that either $a(x_0,y_0) \neq 0$ or $c(x_0,y_0) \neq 0$. For, if both $a(x_0,y_0) = 0$ and $c(x_0,y_0) = 0$ holds, then it follows that $b(x_0,y_0) = 0$ since the equation (3.33) is of parabolic type in Ω. But we assumed that (a,b,c) is non-zero at every point (x,y). Thus without loss of generality, we may assume that $a(x_0,y_0) \neq 0$. By continuity of the function a, it follows that $a(x,y) \neq 0$ in some open set U containing the point (x_0,y_0). As a consequence of parabolicity of the equation (3.33), we conclude that $b(x,y) \neq 0$ in U.

We know that under a change of coordinates the equation (3.33) transforms to (3.34). Thus for proving the theorem, it is sufficient to find a system of coordinates (ξ,η) so that $A(\xi,\eta) = B(\xi,\eta) = 0$ where A, B are given by (3.35). Thus we need to find ξ,η satisfying

$$a\left(\xi_x(x,y)\right)^2 + 2b\xi_x(x,y)\xi_y(x,y) + c\left(\xi_y(x,y)\right)^2 = 0, \tag{3.53a}$$

$$a\xi_x(x,y)\eta_x(x,y) + b\left(\xi_x(x,y)\eta_y(x,y) + \xi_y(x,y)\eta_x(x,y)\right) + c\xi_y(x,y)\eta_y(x,y) = 0. \tag{3.53b}$$

Since the equation (3.53a) involves only ξ, we may solve for ξ as in the proof of Theorem 3.10 to get that ξ is constant along the characteristic curves of the equation (3.42a), where the characteristic curves are given by the ODE (on eliminating the parameter t)

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}.$$

Thus whatever may be the choice of η, we have $A(\xi,\eta) \equiv 0$ and consequently $B(\xi,\eta) \equiv 0$ due to the invariance of type of the equation under change of coordinates.

Thus we need to choose η so that $(x,y) \mapsto (\xi(x,y), \eta(x,y)$ gives rise to a nonsingular transformation. Applying inverse function theorem, we conclude that (ξ,η) defines

a new coordinate system near (x_0, y_0). To achieve this we need ξ, η to satisfy the corresponding jacobian to be non-zero, *i.e.*,

$$\begin{vmatrix} \xi_x(x,y) & \xi_y(x,y) \\ \eta_x(x,y) & \eta_y(x,y) \end{vmatrix} \neq 0.$$

In other words, the quantities $\frac{\xi_x}{\xi_y}(x,y)$ and $\frac{\eta_x}{\eta_y}(x,y)$ are not equal. Note that there are infinitely many choices for η.

Since we know that

$$\frac{\xi_x}{\xi_y}(x,y) = -\frac{b(x,y)}{a(x,y)},$$

we choose η satisfying

$$\frac{\eta_x}{\eta_y}(x,y) = \frac{a(x,y)}{b(x,y)},$$

and this means that the curves corresponding to $\xi =$ constant and $\eta =$ constant are orthogonal families of curves. Thus (ξ, η) defines a coordinate transformation near the point (x_0, y_0), by inverse function theorem. The equation (3.52) is known as the canonical form of a parabolic equation. □

Example 3.14. Consider the following linear PDE

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} = 0. \tag{3.54}$$

For this equation $a = x^2$, $b = -xy$, $c = y^2$. Thus $b^2 - ac = 0$. Thus the equation (3.54) is of parabolic type at every point $(x,y) \in \mathbb{R}^2$.

Note that at the point $(x,y) = (0,0)$ the equation reduces to $0 = 0$ due to vanishing of all the coefficients of derivatives of the unknown u, and thus we can determine canonical form in any domain not containing the origin.

Let us transform the equation (3.54) into its canonical form in left and right upper half planes. It is possible to derive a canonical form in upper and lower half planes, which is left to the reader as an exercise. In order to find the new coordinate system (ξ, η), we need to solve the ODE

$$\frac{dy}{dx} = \frac{b(x,y) \pm \sqrt{b^2(x,y) - a(x,y)c(x,y)}}{a(x,y)} = -\frac{y}{x}.$$

to find one ξ, and then we need to choose η so that (ξ, η) represent a coordinate system. Thus we need to solve the ODE

$$\frac{dy}{dx} = -\frac{y}{x},$$

whose solution is given by $xy =$ constant. We choose $\xi(x,y) = xy$, and $\eta(x,y) = x$ so that the jacobian

$$\begin{vmatrix} \xi_x(x,y) & \xi_y(x,y) \\ \eta_x(x,y) & \eta_y(x,y) \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = x \neq 0.$$

