Chapter 4

Wave equation I: Well-posedness of Cauchy problem

In this chapter, we prove that Cauchy problem for Wave equation is well-posed (see Appendix A for a detailed account of well-posedness) by proving the existence of a solution by explicitly exhibiting a formula, followed by uniqueness of solutions to Cauchy problem. The third part of the Hadamard's criterion of a well-posed problem, which is referred to as continuous dependence on parameters in general, is proved in the form of stability of solutions.

Cauchy problem

Given three functions f, φ, and ψ from appropriate classes of functions, Cauchy problem for wave equation is to find a solution of

$$\Box u \equiv u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_dx_d}\right) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \mathbb{R}^d,\ t > 0. \tag{4.1a}$$

$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{4.1b}$$

$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{4.1c}$$

where $\boldsymbol{x}$ denotes the vector $(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $c > 0$.

Definition 4.1 (Classical Solution).

(i) *A function $u \in C^2\left(\mathbb{R}^d \times (0,\infty)\right) \cap C^1\left(\mathbb{R}^d \times [0,\infty)\right)$ is said to be a classical solution of the Cauchy problem (4.1) if u satisfies the wave equation (4.1a), and the initial conditions (4.1b) and (4.1c) are satisfied.*

(ii) *Let $T > 0$ be fixed. When the Cauchy problem (4.1) is posed for $\boldsymbol{x} \in \mathbb{R}^d$, $0 < t < T$, then the corresponding notion of a classical solution of the Cauchy problem (4.1) is defined by replacing ∞ with T in (i) above.*

Owing to the linearity of the d'Alembertian operator $\Box$, we reduce the problem of solving (4.1) to solving two simpler problems: in one of them $f = 0$, and in the other $\varphi = \psi = 0$.

Let v be a solution of the homogeneous wave equation satisfying the given initial conditions, *i.e.*,

$$\Box v \equiv v_{tt} - c^2\left(v_{x_1x_1} + v_{x_2x_2} + \cdots + v_{x_dx_d}\right) = 0, \quad \boldsymbol{x} \in \mathbb{R}^d,\ t > 0. \tag{4.2a}$$

$$v(\boldsymbol{x},0)=\varphi(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^d, \tag{4.2b}$$
$$v_t(\boldsymbol{x},0)=\psi(\boldsymbol{x}), \quad \boldsymbol{x}\in\mathbb{R}^d, \tag{4.2c}$$

and $\tilde{v}$ be a solution of the nonhomogeneous wave equation with zero initial conditions, *i.e.*,

$$\Box\tilde{v}\equiv\tilde{v}_{tt}-c^2\left(\tilde{v}_{x_1x_1}+\tilde{v}_{x_2x_2}+\cdots+\tilde{v}_{x_dx_d}\right)=f(\boldsymbol{x},t), \quad \boldsymbol{x}\in\mathbb{R}^d,\ t>0. \tag{4.3a}$$
$$\tilde{v}(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\mathbb{R}^d, \tag{4.3b}$$
$$\tilde{v}_t(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\mathbb{R}^d. \tag{4.3c}$$

Letting $u:=v+\tilde{v}$, it is easy to see that u solves the Cauchy problem (4.1).

We study Cauchy problem (4.1) in the space dimensions $d=1,2,3$.

The organization of this chapter is as follows: Existence of solutions to Cauchy problem is proved in Section 4.1, uniqueness of solutions is proved in Section 4.2, and stability of solutions is established in Section 4.3 and thereby completing the proof of well-posedness of Cauchy problem.

In Section 4.4, initial boundary value problems are considered for one dimensional wave equation.

4.1 ▪ Existence of solutions

In this section, we derive an expression for solution to the homogeneous Cauchy problem (4.2) in Subsection 4.1.1 (for $d=1$), Subsection 4.1.2 (for $d=3$), Subsection 4.1.3 (for $d=2$). The nonhomogeneous Cauchy problem (4.3) will be solved in Subection 4.1.4 using Characteristic coordinates for $d=1$, and using Duhamel's principle for a general d.

We also prove that the derived expression is indeed a classical solution, when the data f,φ,ψ are sufficiently smooth.

4.1.1 ▪ Case of one dimensional wave equation

Cauchy problem for one dimensional wave equation takes the form

$$u_{tt}=c^2u_{xx}, \quad x\in\mathbb{R},\ t>0, \tag{4.4a}$$
$$u(x,0)=\varphi(x), \qquad x\in\mathbb{R}, \tag{4.4b}$$
$$u_t(x,0)=\psi(x), \qquad x\in\mathbb{R}. \tag{4.4c}$$

Main steps in solving the above Cauchy problem are

(i) The wave equation (4.4a) is transformed into its canonical form by changing the independent variables from (x,t) to the characteristic coordinates. The resultant equation is then solved.

(ii) The general solution of the wave equation (4.4a) is obtained by change of variables in the solution of the transformed equation.

(iii) A solution of Cauchy problem is then obtained by imposing the initial conditions (4.4b) and (4.4c), which results in the d'Alembert's formula.

Canonical form of Wave operator

In this section, we will deduce a canonical form of the wave equation (4.4a). To this end, let us analyze how the wave equation transforms when a new coordinate system is introduced

on $\mathbb{R}\times\mathbb{R}$ *i.e.*, the coordinate system (x,t) undergoes a change of variables to (ξ,η), where $(\xi,\eta)=(\xi(x,t),\eta(x,t))$.

We have the following identites:

$$w(\xi,\eta)=u\left(x(\xi,\eta),t(\xi,\eta)\right),\ \text{and}\ u(x,t)=w\left(\xi(x,t),\eta(x,t)\right). \tag{4.5}$$

We will compute u_{tt} and u_{xx} by choosing an appropriate identity from (4.5). Applying one of the most fundamental and important results in differential calculus, namely, chain rule, to $u(x,t)=w(\xi(x,t),\eta(x,t))$ we get

$$\begin{aligned}
u_t(x,t)&=w_\xi\left(\xi(x,t),\eta(x,t)\right)\xi_t(x,t)+w_\eta\left(\xi(x,t),\eta(x,t)\right)\eta_t(x,t)\\
u_x(x,t)&=w_\xi\left(\xi(x,t),\eta(x,t)\right)\xi_x(x,t)+w_\eta\left(\xi(x,t),\eta(x,t)\right)\eta_x(x,t)
\end{aligned}$$

Differentiating the above set of equations once more, we get

$$\begin{aligned}
u_{tt}(x,t)=&\,w_{\xi\xi}\left(\xi(x,t),\eta(x,t)\right)(\xi_t(x,t))^2+2w_{\xi\eta}\left(\xi(x,t),\eta(x,t)\right)\xi_t(x,t)\eta_t(x,t)+\\
&\qquad w_{\eta\eta}\left(\xi(x,t),\eta(x,t)\right)(\eta_t(x,t))^2,\\
u_{xx}(x,t)=&\,w_{\xi\xi}\left(\xi(x,t),\eta(x,t)\right)(\xi_x(x,t))^2+2w_{\xi\eta}\left(\xi(x,t),\eta(x,t)\right)\xi_x(x,t)\eta_x(x,t)+\\
&\qquad w_{\eta\eta}\left(\xi(x,t),\eta(x,t)\right)(\eta_x(x,t))^2.
\end{aligned}$$

Substituting the values of $u_{tt}(x,t)$ and $u_{xx}(x,t)$ into (4.4a), we get

$$\begin{aligned}
\Box u(x,t)=&\,w_{\xi\xi}\left(\xi(x,t),\eta(x,t)\right)\left(\xi_t^2-c^2\xi_x^2\right)(x,t)+2w_{\xi\eta}\left(\xi(x,t),\eta(x,t)\right)\left(\xi_t\eta_t-c^2\xi_x\eta_x\right)(x,t)\\
&+w_{\eta\eta}\left(\xi(x,t),\eta(x,t)\right)\left(\eta_t^2-c^2\eta_x^2\right)(x,t)
\end{aligned} \tag{4.6}$$

To obtain a canonical form of Wave equation, which is an hyperbolic equation, we require that the coefficients of $w_{\xi\xi}$ and $w_{\eta\eta}$ be zero. That is,

$$\left(\xi_t^2-c^2\xi_x^2\right)(x,t)=0,\quad \left(\eta_t^2-c^2\eta_x^2\right)(x,t)=0. \tag{4.7}$$

Note that both ξ and η satisfy the same equation. Since we are interested in (ξ,η) to be a change of coordinates from (x,t), we must choose two functionally independent (Jacobian non-zero) solutions of (4.7). Let us consider the equation satisfied by ξ, which may be factored as

$$(\xi_t-c\xi_x)(\xi_t+c\xi_x)=0. \tag{4.8}$$

The factorization (4.8) allows us to find ξ,η satisfying

$$\xi_t+c\xi_x=0,\ \text{and}\ \eta_t-c\eta_x=0. \tag{4.9}$$

From the last equation, we conclude that ξ and η are arbitrary differentiable functions depending solely on $x-ct$ and $x+ct$ respectively. For simplicity, we choose

$$\xi(x,t)=x-ct,\ \text{and}\ \eta(x,t)=x+ct. \tag{4.10}$$

The new coordinate system found in this way is called system of characteristic coordinates, in view of the following definition

Definition 4.2 (characteristics). *The two families of lines*

$$x-ct=\text{constant},\quad x+ct=\text{constant}$$

are called the **characteristics** *of the one dimensional wave equation.*

Note that characteristics consist of two families of straight lines with slopes $\pm\frac{1}{c}$. When $c=1$, the two families of characterisitcs are orthogonal.