Thus we introduce the following change of coordinates

$$\xi = \xi(x,y) = xy, \quad \text{and} \quad \eta = \eta(x,y) = x.$$

On differentiating the equation

$$u(x,y) = w(\xi(x,y), \eta(x,y)) = w(xy, x) \tag{3.55}$$

w.r.t. x and y we obtain

$$\begin{aligned} u_x &= y w_\xi + w_\eta, \\ u_{xx} &= y^2 w_{\xi\xi} + 2y w_{\xi\eta} + w_{\eta\eta}, \\ u_{xy} &= xy w_{\xi\xi} + x w_{\xi\eta} + w_\xi, \\ u_y &= x w_\xi, \\ u_{yy} &= x^2 w_{\xi\xi}. \end{aligned}$$

On substituting these values in the equation (3.54), we get

$$x^2 w_{\eta\eta} - 2xy w_\xi = 0. \tag{3.56}$$

The last equation can be written in the variables ξ, η completely, on expressing x and y as functions of ξ, η. Indeed we have

$$x = x(\xi,\eta) = \eta, \quad y = y(\xi,\eta) = \frac{\xi}{\eta}.$$

Thus the equation (3.56) becomes

$$\eta^2 w_{\eta\eta} - 2\xi w_\xi = 0. \tag{3.57}$$

On simplification, the last equation (3.57) takes the form

$$w_{\eta\eta} - 2\frac{\xi}{\eta^2} w_\xi = 0,$$

which is a canonical form of the given PDE. ∎

3.4.3 ▪ Elliptic equations

Theorem 3.15. *Let the equation (3.33) be elliptic in a region Ω of the xy-plane, and the coefficients a, b, c be real analytic functions in Ω. Let $(x_0, y_0) \in \Omega$. Then there exists a change of coordinates $(x,y) \mapsto (\xi,\eta)$ in an open set containing the point (x_0, y_0) such that the equation (3.33) is transformed in the (ξ,η) variables into*

$$w_{\xi\xi} + w_{\eta\eta} + 2D(\xi,\eta) w_\xi + 2E(\xi,\eta) w_\eta + F(\xi,\eta) w + G(\xi,\eta) = 0, \tag{3.58}$$

Proof. Since $(b^2 - ac)(x_0, y_0) < 0$, both $a(x_0, y_0)$ and $c(x_0, y_0)$ are non-zero. By continuity of the function a, there exists an open set U containing (x_0, y_0) such that $a(x,y) \neq 0$

for all $(x,y) \in U$. The desired canonical form (3.58) requires the change of coordinates transformation $(x,y) \mapsto (\xi,\eta)$ to satisfy $A = C$, and $B = 0$, *i.e.*,

$$a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_y^2 + 2b\eta_x\eta_y + c\eta_y^2 \tag{3.59a}$$

$$a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + c\xi_y\eta_y = 0. \tag{3.59b}$$

Note that the two equations (3.59) are coupled system of first order nonlinear PDEs. Recall that in the hyperbolic and elliptic cases, the equations for ξ and η were decoupled. We can overcome this difficulty by using the assumption of real analyticity of a,b,c and complex variables. The system of equations (3.59) may be rewritten as

$$a(\xi_x^2 - \eta^2) + 2b(\xi_x\xi_y - \eta_x\eta_y) + c(\xi_y^2 - \eta_y^2) = 0, \tag{3.60a}$$

$$ia\xi_x\eta_x + ib\left(\xi_x\eta_y + \xi_y\eta_x\right) + ic\xi_y\eta_y = 0, \tag{3.60b}$$

where $i = \sqrt{-1}$. Define a complex valued function $\Phi = \xi + i\eta$. The system (3.60) is equivalent to

$$a\Phi_x^2 + 2b\Phi_x\Phi_y + c\Phi_y^2 = 0. \tag{3.61}$$

Since $b^2 - ac < 0$, the equation (3.61) does not admit real valued solutions. Note that we came across the same equation while determining the canonical form for hyperbolic equations. Factorizing the equation (3.61) as was done in the proof of Theorem 3.10, we get the following PDEs in two complex variables x,y

$$a\Phi_x + (b + i\sqrt{ac - b^2})\Phi_y = 0, \tag{3.62a}$$

$$a\Phi_x + (b - i\sqrt{ac - b^2})\Phi_y = 0. \tag{3.62b}$$

Since the coefficients a,b,c are real analytic, it is known (by a complex version of Cauchy-Kowalewski theorem) that the system (3.62) has solutions, and the usual method of solving corresponding real variable equations works. If Φ is a solution of (3.62a), then $\Psi := \overline{\Phi}$ is a solution of (3.62b).