Substituting the expressions for $\xi(x,t)$ and $\eta(x,t)$ from (4.10) into the equation (4.6) yields

$$u_{tt}(x,t)-c^2u_{xx}(x,t)=-4c^2w_{\xi\eta}(\xi(x,t),\eta(x,t)) \tag{4.11}$$

In other words,

$$u_{tt}(x,t)-c^2u_{xx}(x,t)=-4c^2w_{\xi\eta}(\xi(x,t),\eta(x,t)). \tag{4.12}$$

Thus the homogeneous wave equation (4.4a) reduces to

$$w_{\xi\eta}(\xi,\eta)=0 \tag{4.13}$$

in the coordinates ξ and η.

By integrating the equation (4.13) first w.r.t. η, and then w.r.t. ξ, we get

$$w_\xi(\xi,\eta)=f(\xi)$$
$$w(\xi,\eta)=\int^{\xi} f(\xi)\,d\xi+G(\eta)$$

Thus the general solution of (4.13) is of the form

$$w(\xi,\eta)=F(\xi)+G(\eta).$$

Note that the function $w\in C^2(\mathbb{R}\times\mathbb{R})$ if and only if $F\in C^2(\mathbb{R})$ and $G\in C^2(\mathbb{R})$.

Thus solution u of the Wave equation in (x,t) coordinates is given by

$$u(x,t)=F(x-ct)+G(x+ct), \tag{4.14}$$

where $F\in C^2(\mathbb{R})$ and $G\in C^2(\mathbb{R})$. If F and G are twice continuously differentiable, then the function u defined above is easily seen to be a classical solution; which we record below in the form of a lemma whose proof is left to the reader.

Lemma 4.3 (Classical solution). *Let $F\in C^2(\mathbb{R})$ and $G\in C^2(\mathbb{R})$. Then the function u defined by the formula (4.14) is a classical solution of the homogeneous wave equation (4.4a).* ∎

Remark 4.4 (Geometric interpretation: Wave propagation). To present a geometric interpretation, let us fix a time instant $t=t_0$.

(i) The graph of $F(x+ct_0)$ is precisely the graph of $F(x)$ shifted by ct_0 units towards the left. Thus $F(x+ct)$ represents a *backward* moving wave with speed c.

(ii) The graph of $G(x-ct_0)$ is precisely the graph of $G(x)$ shifted by ct_0 units towards the right. Thus $G(x-ct)$ represents a *forward* moving wave with speed c.

Thus we conclude that any solution of the wave equation is a superposition of forward, and backward moving waves. ∎

Solution of Cauchy problem for homogeneous Wave equation: formula of d'Alembert

Recall from (4.14) that the general solution of the wave equation is given by

$$u(x,t) = F(x-ct) + G(x+ct).$$

Imposing the initial condition $u(x,0) = \varphi(x)$ yields the equation

$$F(x) + G(x) = \varphi(x).$$

Imposing the other initial condition $u_t(x,0) = \psi(x)$ gives

$$-cF'(x) + cG'(x) = \psi(x).$$

Integrating the last equation over the interval $[0,x]$ yields

$$-F(x) + G(x) = \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0)$$

Solving for F and G, we get

$$F(\xi) = \frac{1}{2}\varphi(\xi) - \frac{1}{2c}\int_0^\xi \psi(s)\,ds + \frac{F(0)-G(0)}{2} \tag{4.15}$$

$$G(\eta) = \frac{1}{2}\varphi(\eta) + \frac{1}{2c}\int_0^\eta \psi(s)\,ds - \frac{F(0)-G(0)}{2} \tag{4.16}$$

Substituting these values in the general solution, we get the following expression for u,

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds, \tag{4.17}$$

which is known as d'Alembert's formula.

We now have the following result whose proof is left as an exercise to the reader.

Theorem 4.5 (Classical solution). *Let the Cauchy data be such that $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. Then the function u defined by the formula (4.17) is a classical solution of the Cauchy problem for homogeneous wave equation (4.4).* ∎

4.1.2 ▪ Case of three dimensional wave equation

Cauchy problem for three dimensional wave equation takes the form

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}\right) = 0, \quad \boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3,\ t > 0, \tag{4.18a}$$

$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3, \tag{4.18b}$$

$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3. \tag{4.18c}$$

Main steps in solving the above Cauchy problem are

(i) Note that the wave equation (4.18a) is already in a canonical form. In the case of one dimensional wave equation, the wave equation was reduced to a simpler canonical form $w_{\xi\eta}$ and a general solution was determined. As discussed in Remark 3.25, it is very cumbersome, and perhaps impossible to derive such a simpler a canonical form as in the case of one dimensional wave equation. Thus following the procedure implemented for wave equation in one space dimension is ruled out.

(ii) The Cauchy problem (4.18) will be reduced to an equivalent Cauchy problem for a one dimensional wave equation by following *the method of spherical means*. Formula of d'Alembert will be used to solve the resultant one dimensional wave equation.

(iii) A solution of Cauchy problem (4.18) is then retrieved from the d'Alembert formula so-obtained.

Spherical means and the Darboux formula

In this paragraph, we deal with functions of d real variables (not necessarily $d = 3$). Let $B(\boldsymbol{x};\rho)$ and $S(\boldsymbol{x};\rho)$ denote the open ball of radius $\rho > 0$ and sphere of radius $\rho > 0$, with center at $\boldsymbol{x} \in \mathbb{R}^d$ respectively. That is, denoting the Euclidean norm of $\boldsymbol{x} \in \mathbb{R}^d$ by $||\boldsymbol{x}||$, we have

$$B(\boldsymbol{x};\rho) := \{\boldsymbol{y} : ||\boldsymbol{x}-\boldsymbol{y}|| < \rho\} \text{ and } S(\boldsymbol{x};\rho) := \{\boldsymbol{y} : ||\boldsymbol{x}-\boldsymbol{y}|| = \rho\}. \tag{4.19}$$

Let ω_d denote the measure of the unit sphere $S(0;1)$ in $\mathbb{R}^d$, and $d\omega$ denote the surface measure on $S(0;1)$.

Definition 4.6 (Spherical means). *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. The spherical mean of g, denoted by $M_g(\boldsymbol{x},\rho)$, is a function defined on $\mathbb{R}^d \times (0,\infty)$, given by*

$$M_g(\boldsymbol{x},\rho) := \frac{1}{|S(\boldsymbol{x};\rho)|}\int_{S(\boldsymbol{x};\rho)} g(\boldsymbol{y})\,d\sigma \tag{4.20}$$

where $d\sigma$ is the surface measure on the sphere $S(\boldsymbol{x};\rho)$, and $|S(\boldsymbol{x};\rho)|$ denotes the measure of $S(\boldsymbol{x};\rho)$ w.r.t. $d\sigma$.

Remark 4.7 (on spherical means). The formula (4.20) for spherical means $M_g(\boldsymbol{x},\rho)$ may be written as

$$M_g(\boldsymbol{x},\rho) = \frac{1}{|S(\boldsymbol{x};\rho)|}\int_{S(\boldsymbol{x};\rho)} g(\boldsymbol{y})\,d\sigma = \frac{1}{\rho^{d-1}\omega_d}\int_{||\boldsymbol{x}-\boldsymbol{y}||=\rho} g(\boldsymbol{y})\,d\sigma = \frac{1}{\omega_d}\int_{||\nu||=1} g(\boldsymbol{x}+\rho\nu)\,d\omega, \tag{4.21}$$

since $|S(\boldsymbol{x};\rho)| = \rho^{d-1}\omega_d$, and $d\sigma = \rho^{d-1}d\omega$.

Lemma 4.8. *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous function. Let $M_g : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ be as defined by (4.20).*

(i) *The function M_g can be extended to the domain $\mathbb{R}^d \times \mathbb{R}$ such that $\rho \mapsto M_g(\boldsymbol{x},\rho)$ is an even function, for each fixed $\boldsymbol{x}$.*

(ii) *If $g \in C^k(\mathbb{R}^d)$, then so is the function $\boldsymbol{x} \mapsto M_g(\boldsymbol{x},\rho)$ for each ρ.*

(iii) *The function g can be retrieved from $M_g(\boldsymbol{x},\rho)$ in the following sense:*

$$\lim_{\rho\to 0} M_g(\boldsymbol{x},\rho) = g(x) \text{ for all } \boldsymbol{x} \in \mathbb{R}^d. \tag{4.22}$$

Proof. Since the unit sphere is invariant under the mapping $\nu \mapsto -\nu$, the last integral in (4.21) may be re-written as

$$\int_{||\nu||=1} g(\boldsymbol{x}+\rho\nu)\,d\omega = \int_{||\nu||=1} g(\boldsymbol{x}-\rho\nu)\,d\omega. \tag{4.23}$$

Thus we get

$$M_g(\boldsymbol{x},\rho)=\frac{1}{\rho^{d-1}}\int_{\|\nu\|=1} g(\boldsymbol{x}+\rho\nu)\,d\omega=\frac{1}{\rho^{d-1}}\int_{\|\nu\|=1} g(\boldsymbol{x}-\rho\nu)\,d\omega. \tag{4.24}$$

The last equation (4.24) suggests that M_g can be extended to the domain $\mathbb{R}^d\times\mathbb{R}$ such that $\rho\mapsto M_g(\boldsymbol{x},\rho)$ is an even function, for each fixed $\boldsymbol{x}$. Setting $M_g(\boldsymbol{x},0)=g(\boldsymbol{x})$, the function $\rho\mapsto M_g(\boldsymbol{x},\rho)$ is now meaningful for all $\rho\in\mathbb{R}$. This completes the proof of (i).