Thus Φ, Ψ are constant on the two complex characteristics given by the equation(s)

$$\frac{dy}{dx} = \frac{b(x,y) \pm i\sqrt{a(x,y)c(x,y) - b^2(x,y)}}{a(x,y)} \tag{3.63}$$

corresponding to $+$ and $-$ respectively.

Now define φ and ψ by $\varphi(x,y) = \operatorname{Re}\Phi(x,y)$, and $\psi(x,y) = \operatorname{Im}\Phi(x,y)$ where $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$ are real and imaginary parts of the complex valued function Φ, and let $\xi = \varphi(x,y)$, $\eta = \psi(x,y)$ be the new independent variables.

Note that ξ and η satisfy the system of equations (3.59), and thus the equation (3.33) reduces to the required canonical form (3.58).

□

Example 3.16. Consider the following linear PDE

$$u_{xx} + (1+y^2)^2 u_{yy} - 2y(1+y^2)u_y = 0. \tag{3.64}$$

For this equation $a = 1$, $b = 0$, $c = (1+y^2)^2$. Thus $b^2 - ac = -(1+y^2)^2$. Thus the equation (3.64) is of elliptic type at every point $(x,y) \in \mathbb{R}^2$.

Let us transform the equation (3.64) into its canonical form. In order to find the new coordinate system (ξ, η), we need to solve the ODEs

$$\frac{dy}{dx} = \frac{b(x,y) \pm \sqrt{b^2(x,y) - a(x,y)c(x,y)}}{a(x,y)} = \pm i(1+y^2).$$

Thus we need to solve the ODE

$$\frac{dy}{dx} = i(1+y^2),$$

whose solution is given by $\tan^{-1} y - ix =$ constant. We introduce the following change of coordinates

$$\xi = \xi(x,y) = x, \quad \text{and} \quad \eta = \eta(x,y) = \tan^{-1} y.$$

On differentiating the equation

$$u(x,y) = w(\xi(x,y), \eta(x,y)) = w(x, \tan^{-1} y) \tag{3.65}$$

w.r.t. x and y we obtain

$$\begin{aligned} u_x &= w_\xi, \\ u_{xx} &= w_{\xi\xi}, \\ u_y &= \frac{1}{1+y^2} w_\eta, \\ u_{yy} &= \frac{1}{(1+y^2)^2} w_{\eta\eta} - \frac{2y}{(1+y^2)^2} w_\eta. \end{aligned}$$

On substituting these values in the equation (3.64), we get

$$w_{\xi\xi} + w_{\eta\eta} - 4(\tan\eta)\, w_\eta = 0,$$

which is a canonical form of the given PDE. ∎

3.5 ▪ Classification of second order linear PDE in several variables

In this section, we turn our attention to classification of second order linear PDEs in n independent variables, in a manner that is consistent with the classification for quasilinear PDE in two independent variables (see Section 3.2).

Recall the important role played by characteristics in the classification of second order linear PDEs in two independent variables (see Definition 3.1). According to this definition, a curve Γ is a characteristic if and only if we are unable to determine all the second order partial derivatives of the unknown function u along Γ_0 from the equation, and the Cauchy data. It also turned out that characteristics are precisely the curves along which discontinuities in second order partial derivatives propagate (see Section 3.3 of this chapter for details). In Chapter 2 we saw that when the Cauchy data is a characteristic curve, then

the Cauchy problem either admits infinitely many solutions or none. These two properties of characteristics are seemingly unrelated, but they are indeed related as illustrated in Example 3.18.

A second order linear PDE in n independent variables is of the form

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x})\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\boldsymbol{x})\frac{\partial u}{\partial x_i} + c(\boldsymbol{x})u + d(\boldsymbol{x}) = 0, \tag{3.66}$$

where a_{ij}, b_i, c, d are functions defined on a domain Ω in $\mathbb{R}^n$, and $\boldsymbol{x}$ denotes $\boldsymbol{x} := (x_1, x_2, \cdots, x_n)$.