Assertion (ii) of lemma follows by differentiating under the integral sign. Assertion (iii) is an easy consequence of the continuity of g. □

Lemma 4.9 (Darboux formula). *Let $g\in C^2(R^d)$. Then M_g satisfies the partial differential equation*

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{d-1}{\rho}\frac{\partial}{\partial\rho}\right)M_g(\boldsymbol{x},\rho)=\Delta M_g(\boldsymbol{x},\rho), \tag{4.25}$$

where Δ stands for the Laplacian in the variables $\boldsymbol{x}=(x_1,x_2,\cdots,x_d)\in\mathbb{R}^d$.

Proof. Using integration by parts formula (see Appendix C), we get

$$\int_{\|\boldsymbol{x}-\boldsymbol{y}\|<\rho}\Delta g(\boldsymbol{y})\,d\boldsymbol{y}=\int_{\|\boldsymbol{x}-\boldsymbol{y}\|=\rho}\nabla g(\boldsymbol{y}).\nu(\boldsymbol{y})\,d\sigma(\boldsymbol{y}) \tag{4.26}$$

$$=\rho^{d-1}\int_{\|\nu\|=1}\nabla g(\boldsymbol{x}+\rho\nu).\nu\,d\omega \tag{4.27}$$

$$=\rho^{d-1}\frac{\partial}{\partial\rho}\int_{\|\nu\|=1} g(\boldsymbol{x}+\rho\nu)\,d\omega. \tag{4.28}$$

Also

$$\frac{\partial}{\partial\rho}M_g(\boldsymbol{x},\rho)=\frac{1}{\rho^{d-1}\omega_d}\int_{\|\boldsymbol{x}-\boldsymbol{y}\|<\rho}\Delta g(\boldsymbol{y})\,d\boldsymbol{y} \tag{4.29}$$

$$=\frac{1}{\rho^{d-1}\omega_d}\int_0^\rho r^{d-1}\Delta\int_{\|\nu\|=1} g(\boldsymbol{x}+r\nu)\,d\omega\,dr. \tag{4.30}$$

Multiply the equation (4.30) with ρ^{d-1}, and then differentiate w.r.t. ρ, to get

$$\frac{\partial}{\partial\rho}\left(\rho^{d-1}\frac{\partial}{\partial\rho}M_g(\boldsymbol{x},\rho)\right)=\Delta\left(\rho^{d-1}M_g(\boldsymbol{x},\rho)\right). \tag{4.31}$$

Carrying out the differentiation on both sides of the equation (4.31), and then cancelling the factor ρ^{d-1} yields the Darboux formula (4.25). □

Reduction of the Cauchy problem (4.18) to an equivalent Cauchy problem

Let $u\in C^2(\mathbb{R}^d\times\mathbb{R})$ be a solution of (4.2). Then for each fixed $t\in\mathbb{R}$, we have

$$M_u(\boldsymbol{x},t,\rho)=\frac{1}{\omega_d}\int_{\|\nu\|=1} u(\boldsymbol{x}+\rho\nu,t)\,d\omega. \tag{4.32}$$

The following equalities follow by applying Laplacian operator to the last equation:

$$\begin{aligned}\Delta M_u(\boldsymbol{x},t,\rho) &= \frac{1}{\omega_d}\int_{\|\nu\|=1} \Delta u(\boldsymbol{x}+\rho\nu,t)\,d\omega \\ &= \frac{1}{c^2\omega_d}\int_{\|\nu\|=1} \frac{\partial^2}{\partial t^2}u(\boldsymbol{x}+\rho\nu,t)\,d\omega \\ &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}M_u(\boldsymbol{x},t,\rho)\end{aligned}$$

Thus we are led to define, for each fixed $\boldsymbol{x}\in\mathbb{R}^d$,

$$M(\rho,t) := \frac{1}{\omega_d}\int_{\|\nu\|=1} u(\boldsymbol{x}+\rho\nu,t)\,d\omega. \tag{4.34}$$

Lemma 4.10 (Equivalent Cauchy problems). *The following statements concerning an $u\in C^2(\mathbb{R}^d\times\mathbb{R})$ are equivalent.*

1. *The function u is a solution of the Cauchy problem (4.2).*
2. *The function $M(\rho,t)$ solves the following Cauchy problem*

$$\frac{\partial^2}{\partial t^2}M(\rho,t) = c^2\left(\frac{\partial^2}{\partial\rho^2}+\frac{d-1}{\rho}\frac{\partial}{\partial\rho}\right)M(\rho,t), \quad \rho\in\mathbb{R},\ t>0, \tag{4.35a}$$

$$M(\rho,0) = M_\varphi(\boldsymbol{x},\rho), \qquad \rho\in\mathbb{R}, \tag{4.35b}$$

$$\frac{\partial M}{\partial t}(\rho,0) = M_\psi(\boldsymbol{x},\rho), \qquad \rho\in\mathbb{R}. \tag{4.35c}$$

Proof. Statement 2 follows from Statement 1 by direct computation using the definition of $M(\rho,t)$, computation in the equations (4.33), and Darboux formula, which is left as an exercise to the reader.

Statement 1 follows from Statement 2 in view of Lemma 4.8, the reader is urged to fill in the details of its proof. □

Solution of Cauchy problem (4.18): Formulae of Poisson

From now on we consider the case $d=3$, till the end of this subsection.

Setting $L(\rho,t) := \rho M(\rho,t)$, we see that L satisfies the following Cauchy problem

$$\frac{\partial^2}{\partial t^2}L(\rho,t) = c^2\frac{\partial^2}{\partial\rho^2}(L(\rho,t)), \quad \rho\in\mathbb{R},\ t>0, \tag{4.36a}$$

$$L(\rho,0) = \rho M_\varphi(\boldsymbol{x},\rho), \qquad \rho\in\mathbb{R}, \tag{4.36b}$$

$$\frac{\partial L}{\partial t}(\rho,0) = \rho M_\psi(\boldsymbol{x},\rho), \qquad \rho\in\mathbb{R}. \tag{4.36c}$$

Using the d'Alembert formula (4.17), we get

$$L(\rho,t) = \rho M(\rho,t) = \frac{1}{2}\Big[(\rho-ct)M_\varphi(\boldsymbol{x},\rho-ct)+(\rho+ct)M_\varphi(\boldsymbol{x},\rho+ct)\Big]+$$

$$\frac{1}{2c}\int_{\rho-ct}^{\rho+ct} sM_{\psi}(\boldsymbol{x},s)\,ds. \tag{4.37}$$

We cannot retrieve u from the equation (4.37) easily, by dividing both the sides with ρ and then passing to the limit as $\rho \to 0$. We work our way around this difficulty by differentiating w.r.t. ρ the equation (4.37), and use the resultant equation to pass to the limit as $\rho \to 0$.

Differentiating the equation (4.37) w.r.t. ρ, we get

$$\begin{aligned} M(\rho,t)+\rho\frac{\partial M}{\partial \rho}(\rho,t) &= \frac{1}{2}\Big[M_{\varphi}(\boldsymbol{x},\rho-ct)+M_{\varphi}(\boldsymbol{x},\rho+ct)\Big] \\ &\quad + \frac{1}{2}\left[(\rho-ct)\frac{\partial}{\partial\rho}M_{\varphi}(\boldsymbol{x},\rho-ct)+(\rho+ct)\frac{\partial}{\partial\rho}M_{\varphi}(\boldsymbol{x},\rho+ct)\right] \\ &\quad + \frac{1}{2c}\Big[(\rho+ct)M_{\psi}(\boldsymbol{x},\rho+ct)-(\rho-ct)M_{\psi}(\boldsymbol{x},\rho-ct)\Big]. \end{aligned} \tag{4.38}$$

Expanding the RHS of (4.38), we get

$$\begin{aligned} M(\rho,t)+\rho\frac{\partial M}{\partial \rho}(\rho,t) &= \frac{1}{8\pi}\left[\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+(\rho-ct)\nu)\,d\omega+\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+(\rho+ct)\nu)\,d\omega\right] \\ &\quad + \frac{1}{8\pi}\left[(\rho-ct)\int_{\|\nu\|=1}\nabla\varphi(\boldsymbol{x}+(\rho-ct)\nu).\nu\,d\omega+(\rho+ct)\int_{\|\nu\|=1}\nabla\varphi(\boldsymbol{x}+(\rho+ct)\nu).\nu\,d\omega\right] \\ &\quad + \frac{1}{8c\pi}\left[(\rho+ct)\int_{\|\nu\|=1}\psi(\boldsymbol{x}+(\rho+ct)\nu)\,d\omega-(\rho-ct)\int_{\|\nu\|=1}\psi(\boldsymbol{x}+(\rho-ct)\nu)\,d\omega\right]. \end{aligned} \tag{4.39}$$