We may assume that the matrix (a_{ij}) in the equation (3.66) is symmetric, since mixed partial derivatives of u are expected to be equal we may adjust the coefficients a_{ij} and a_{ji} to be equal.

Cauchy problem

Cauchy problem for the equation (3.66) consists of finding a twice continuously differentiable function u that solves the equation (3.66), and satisfies the Cauchy data prescribed on a hypersurface Γ. The Cauchy problem consists of prescribing the values of u and all its first order partial derivatives along the hypersurface Γ. We assume that Γ is the level set of a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$. That is,

$$\Gamma = \{\, \boldsymbol{x} \in \mathbb{R}^n \,:\, \varphi(\boldsymbol{x}) = 0 \,\}.$$

Following the ideas of Section 3.1 to solve Cauchy problem, we would like to solve for partial derivatives of all order of the unknown function u on the hypersurface Γ using the given Cauchy data and the equation (3.66), and then formally propose a power series solution at least when the coefficients a_{ij}, b_i, c, d are analytic functions. The first step is to solve for all second order partial derivatives of u along Γ, but this may not be always possible as observed in the case of two variables in Section 3.1. This kind of possible obstructions in finding second order partial derivatives motivates the following definition, which is consistent with Definition 3.1.

Definition 3.17. *A hypersurface Γ is said to be a characteristic surface with respect to a second order linear PDE if at least one of the second order partial derivatives cannot be determined from the PDE and the Cauchy data.*

Example 3.18. Consider the following Cauchy problem

$$\begin{gathered} u_{xy} = 0, \\ u(x,0) = f(x), \quad u_y(x,0) = g(x). \end{gathered}$$

Note that the general solution of the PDE is given by $u(x,y) = F(x) + G(y)$. This u satisfies the Cauchy data if and only if the following equalities hold:

$$F(x) + G(0) = f(x), \quad G'(0) = g(x). \tag{3.67}$$

But the equalities (3.67) are generally inconsistent, and hence the given Cauchy problem does not possess a solution. Also note that whenever the equalities (3.67) hold, then the Cauchy problem admits infinitely many solutions. Proving this assertion is left to the reader as an exercise.

Note that Cauchy data was prescribed on the line $y = 0$ which is a characteristic curve w.r.t. the given PDE. Also note that u_{yy} cannot be determined along $y = 0$ using the equation and the Cauchy data. Also note that the equation does not feature u_{yy}. ∎

We have the following result which asserts that a surface along which a solution can be singular must be a characteristic surface.

Theorem 3.19. *Let u be continuously differentiable function on a domain $\Omega \subseteq \mathbb{R}^n$. Let u be twice continuously differentiable on Ω except for a hypersurface Γ, and solves the PDE (3.66). Then the hypersurface Γ is a characteristic surface.*

Proof. If Γ is not a characteristic surface, then all second order partial derivatives of u can be determined on Γ using the PDE and the Cauchy data. This means that all second order partial derivatives of u are continuous. This proves that Γ is a characteristic surface if some second order partial derivative of u has jump across Γ. □

Note that characteristic surface was defined using qualitative means in Definition 3.17. We wish to find quantitative characterization of characteristic surface, which will be helpful in searching and finding characteristic surfaces. Example 3.18 suggests that if a certain second order partial derivative $\frac{\partial^2 u}{\partial x_k^2}$ does not feature in the equation, then $x_k = 0$ would be a characteristic surface. This discussion yields a sufficient condition for a hypersurface to be a characteristic surface which is presented in the following lemma.

Lemma 3.20. *Let Γ be a hypersurface given by $\varphi_1(\boldsymbol{x}) = 0$. Then Γ is a characteristic surface if φ_1 satisfies*

$$\nabla\varphi_1.(A\nabla\varphi_1) = 0, \tag{3.68}$$

where A is the matrix $A = (a_{ij})$.

Proof. Consider a non-singular coordinate transformation given by

$$(x_1, x_2, \cdots, x_n) \mapsto (\eta_1, \eta_2, \cdots, \eta_n),$$

where $\eta_i = \varphi_i(\boldsymbol{x})$ for $i = 1, 2, \cdots, n$. We now transform the given PDE (3.66) into the new coordinate system, by setting

$$u(x_1, x_2, \cdots, x_n) = w(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \cdots, \varphi_n(\boldsymbol{x})) \tag{3.69}$$

and finding conditions under which the partial derivative $\frac{\partial^2 w}{\partial \eta_1^2}$ does not appear in the transformed equation. Then Γ would be a characteristic surface.

Since the term $\frac{\partial^2 w}{\partial \eta_1^2}$ would arise only from the principal part of the given PDE, namely

$$\sum_{i,j=1}^{n} a_{ij}(\boldsymbol{x}) \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{3.70}$$

we will find an expression for (3.70) in the new coordinate system. Indeed, the principal part gives rise to

$$\sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 w}{\partial \eta_i \partial \eta_j}, \text{ where } A_{ij} = \sum_{k,l=1}^{n} a_{kl} \frac{\partial \varphi_i}{\partial x_k} \frac{\partial \varphi_j}{\partial x_l}.$$

Thus the analytic characterization of a characteristic surface Γ is obtained by setting the coefficient of $\frac{\partial^2 w}{\partial \eta_1^2}$ to zero, which is

$$\sum_{k,l=1}^{n} a_{kl} \frac{\partial \varphi_1}{\partial x_k} \frac{\partial \varphi_1}{\partial x_l} = 0 \tag{3.71}$$

The equation (3.71) is nothing but the desired equation (3.68). □

Remark 3.21. The previous lemma asserts that a hypersurface $\varphi_1(\boldsymbol{x}) = 0$ is a characteristic surface if the equation (3.68) is satisfied. We are interested in knowing whether the equation (3.66) has any characteristic surfaces. A partial answer is provided by the above lemma, when the equation (3.66) does not feature second order derivative w.r.t. one of the independent variables in its principal part given by (3.70). If the missing variable is x_k, then $x_k = 0$ is a characteristic surface. ■

Definition 3.22. *A linear second order PDE is called*

(i) *elliptic if it has no characteristics.*

(ii) *parabolic if there exists a coordinate system such that one of the independent variables does not appear at all in the principal part, and the principal part is elliptic w.r.t. all the variables that appear in it.*

(iii) *hyperbolic if it is neither elliptic nor parabolic.*

Remark 3.23. (i) Though we discussed canonical forms for equations in two independent variables in Section 3.4, we never defined what is meant by a canonical form. Some authors define a canonical form by saying that a PDE is in canonical form if there are no mixed partial derivatives in the principal part of the differential operator. If we adopt this definition, then $w_{\xi\eta} = 0$ cannot be a canonical form of a hyperbolic equation, as obtained in Section 3.4.

(ii) Even if we adopt the definition of a canonical form as in (i) above, it is a cumbersome task to transform a given equation into its canonical form, as it involves solving several nonlinear PDEs which are obtained by setting the coefficients A_{ij} equal to zero for $i \neq j$. That is,

$$A_{ij} = \sum_{k,l=1}^{n} a_{kl} \frac{\partial \varphi_i}{\partial x_k} \frac{\partial \varphi_j}{\partial x_l} = 0 \text{ for all } i \neq j. \tag{3.72}$$

Note that when $n = 2$ there is only one mixed second order partial derivative (and we need to determine φ_1 and φ_2), and for a general n the number of such derivatives is $\frac{n(n-1)}{2}$ (and we need to determine $\varphi_1, \varphi_2, \cdots, \varphi_n$) thus making it impossible to determine canonical forms from $n = 4$ onwards as (3.72) is an over-determined system of PDEs.

(iii) Thus we abandon the idea of finding a canonical form after determining the type of the equation from $n = 3$ onwards. ■

3.5.1 ▪ Canonical forms for equations with constant coefficients

Though in Remark 3.23 we observed that it is in general difficult and near-impossible to determine canonical forms for equations in more than two independent variables, we are better placed while dealing with equations having constant coefficients.

A second order linear PDE with constant coefficients, in n independent variables is of the form

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu + d = 0, \tag{3.73}$$

and we assume that the matrix $A = (a_{ij})$ is a symmetric matrix.

Since the matrix A is symmetric, it is diagonalizable, and thus there exists an orthogonal matrix Q such that

$$Q^t A Q = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{3.74}$$

where $\text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ represents a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_n$ which are the eigenvalues of A. Since A is symmetric, all the eigenvalues are known to be real numbers.