Now taking the limit as $\rho \to 0$ in the equation (4.39), we get

$$\begin{aligned} u(\boldsymbol{x},t) &= \frac{1}{8\pi}\left[\int_{\|\nu\|=1}\varphi(\boldsymbol{x}-ct\nu)\,d\omega+\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+ct\nu)\,d\omega\right] \\ &\quad + \frac{1}{8\pi}\left[-ct\int_{\|\nu\|=1}\nabla\varphi(\boldsymbol{x}-ct\nu).\nu\,d\omega+ct\int_{\|\nu\|=1}\nabla\varphi(\boldsymbol{x}+ct\nu).\nu\,d\omega\right] \\ &\quad + \frac{1}{8\pi}\left[t\int_{\|\nu\|=1}\psi(\boldsymbol{x}+ct\nu)\,d\omega+t\int_{\|\nu\|=1}\psi(\boldsymbol{x}-ct\nu)\,d\omega\right]. \end{aligned} \tag{4.40}$$

By Lemma 4.8, the equation (4.40) reduces to

$$\begin{aligned} u(\boldsymbol{x},t) &= \frac{1}{4\pi}\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+ct\nu)\,d\omega+\frac{1}{4\pi}ct\int_{\|\nu\|=1}\nabla\varphi(\boldsymbol{x}+ct\nu).\nu\,d\omega \\ &\quad + \frac{1}{4\pi}t\int_{\|\nu\|=1}\psi(\boldsymbol{x}+ct\nu)\,d\omega. \end{aligned} \tag{4.41}$$

The last equation (4.41) may be written as

$$u(\boldsymbol{x},t)=\frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{\|\nu\|=1}\varphi(\boldsymbol{x}+ct\nu)\,d\omega\right)+\frac{1}{4\pi}t\int_{\|\nu\|=1}\psi(\boldsymbol{x}+ct\nu)\,d\omega. \tag{4.42}$$

In the formula (4.42), converting the integrals on the unit sphere to those on $S(\boldsymbol{x}, ct)$ yields us another formula for the solution of the Cauchy problem as

$$4\pi c^2 u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{t}\int_{S(\boldsymbol{x},ct)} \varphi(\boldsymbol{y})\, d\sigma\right) + \frac{1}{t}\int_{S(\boldsymbol{x},ct)} \psi(\boldsymbol{y})\, d\sigma \tag{4.43}$$

On the other hand, the formula (4.41) gives us

$$4\pi c^2 u(\boldsymbol{x}, t) = \frac{1}{t^2}\int_{S(\boldsymbol{x},ct)} \{t\psi(\boldsymbol{y}) + \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}).(\boldsymbol{y}-\boldsymbol{x})\}\, d\sigma \tag{4.44}$$

The formulae (4.41), (4.42), (4.43), (4.44) are known as Poisson's formulae, which are also called sometimes as Kirchoff's formulae.

This finishes the derivation of a formula for the solution of Cauchy problem (4.18).

The following results says that all these formulae represent a classical solution of the Cauchy problem, if the Cauchy data is sufficiently smooth, the proof is left as an exercise to the reader.

Theorem 4.11 (Classical solution). *Let $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^2(\mathbb{R}^3)$. Then a classical solution of the Cauchy problem is given by*

$$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{S(\boldsymbol{x},ct)} \varphi(\boldsymbol{y})\, d\sigma\right) + \frac{1}{4\pi c^2 t}\int_{S(\boldsymbol{x},ct)} \psi(\boldsymbol{y})\, d\sigma. \tag{4.45}$$

4.1.3 ▪ Case of two dimensional wave equation via Hadamard's method of descent

Cauchy problem for two dimensional wave equation takes the form

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2}\right) = 0, \quad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2,\ t > 0, \tag{4.46a}$$

$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2, \tag{4.46b}$$

$$u_t(\boldsymbol{x}, 0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2. \tag{4.46c}$$

Main steps in solving the above Cauchy problem are

(i) All the hard work was already done in deriving a formula for solution of Cauchy problem in three space dimensions. To find a solution in two space dimensions, we follow Hadamard's method of descent, which presumes that solutions of Cauchy problem (4.46) are actually special solutions of the Cauchy problem (4.18) in three space dimensions that do not depend on the x_3 variable.

(ii) We then use one of the Poisson's formulae (4.41)-(4.44), and reduce the integrals on the spheres in three dimensions to those on disks in two dimensions. The resultant formulae yield a solution for the Cauchy problem (4.46) for two dimensional wave equation.

Let S denote the sphere with center at $(x_1, x_2, 0)$ and radius ct, *i.e.*,

$$S = \{\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3 : (x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2 = c^2t^2\}. \tag{4.47}$$

From the formula (4.45), we get

$$\begin{aligned} u(x_1,x_2,t) &= u(x_1,x_2,0,t) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_S \varphi(y_1,y_2)\,d\sigma\right) + \frac{1}{4\pi c^2 t}\int_S \psi(y_1,y_2)\,d\sigma. \end{aligned} \tag{4.48}$$

Let us compute the integral $\int_S \varphi(y_1,y_2)\,d\sigma$, and computation of the other integral is exactly similar.

Let S^+ denote the upper half of the sphere S. Since integrand is independent of the y_3 variable, we have

$$\int_S \varphi(y_1,y_2)\,d\sigma = 2\int_{S^+} \varphi(y_1,y_2)\,d\sigma. \tag{4.49}$$

Note that the upper half of the sphere is indeed the graph of the function $h : D((x_1,x_2),ct) \to \mathbb{R}$ given by

$$h(y_1,y_2) = \sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}, \tag{4.50}$$

where $D((x_1,x_2),ct)$ denotes the open disk with center at (x_1,x_2) having radius ct.

Hence the surface measure $d\sigma$ on S^+ is thus given by

$$d\sigma = \sqrt{1+\|\nabla h(y_1,y_2)\|^2}\,dy_1dy_2, \tag{4.51}$$

which becomes

$$d\sigma = \frac{ct}{\sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2, \tag{4.52}$$

Thus the formula (4.48) simplifies to

$$\begin{aligned} u(x_1,x_2,t) = \frac{\partial}{\partial t}&\left(\frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\varphi(y_1,y_2)}{\sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2\right) \\ &+ \frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\psi(y_1,y_2)}{\sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2 \end{aligned} \tag{4.53}$$

Carrying out the differentiation in equation (4.53), we get

$$u(\boldsymbol{x},t) = \frac{1}{2\pi ct^2}\int_{D(\boldsymbol{x},ct)} \frac{t\varphi(\boldsymbol{y})+t\nabla\varphi(\boldsymbol{y}).(\boldsymbol{y}-\boldsymbol{x})+t^2\psi(\boldsymbol{y})}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}. \tag{4.54}$$

Re-writing the integral in the equation (4.54) on the unit disk, we get

$$\begin{aligned} u(\boldsymbol{x},t) = \frac{1}{2\pi}&\int_{D(0,1)} \frac{\varphi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z} + \frac{ct}{2\pi}\int_{D(0,1)} \frac{\nabla\varphi(\boldsymbol{x}+ct\boldsymbol{z}).\boldsymbol{z}}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z} \\ &+ \frac{t}{2\pi}\int_{D(0,1)} \frac{\psi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z}. \end{aligned} \tag{4.55}$$

The following result says that the formula (4.53) represents a classical solution of the Cauchy problem, if the Cauchy data is sufficiently smooth, the proof is left as an exercise to the reader.

Theorem 4.12 (Classical solution). *Let $\varphi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$. Then a classical solution of the Cauchy problem is given by*

$$u(x_1,x_2,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\varphi(y_1,y_2)}{\sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2\right) + \frac{1}{2\pi c}\int_{D((x_1,x_2),ct)} \frac{\psi(y_1,y_2)}{\sqrt{c^2t^2-(x_1-y_1)^2-(x_2-y_2)^2}}\,dy_1dy_2, \tag{4.56}$$

where $D((x_1,x_2),ct)$ denotes the open disk with center at (x_1,x_2) having radius ct.

The formulae (4.56), (4.54), (4.55) are known as Poisson's formulae, and sometimes also as Kirchhoff's formulae.

4.1.4 ▪ Nonhomogeneous problem

As discussed at the beginning of this section, a solution of (4.1) can be obtained by adding a solution of (4.2) to a solution of (4.3). In the previous subsections, we already solved the Cauchy problem (4.2). In this paragraph we concentrate on finding a solution to (4.3), and then find a solution to (4.1).

We find a solution of (4.3) by a general principle called *Duhamel's principle* for all dimensions d. However note that wave equation in one space dimension is special when compared to wave equation in more than one space dimensions, in the sense that a canonical form exists which has a very simple form (in the characterisitcs coordinates) and hence is easier to solve. Hence we present a method based on characteristic coordinates and solve (4.3) for $d = 1$.

Existence of solutions via Characteristic Coordinates

The Cauchy problem (4.3) (for $d = 1$), in view of (4.12), is equivalent to the following problem in the characterisitcs coordinates given by (4.10):

$$w_{\xi\eta}(\xi,\eta) = -\frac{1}{4c^2} f\left(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c}\right) \tag{4.57a}$$

$$w(\xi,\xi) = w_\xi(\xi,\xi) = w_\eta(\xi,\xi) = 0 \tag{4.57b}$$

While the equation (4.57a) follows from (4.12), the equations (4.57b) follow from

$$u(x,0) = 0, u_x(x,0) = 0, u_t(x,0) = 0.$$

Integrating (4.57a) w.r.t. η, from $\eta = \xi$ to $\eta = \eta_0$, we get

$$w_\xi(\xi,\eta_0) - w_\xi(\xi,\xi) = -\frac{1}{4c^2}\int_\xi^{\eta_0} f\left(\frac{\xi+s}{2}, \frac{s-\xi}{2c}\right) ds, \tag{4.58}$$

which, in view of the condition (4.57b), reduces to

$$w_\xi(\xi,\eta_0) = -\frac{1}{4c^2}\int_\xi^{\eta_0} f\left(\frac{\xi+s}{2}, \frac{s-\xi}{2c}\right) ds \tag{4.59}$$

Integrating (4.59) w.r.t. ξ, from $\xi = \xi_0$ to $\xi = \eta_0$, we get

$$w(\eta_0,\eta_0) - w(\xi_0,\eta_0) = -\frac{1}{4c^2}\int_{\xi_0}^{\eta_0}\int_z^{\eta_0} f\left(\frac{z+s}{2}, \frac{s-z}{2c}\right) ds\,dz, \tag{4.60}$$

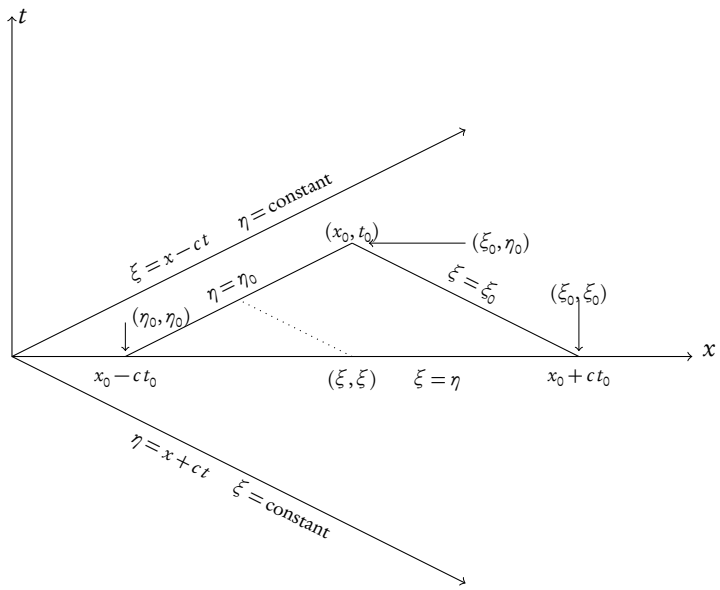

Figure 4.1. *Domain of integration in (x,t) and (ξ,η) coordinates.*

which, in view of the condition (4.57b), reduces to

$$w(\xi_0,\eta_0) = \frac{1}{4c^2}\int_{\xi_0}^{\eta_0}\int_{z}^{\eta_0} f\left(\frac{z+s}{2}, \frac{s-z}{2c}\right) ds\, dz. \tag{4.61}$$

Let us now get an expression for $u(x,t)$ by making the following change of variables

$$s = x + ct,\ z = x - ct. \tag{4.62}$$

Note that the absolute value of Jacobian of this transformation is $2c$. Under this change of variables, the domain of integration in (4.61)

$$D = \{(s,z) : z \le s \le \eta_0,\ \xi_0 \le z \le \eta_0\} \tag{4.63}$$

transforms into

$$\{(x,t) : x_0 - c(t_0 - t) \le x \le x_0 + c(t_0 - t),\ 0 \le t \le t_0\}. \tag{4.64}$$

Note that $z \le s \le \eta_0$ implies $x - ct \le x + ct \le x_0 + ct_0$, from which we get $x_0 - c(t_0 - t) \le x \le x_0 + c(t_0 - t)$. Also note that $\xi_0 \le z \le \eta_0$ implies $x_0 - ct_0 \le x - ct \le x_0 + ct_0$, which in view of $x - ct \le x + ct \le x_0 + ct_0$, will yield the inequalities

$$x_0 - ct_0 \le x - ct \le x + ct \le x_0 + ct_0.$$

From the above inequalities, we conclude that $0 \le 2ct \le 2ct_0$, from which we get $0 \le t \le t_0$. Thus the equation (4.61) transforms to

$$u(x_0,t_0) = \frac{1}{2c}\int_0^t \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x,t)\, dx\, dt = \frac{1}{2c}\int_T f(x,t)\, dx\, dt, \tag{4.65}$$

where T is the characteristic triangle having vertices at (x_0,t_0), $(x_0 - ct_0, 0)$, and $(x_0 + ct_0, 0)$.

Solution via Duhamel's principle

Duhamel principle gives us a way to solve nonhomogeneous problems corresponding to a linear differential operator, by superposition of solutions of a family of corresponding homogeneous problems. In this subsection we motivate Duhamel principle using an explicit example of an ordinary differential equation, and then compute solutions of Cauchy problem for nonhomogeneous Wave equation given by (4.3) for $d = 1, 2, 3$.

Motivation for Duhamel principle: Method of variation of parameters for Ordinary differential equations

Let $\mu > 0$, $y_0, y_1 \in \mathbb{R}$. Consider the following initial value problem

$$\frac{d^2y}{dx^2} + \mu^2 y = f(t), \tag{4.66a}$$

$$y(0) = y_0,\ y'(0) = y_1. \tag{4.66b}$$

We will solve the IVP (4.66) using method of variation of parameters. A general solution of the homogeneous equation

$$y'' + \mu^2 y = 0 \tag{4.67}$$

is $C_1 \cos \mu t + C_2 \sin \mu t$, where C_1, C_2 are arbitrary constants. Method of variation of parameters attempts to find a solution of the nonhomogeneous ODE (4.66a) of the form

$$y(t) = C_1(t) \sin \mu t + C_2(t) \cos \mu t. \tag{4.68}$$

Differentiating the equation (4.68), we get

$$y'(t) = \mu C_1(t) \cos \mu t + C_1'(t) \sin \mu t - \mu^2 C_2(t) \sin \mu t + C_2'(t) \cos \mu t. \tag{4.69}$$

Method of variaiton of parameters assumes that

$$C_1'(t) \sin \mu t + C_2'(t) \cos \mu t = 0 \tag{4.70}$$

In view of the equation (4.70), the expression for $y'(t)$ in (4.69) reduces to

$$y'(t) = \mu C_1(t) \cos \mu t - \mu C_2(t) \sin \mu t. \tag{4.71}$$

Differentiating the equation (4.71), we get

$$y''(t) = -\mu^2 C_1(t) \sin \mu t + \mu C_1'(t) \cos \mu t - \mu^2 C_2(t) \cos \mu t - \mu C_2'(t) \sin \mu t. \tag{4.72}$$

From the equation (4.72), we get

$$y''(t) + \mu^2 y(t) = \mu C_1'(t) \cos \mu t - \mu C_2'(t) \sin \mu t = f(t). \tag{4.73}$$

Solving for C_1' and C_2' from equations (4.70) and (4.73), we get

$$C_1'(t) = \frac{1}{\mu} f(t) \cos \mu t,\ C_2'(t) = -\frac{1}{\mu} f(t) \sin \mu t. \tag{4.74}$$

Substituting expressions for $C_1(t), C_2(t)$ in the formula for $y(t)$, and using the initial conditions (4.66b), we get

$$y(t) = y_1 \frac{\sin \mu t}{\mu} + y_0 \cos \mu t + \frac{1}{\mu} \int_0^t \sin \mu (t - \tau) f(\tau)\, d\tau. \tag{4.75}$$

Let us denote by $S(t)y_1$, called *source operator*, the solution of the Cauchy problem

$$\frac{d^2y}{dx^2}+\mu^2 y=0,$$
$$y(0)=0,\ y'(0)=y_1.$$

In terms of the source operator, the solution of (4.66) is given by

$$y(t)=S'(t)y_0+S(t)y_1+\int_0^t S(t-\tau)f(\tau)\,d\tau. \tag{4.76}$$

Application to one dimensional wave equation

The Cauchy problem for $d=1$ is

$$u_{tt}-c^2u_{xx}=f(x,t),\quad x\in\mathbb{R},\ t>0, \tag{4.77a}$$
$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x)\quad\text{for}\quad x\in\mathbb{R}. \tag{4.77b}$$

Let $S(t)\psi$ denote the solution of the Cauchy problem

$$u_{tt}-c^2u_{xx}=0,\quad x\in\mathbb{R},\ t>0,$$
$$u(x,0)=0,\ u_t(x,0)=\psi(x)\quad\text{for}\quad x\in\mathbb{R}.$$

Taking cue from (4.76), we expect the following formula to yield a solution of the Cauchy problem (4.77)

$$u(x,t)=\frac{\partial}{\partial t}(S(t)\varphi)(x)+S(t)\psi(x)+\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau, \tag{4.78}$$

where $f_\tau(x):=f(x,\tau)$.