Denote the ith column of the matrix Q by $\boldsymbol{q}_i$, and introduce the following change of variables:

$$\eta_i(x_1, x_2, \cdots, x_n) = \boldsymbol{q}_i.\boldsymbol{x} \quad \text{for } i = 1, 2, \cdots, n. \tag{3.75}$$

Since the matrix Q is invertible, the linear transformation in (3.75) is indeed invertible. With respect to the new coordinates the principal part of the PDE (3.73) takes the following form

$$\sum_{i=1}^{n} \lambda_i \frac{\partial^2 w}{\partial \eta_i^2}. \tag{3.76}$$

The quantity (3.76) represents the principal part of the partial differential operator in the canonical form for the equation (3.73).

Remark 3.24 (Classification of constant coefficient PDEs). We classified equations with variable coefficients in Definition 3.22. Since equation for a characteristic surface given by (3.68) simplifies for equations in constant coefficients, particularly when expressed in coordinates (3.75).

Thus we have the following consequences of Definition 3.22 for equations with constant coefficients. The PDE (3.73)

(i) is elliptic if all the eigenvalues of A are of same sign.
(ii) is parabolic if atleast one of the eigenvalues of A is zero, and all other eigenvalues are of same sign.
(iii) is hyperbolic if it is neither elliptic nor parabolic. This is the case when the matrix A has a positive eigenvalue and a negative eigenvalue.

Note that if the PDE (3.73) is parabolic, and if the eigenvalue λ_k of A is equal to zero, then the second order derivative $\frac{\partial^2 w}{\partial \eta_k^2}$ does not appear in the principal part (3.76) (of the PDE in the new coordinates (3.75)). ∎

According to the above remark, the equation

$$u_{x_1 x_1} + u_{x_2 x_2} - u_{x_3 x_3} - u_{x_4 x_4} = 0$$

is a hyperbolic equation, with two 'time-like' variables. Some authors call such equations as *ultra hyperbolic*. More percisely, *ultra hyperbolic* equations are those for which the matrix A has at least two positive eigenvalues, two negative eigenvalues, and none of the other eigenvalues is zero.

Exercises

Classification of PDEs

3.1. Give an example of a second order linear partial differential equation such that

(a) the equation is of elliptic type at points of a domain Ω and is of parabolic type on $\mathbb{R}^2 \setminus \Omega$.

(b) the equation is of hyperbolic type at points of a domain Ω and is of parabolic type on $\mathbb{R}^2 \setminus \Omega$.

(c) the equation is of elliptic type at points of a domain Ω and is of hyperbolic type on $\mathbb{R}^2 \setminus \Omega$.

Justify each of your answers.

Canonical forms for PDEs in two variables

3.2. Find the canonical forms for the Tricomi equation

$$u_{yy} - y u_{xx} = 0$$

in the upper and lower half planes, where Tricomi equation is of hyperbolic and elliptic types respectively. (**Answer:** For the upper half plane, with $\xi = 3x - 2y^{3/2}$ and $\eta = 3x + 2y^{3/2}$, the canonical form is

$$w_{\xi\eta} - \frac{1}{6(\eta - \xi)}(w_\xi + w_\eta) = 0.$$

For the lower half plane, with $\xi = \frac{3}{2}x$ and $\eta = (-y)^{\frac{3}{2}}$, the canonical form is

$$w_{\xi\xi} + w_{\eta\eta} + \frac{1}{3\eta} w_\eta = 0.)$$

3.3. [24] Find the general solution of the PDE

$$x^2 u_{xx} - 2xy u_{xy} - 3y^2 u_{yy} = 0$$

by first reducing it to a canonical form. Then find a particular solution that satisfies $u(x,1) = \varphi(x)$ and $u_y(x,1) = \psi(x)$ where φ, ψ are smooth functions.

3.4. [24] Solve the following Cauchy problem

$$u_{xx} + 2\cos x\, u_{xy} - \sin^2 x\, u_{yy} - \sin x\, u_y = 0,$$
$$u(x, \sin x) = \varphi(x), \quad u_y(x, \sin x) = \psi(x),$$

by first reducing the PDE to a canonical form. Here assume that φ, ψ are smooth functions.

Canonical forms for PDEs in several variables

3.5. Let $\Gamma : \varphi(\boldsymbol{x}) = 0$ be a hypersurface. Find the equation satisfied by φ so that Γ is a characteristic surface for the wave equation in n space variables given by

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}\right) = 0.$$

Find a solution of the equation thus obtained. (**Hint:** Guessing a solution is easy and then verify that the guess was correct.)