We will now check that the last term on the right hand side of (4.78) solves the non-homogeneous wave equation, as we will see later that first two terms therein correspond to solution of the homogeneous wave equation satisfying the given Cauchy data.

Indeed, differentiating the equation (4.78) w.r.t. t gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau\right)&=S(0)f_t+\int_0^t \frac{\partial}{\partial t}(S(t-\tau)f_\tau)(x)\,d\tau\\&=\int_0^t \frac{\partial}{\partial t}(S(t-\tau)f_\tau)(x)\,d\tau.\end{aligned}$$

Differentiating w.r.t. t once again, we get

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left(\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau\right)&=\left(\frac{\partial}{\partial t}S\right)(0)f_t(x)+\int_0^t \frac{\partial^2}{\partial t^2}(S(t-\tau)f_\tau)(x)\,d\tau\\&=f(x,t)+c^2\int_0^t \frac{\partial^2}{\partial x^2}(S(t-\tau)f_\tau)(x)\,d\tau\\&=f(x,t)+c^2\frac{\partial^2}{\partial x^2}\left(\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau\right).\end{aligned}$$

Thus we have

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right)\left(\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau\right) = f(x,t). \tag{4.79}$$

Since $S(t)$ is explicitly known via d'Alembert formula, let us substitute in the above expression and get an explicit formula for solution of the nonhomogeneous Cauchy problem in terms of the given data f, φ, ψ. Note that

$$(S(t)\psi)(x) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,d\sigma. \tag{4.80}$$

Thus the formula (4.78) takes the form

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} \varphi(\sigma)\,d\sigma\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,d\sigma + \int_0^t \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma\,d\tau \\ &= \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,d\sigma + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma\,d\tau. \end{aligned}$$

Thus we obtain a solution of the Cauchy problem

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,d\sigma + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma\,d\tau. \tag{4.81}$$

Remark 4.13. Note that $w(x,t;\tau) := S(t-\tau)f_\tau$ solves the following problem

$$w_{tt} - c^2 w_{xx} = 0, \quad x \in \mathbb{R},\ t > \tau \tag{4.82a}$$

and satisfies the conditions

$$w(x,\tau;\tau) = 0,\ u_t(x,\tau;\tau) = f(x,\tau) \quad \text{for} \quad x \in \mathbb{R}. \tag{4.82b}$$

Thus the integral

$$\int_0^t (S(t-\tau)f_\tau)(x)\,d\tau$$

in the Duhamel's formula has the form

$$\int_0^t w(x,t;\tau)\,d\tau$$

∎

Application to two dimensional wave equation

In the case of two dimensional wave equation, the source operator $S(t)$ is given by

$$S(t)\psi = \frac{1}{2\pi c}\int_{D(\boldsymbol{x},ct)} \frac{\psi(\boldsymbol{y})}{\sqrt{c^2t^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y} \tag{4.83}$$

Thus Duhamel principle gives the following solution of nonhomogeneous problem as

$$u(\boldsymbol{x},t)=\int_0^t (S(t-\tau)f_\tau)(\boldsymbol{x})\,d\tau=\int_0^t \frac{1}{2\pi c}\int_{D(\boldsymbol{x},c(t-\tau))}\frac{f(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\,d\tau \quad (4.84)$$

That is,

$$u(\boldsymbol{x},t)=\frac{1}{2\pi c}\int_0^t\int_{D(\boldsymbol{x},c(t-\tau))}\frac{f(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\,d\tau. \quad (4.85)$$

Thus a solution of the Cauchy problem (4.1) for $d=2$ is given by

$$\begin{aligned}u(\boldsymbol{x},t)=&\frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{D(\boldsymbol{x},ct)}\frac{\varphi(\boldsymbol{y})}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\right)+\frac{1}{2\pi c}\int_{D(\boldsymbol{x},ct)}\frac{\psi(\boldsymbol{y})}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\\&+\frac{1}{2\pi c}\int_0^t\int_{D(\boldsymbol{x},c(t-\tau))}\frac{f(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\,d\tau,\end{aligned} \quad (4.86)$$

where $D(\boldsymbol{x},ct)$ denotes the open disk with center at $\boldsymbol{x}$ having radius ct.

If we use the formula (4.55) for the solution of homogeneous wave equation, a solution of the Cauchy problem (4.1) for $d=2$ is

$$\begin{aligned}u(\boldsymbol{x},t)=&\frac{1}{2\pi}\int_{D(0,1)}\frac{\varphi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z}+\frac{ct}{2\pi}\int_{D(0,1)}\frac{\nabla\varphi(\boldsymbol{x}+ct\boldsymbol{z}).\boldsymbol{z}}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z}\\&+\frac{t}{2\pi}\int_{D(0,1)}\frac{\psi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-\|\boldsymbol{z}\|^2}}\,d\boldsymbol{z}+\frac{1}{2\pi c}\int_0^t\int_{D(\boldsymbol{x},c(t-\tau))}\frac{f(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-\|\boldsymbol{x}-\boldsymbol{y}\|^2}}\,d\boldsymbol{y}\,d\tau,\end{aligned} \quad (4.87)$$

Application to three dimensional wave equation

In the case of three dimensional wave equation, the source operator $S(t)$ is given by

$$S(t)\psi=\frac{1}{4\pi c^2t}\int_{S(\boldsymbol{x},ct)}\psi(\boldsymbol{y})\,d\sigma \quad (4.88)$$

Thus Duhamel principle gives the following solution of nonhomogeneous problem as

$$u(\boldsymbol{x},t)=\int_0^t (S(t-\tau)f_\tau)(\boldsymbol{x})\,d\tau=\int_0^t \frac{1}{4\pi c^2(t-\tau)}\int_{S(\boldsymbol{x},c(t-\tau))}f(\boldsymbol{y},\tau)\,d\sigma(\boldsymbol{y})\,d\tau \quad (4.89)$$

That is,

$$u(\boldsymbol{x},t)=\frac{1}{4\pi c^2}\int_0^t\frac{1}{(t-\tau)}\int_{S(\boldsymbol{x},c(t-\tau))}f(\boldsymbol{y},\tau)\,d\sigma(\boldsymbol{y})\,d\tau. \quad (4.90)$$

Thus a solution of the Cauchy problem (4.1) for $d=3$ is given by

$$\begin{aligned}u(\boldsymbol{x},t)=&\frac{1}{4\pi c^2t^2}\int_{S(\boldsymbol{x},ct)}\{t\psi(\boldsymbol{y})+\varphi(\boldsymbol{y})+\nabla\varphi(\boldsymbol{y}).(\boldsymbol{x}-\boldsymbol{y})\}\,d\sigma\\&+\frac{1}{4\pi c^2}\int_0^t\frac{1}{(t-\tau)}\int_{S(\boldsymbol{x},c(t-\tau))}f(\boldsymbol{y},\tau)\,d\sigma(\boldsymbol{y})\,d\tau.\end{aligned} \quad (4.91)$$

Since $\boldsymbol{y} \in S(\boldsymbol{x}, c(t-\tau))$ means that $||\boldsymbol{y}-\boldsymbol{x}|| = c(t-\tau)$, the last formula (4.91) may be re-written as

$$u(\boldsymbol{x},t) = \frac{1}{4\pi c^2 t^2}\int_{S(\boldsymbol{x},ct)} \{t\psi(\boldsymbol{y}) + \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}).(\boldsymbol{x}-\boldsymbol{y})\}\, d\sigma$$
$$+ \frac{1}{4\pi c}\int_0^t \int_{S(\boldsymbol{x},c(t-\tau))} \frac{f\left(\boldsymbol{y}, t - \frac{||\boldsymbol{y}-\boldsymbol{x}||}{c}\right)}{||\boldsymbol{y}-\boldsymbol{x}||}\, d\sigma(\boldsymbol{y})\, d\tau. \tag{4.92}$$

Note that the last integral on the RHS of (4.92) is on the surface of the back-ward cone with vertex at $(\boldsymbol{x}, t)$, and is the domain of dependence for the solution at $(\boldsymbol{x}, t)$. The last integral reduces to an integral on $B(\boldsymbol{x}, ct)$, and thus the formula (4.92) takes the form

$$u(\boldsymbol{x},t) = \frac{1}{4\pi c^2 t^2}\int_{S(\boldsymbol{x},ct)} \{t\psi(\boldsymbol{y}) + \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}).(\boldsymbol{x}-\boldsymbol{y})\}\, d\sigma$$
$$+ \frac{1}{4\pi c^2}\int_{B(\boldsymbol{x},ct)} \frac{f\left(\boldsymbol{y}, t - \frac{||\boldsymbol{y}-\boldsymbol{x}||}{c}\right)}{||\boldsymbol{y}-\boldsymbol{x}||}\, d\boldsymbol{y}. \tag{4.93}$$

Note that the solution at the point $(\boldsymbol{x}, t)$ requires the values of the source term f at earlier times (retarded or delayed times) given by $t - \frac{||\boldsymbol{y}-\boldsymbol{x}||}{c}$. Thus the last term on RHS of (4.93) is called retarded potential.

4.2 ▪ Uniqueness of solutions

In this section, we will prove the uniqueness of solutions to Cauchy problem (4.1) for $d = 1$, $d = 2$, and $d = 3$. Uniqueness of solutions also follow from Causality prinicple, and also from energy considerations which will be discussed in Chapter 5.

Case of one dimensional Wave equation

We have in fact shown, while deriving d'Alembert formula, that any solution of the wave equation must be equal to the d'Alembert solution, thereby proving the uniquenesss of solutions.

Case of three dimensional Wave equation

If u and v are solutions of the Cauchy problem (4.1) for $d = 3$, then the spherical mean $L(\rho, t) := \rho M_w(\rho, t)$, of the function $w := u - v$ solves the Cauchy problem (4.36) with zero Cauchy data. Since (4.36) is a Cauchy problem for one dimensional wave equations, and by the uniqueness result that was established, it follows that $M_w \equiv 0$. By passing to the limit as $\rho \to 0$, we get that the function $w(\boldsymbol{x}, t) \equiv 0$. Thus uniqueness of solutions for Cauchy problem in the case of two dimensional wave equation is proved.

Case of two dimensional Wave equation

As was already observed, solutions of Cauchy problem for wave equation in two space dimensions are also solutions of a Cauchy problem for wave equation in three space dimensions. By uniqueness result in the case of three space dimensions, the corresponding uniqueness result follows for two space dimensions.

4.3 ▪ Stability of solutions and Well-posedness of Cauchy Problem

In this section we prove stability (in an appropriate sense) of solutions to Cauchy problem for $d = 1$, $d = 2$, and $d = 3$, thereby establishing the well-posedness of the Cauchy problem. Indeed, we state and prove stability of solutions as part of the well-posedness theorems in each of the dimensions.

4.3.1 ▪ Case of one dimensional Wave equation

We are going to prove that Cauchy problem is well-posed for wave equation in one space dimension.

Theorem 4.14 (Well-posedness). *Let $T > 0$. For $f \in C(\mathbb{R} \times [0, T])$ such that $f_x \in C(\mathbb{R} \times [0, T])$, and for Cauchy data $\varphi \in C^2(\mathbb{R})$, and $\psi \in C^1(\mathbb{R})$, the Cauchy problem*

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in \mathbb{R},\ t \in (0,T), \tag{4.94a}$$

$$u(x,0) = \varphi(x), \qquad x \in \mathbb{R}, \tag{4.94b}$$

$$u_t(x,0) = \psi(x), \qquad x \in \mathbb{R}. \tag{4.94c}$$

is well-posed in the following sense:

(i) *There exists a classical solution to the Cauchy problem (4.94). That is, there exists an $u \in C^2(\mathbb{R} \times (0,T)) \cap C^1(\mathbb{R} \times [0,T])$ such that u solves the wave equation (4.94a), and satisfies the Cauchy data (4.94b) and (4.94c).*

(ii) *There exists at most one classical solution to the Cauchy problem (4.94).*

(iii) *Given $\epsilon > 0$, there exists a $\delta > 0$ such that for every pair of data triples (f_1, φ_1, ψ_1) and (f_2, φ_2, ψ_2) having the smoothness as stated above, satisfying*

$$|\varphi_1(x) - \varphi_2(x)| < \delta,\ |\psi_1(x) - \psi_2(x)| < \delta,\ \forall x \in \mathbb{R}, \tag{4.95a}$$

and

$$|f_1(x,t) - f_2(x,t)| < \delta\ \forall (x,t) \in \mathbb{R} \times [0,T], \tag{4.95b}$$

the corresponding solutions u_1 and u_2 of the Cauchy problem (4.94) satisfy the estimate

$$|u_1(x,t) - u_2(x,t)| < \epsilon,\ \forall (x,t) \in \mathbb{R} \times [0,T] \tag{4.96}$$

Proof.
Proof of (i)*:* We derived d'Alembert formula for the solution of Cauchy problem (4.94), and let us recall the formula (4.81) here.

$$u(x,t) = \frac{\varphi(x-ct) + \varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\sigma)\,d\sigma + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)\,d\sigma\,d\tau. \tag{4.97}$$

The above formula gives us a classical solution can be verfied by using the smoothness of the data triple (f, φ, ψ), and is left as an exercise.

Proof of (ii)*:* We established uniqueness of solutions to the Cauchy problem in the Section 4.2 already.

Proof of (iii)*:* Let u_1 and u_2 be solutions of the Cauchy problem with the data (f_1, φ_1, ψ_1) and (f_2, φ_2, ψ_2) respectively. By d'Alembert's formula, we have

$$(u_1 - u_2)(x,t) = \frac{\varphi_1(x+ct) - \varphi_2(x+ct)}{2} + \frac{\varphi_1(x-ct) - \varphi_2(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} (\psi_1 - \psi_2)(s)\,ds$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \{f_1(\sigma,\tau) - f_2(\sigma,\tau)\}\,d\sigma\,d\tau.$$

If $|\varphi_1(x) - \varphi_2(x)| < \delta$, $|\psi_1(x) - \psi_2(x)| < \delta$, $|f_1(x,t) - f_2(x,t)| < \delta$, then we get

$$|u_1(x,t) - u_2(x,t)| \le \frac{1}{2}(\delta + \delta) + \frac{1}{2c}2ct\delta + \frac{1}{2c}2ct^2\delta \le (1 + T + T^2)\delta. \tag{4.98}$$

Now we will choose δ such that $\delta < \frac{\epsilon}{1+T+T^2}$. This finishes the proof of the theorem. □

Remark 4.15. Note that the well-posedness of Cauchy problem is established by considering distances of data and solutions in the 'uniform sense' (see (iii) of Theorem 4.14). One can also consider questions of stability w.r.t. other notions of distance between functions, in which case the same Cauchy problem could be either well-posed or ill-posed. In any case, in any result of well-posedness one must clearly mention what is really being proved.

4.3.2 ▪ Case of two dimensional Wave equation

We are going to prove that Cauchy problem is well-posed for wave equation in two space dimensions.

Theorem 4.16 (Well-posedness). *Let $T > 0$. For $f \in C(\mathbb{R}^2 \times [0,T])$ such that $\nabla_{\boldsymbol{x}} f \in C(\mathbb{R}^2 \times [0,T])$, and for Cauchy data $\varphi \in C^3(\mathbb{R}^2)$, and $\psi \in C^2(\mathbb{R}^2)$, the Cauchy problem*

$$u_{tt} - c^2\left(u_{x_1x_1} + u_{x_2x_2}\right) = f(\boldsymbol{x},t), \quad \boldsymbol{x} = (x_1,x_2) \in \mathbb{R}^2,\ t > 0, \tag{4.99a}$$
$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2, \tag{4.99b}$$
$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^2. \tag{4.99c}$$

is well-posed in the following sense:

(i) *There exists a classical solution to the Cauchy problem (4.99). That is, there exists an $u \in C^2\left(\mathbb{R}^2 \times (0,T)\right) \cap C^1\left(\mathbb{R}^2 \times [0,T]\right)$ such that u solves the wave equation (4.99a), and satisfies the Cauchy data (4.99b) and (4.99c).*

(ii) *There exists at most one classical solution to the Cauchy problem (4.99).*

(iii) *Given $\epsilon > 0$, there exists a $\delta > 0$ such that for every pair of data triples (f_1, φ_1, ψ_1) and (f_2, φ_2, ψ_2) having the smoothness as stated above, satisfying*

$$|\varphi_1(\boldsymbol{x}) - \varphi_2(\boldsymbol{x})| < \delta,\ ||\nabla\varphi_1(\boldsymbol{x}) - \nabla\varphi_2(\boldsymbol{x})|| < \delta,\ |\psi_1(\boldsymbol{x}) - \psi_2(\boldsymbol{x})| < \delta,\ \forall \boldsymbol{x} \in \mathbb{R}^2, \tag{4.100a}$$

and

$$|f_1(\boldsymbol{x},t)-f_2(\boldsymbol{x},t)|<\delta\ \forall(\boldsymbol{x},t)\in\mathbb{R}^2\times[0,T], \tag{4.100b}$$

the corresponding solutions u_1 and u_2 of the Cauchy problem (4.99) satisfy the estimate

$$|u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)|<\epsilon,\ \forall(\boldsymbol{x},t)\in\mathbb{R}^2\times[0,T] \tag{4.101}$$

Proof.
Proof of (i)*:* We derived d'Alembert formula for the solution of Cauchy problem (4.99), and let us recall the formula (4.87) here.

$$\begin{aligned}u(\boldsymbol{x},t)=&\frac{1}{2\pi}\int_{D(0,1)}\frac{\varphi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}+\frac{ct}{2\pi}\int_{D(0,1)}\frac{\nabla\varphi(\boldsymbol{x}+ct\boldsymbol{z}).\boldsymbol{z}}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}\\&+\frac{t}{2\pi}\int_{D(0,1)}\frac{\psi(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}+\frac{1}{2\pi c}\int_0^t\int_{D(\boldsymbol{x},c(t-\tau))}\frac{f(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-||\boldsymbol{x}-\boldsymbol{y}||^2}}\,d\boldsymbol{y}\,d\tau,\end{aligned} \tag{4.87}$$

where $D(\boldsymbol{x},r)$ denotes the open disk with center at $\boldsymbol{x}$ having radius r.

The above formula gives us a classical solution can be verfied by using the smoothness of the data triple (f,φ,ψ), and is left as an exercise.

Proof of (ii)*:* We established uniqueness of solutions to the Cauchy problem in the Section 4.2 already.

Proof of (iii)*:* Let u_1 and u_2 be solutions of the Cauchy problem with the data (f_1,φ_1,ψ_1) and (f_2,φ_2,ψ_2) respectively. Using the formula (4.87) we write expressions for u_1 and u_2, and then subtract one from another to get

$$\begin{aligned}(u_1-u_2)(\boldsymbol{x},t)=&\frac{1}{2\pi}\int_{D(0,1)}\frac{(\varphi_1-\varphi_2)(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}+\frac{ct}{2\pi}\int_{D(0,1)}\frac{(\nabla\varphi_1-\nabla\varphi_2)(\boldsymbol{x}+ct\boldsymbol{z}).\boldsymbol{z}}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}\\&+\frac{t}{2\pi}\int_{D(0,1)}\frac{(\psi_1-\psi_2)(\boldsymbol{x}+ct\boldsymbol{z})}{\sqrt{1-||\boldsymbol{z}||^2}}\,d\boldsymbol{z}\\&+\frac{1}{2\pi c}\int_0^t\int_{D(\boldsymbol{x},c(t-\tau))}\frac{(f_1-f_2)(\boldsymbol{y},\tau)}{\sqrt{c^2t^2-||\boldsymbol{x}-\boldsymbol{y}||^2}}\,d\boldsymbol{y}\,d\tau,\end{aligned} \tag{4.102}$$

There are four terms on the RHS of the equation (4.102), and let A_1,A_2,A_3,A_4 denote the first, second, third, and fourth terms. On applying triangle equality to RHS of (4.102), we get

$$|(u_1-u_2)(\boldsymbol{x},t)|\le|A_1|+|A_2|+|A_3|+|A_4| \tag{4.103}$$

Using (4.100a) (δ to be determined shortly), we get

$$|A_1|+|A_2|+|A_3|\le\frac{\delta}{2\pi}(1+cT+T)\int_{D(0,1)}\frac{d\boldsymbol{z}}{\sqrt{1-||\boldsymbol{z}||^2}}. \tag{4.104}$$

Using polar coordinates, we compute the integral as

$$\int_{D(0,1)} \frac{d\boldsymbol{z}}{\sqrt{1-||\boldsymbol{z}||^2}} = \int_0^{2\pi}\int_0^1 \frac{r\,dr}{\sqrt{1-r^2}} = 2\pi.$$

Thus the estimate (4.104) reduces to

$$|A_1|+|A_2|+|A_3| \le \delta(1+cT+T). \tag{4.105}$$

Proceeding in a similar manner, we get

$$|A_4| \le 2\pi c\delta T^2. \tag{4.106}$$

Combining all the estimates, we get

$$|u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)| \le \delta(1+(c+1)T+2\pi cT^2) \tag{4.107}$$

For a given $\epsilon > 0$, we will choose δ such that $\delta < \frac{\epsilon}{1+(c+1)T+2\pi cT^2}$. This finishes the proof of the theorem. □

4.3.3 ▪ Case of three dimensional Wave equation

We are going to prove that Cauchy problem is well-posed for wave equation in three space dimensions.

Theorem 4.17 (Well-posedness). *Let $T > 0$. For $f \in C(\mathbb{R}^3 \times [0,T])$ such that $\nabla_{\boldsymbol{x}} f \in C(\mathbb{R}^3 \times [0,T])$, and for Cauchy data $\varphi \in C^3(\mathbb{R}^3)$, and $\psi \in C^2(\mathbb{R}^3)$, the Cauchy problem*

$$u_{tt} - c^2\left(u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3}\right) = f(\boldsymbol{x},t), \quad \boldsymbol{x}=(x_1,x_2,x_3)\in\mathbb{R}^3,\ t>0, \tag{4.108a}$$

$$u(\boldsymbol{x},0) = \varphi(\boldsymbol{x}), \qquad \boldsymbol{x}\in\mathbb{R}^3, \tag{4.108b}$$

$$u_t(\boldsymbol{x},0) = \psi(\boldsymbol{x}), \qquad \boldsymbol{x}\in\mathbb{R}^3. \tag{4.108c}$$

is well-posed in the following sense:

(i) *There exists a classical solution to the Cauchy problem (4.108). That is, there exists an $u \in C^2\left(\mathbb{R}^2\times(0,T)\right)\cap C^1\left(\mathbb{R}^2\times[0,T]\right)$ such that u solves the wave equation (4.108a), and satisfies the Cauchy data (4.108b) and (4.108c).*

(ii) *There exists at most one classical solution to the Cauchy problem (4.108).*

(iii) *Given $\epsilon > 0$, there exists a $\delta > 0$ such that for every pair of data triples (f_1,φ_1,ψ_1) and (f_2,φ_2,ψ_2) having the smoothness as stated above, satisfying*

$$|\varphi_1(\boldsymbol{x})-\varphi_2(\boldsymbol{x})| < \delta,\ ||\nabla\varphi_1(\boldsymbol{x})-\nabla\varphi_2(\boldsymbol{x})|| < \delta,\ |\psi_1(\boldsymbol{x})-\psi_2(\boldsymbol{x})| < \delta,\ \forall \boldsymbol{x}\in\mathbb{R}^3, \tag{4.109a}$$

and

$$|f_1(\boldsymbol{x},t)-f_2(\boldsymbol{x},t)| < \delta\ \forall(\boldsymbol{x},t)\in\mathbb{R}^3\times[0,T], \tag{4.109b}$$

the corresponding solutions u_1 and u_2 of the Cauchy problem (4.108) satisfy the estimate

$$|u_1(\boldsymbol{x},t)-u_2(\boldsymbol{x},t)| < \epsilon,\ \forall(\boldsymbol{x},t)\in\mathbb{R}^3\times[0,T] \tag{4.110}$$

Proof.
Proof of (i)*:* Let us recall the formula (4.91) for a solution of the Cauchy problem (4.108) here.

$$u(\boldsymbol{x},t) = \frac{1}{4\pi c^2 t^2}\int_{S(\boldsymbol{x},ct)} \{t\psi(\boldsymbol{y}) + \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}).(\boldsymbol{x}-\boldsymbol{y})\}\, d\sigma$$
$$+ \frac{1}{4\pi c^2}\int_0^t \frac{1}{(t-\tau)}\int_{S(\boldsymbol{x},c(t-\tau))} f(\boldsymbol{y},\tau)\, d\sigma(\boldsymbol{y})\, d\tau.$$

The above formula gives us a classical solution can be verfied by using the smoothness of the data triple (f,φ,ψ), and is left as an exercise.

Proof of (ii)*:* We established uniqueness of solutions to the Cauchy problem in the Section 4.2 already.

Proof of (iii)*:* Proof of this stability estimate is similar to that of wave equation in two space dimensions, which is left as an exercise to the reader. □

4.4 ▪ Wave equation on subdomains

In this section we study wave equation in bounded domains. We mainly concentrate on $d = 1$, in which case we have explicit formulae for solutions of initial-boundary value problems for wave equation. In Subection 4.4.1, we consider initial-boundary value problems (IBVP) for Wave equation posed on an interval $(0,l)$, and the case of IBVP posed on semi-infinite interval $(0,\infty)$ is similar, which is left as an exercise to the reader. Solution of IBVP posed on a general bounded interval (a,b) is similar to those posed on $(0,l)$, which is also left as an exercise.

4.4.1 ▪ Wave equation in one space dimension: Case of a bounded interval

Let us consider the following initial-boundary value problem (IBVP)

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } 0 < x < l,\ t > 0, \tag{4.111a}$$
$$u(x,0) = \varphi(x) \qquad \text{for } 0 \le x \le l, \tag{4.111b}$$
$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad \text{for } 0 \le x \le l, \tag{4.111c}$$
$$u(0,t) = 0 \qquad \text{for } t \ge 0, \tag{4.111d}$$
$$u(l,t) = 0 \qquad \text{for } t \ge 0. \tag{4.111e}$$

Note that we are dealing with homogeneous Dirichlet boundary conditions where the values of the unknown function u are prescribed to be equal to zero, on the boundary for all positive times. We may consider other boundary conditions (homogeneous and non-homogeneous) such as Neumann conditions (where $\frac{\partial u}{\partial x}(0,t)$ and $\frac{\partial u}{\partial x}(l,t)$ are prescribed), Robin conditions (where $\alpha u(0,t) + \beta\frac{\partial u}{\partial x}(0,t)$ and $\alpha u(l,t) + \beta\frac{\partial u}{\partial x}(l,t)$ are prescribed), or a mix of any of these three boundary conditions in place of Dirichlet conditions. We address some of these problems in exercises.

We derive an expression for the solution of IBVP (4.111) using two different methods. The first of these methods is called *solution via first principles*, and the other method is *method of separation of variables*.

Solution via first principles

Step 1: Solution in the region 0,0

Recall that the general solution of the homogeneous wave equation is given by

$$u(x,t) = F(x-ct) + G(x+ct), \tag{4.112}$$

where $F, G \in C^2(\mathbb{R})$. The boundary condition $u(x,0) = \varphi(x)$ yields the relation

$$F(x) + G(x) = \varphi(x) \qquad \text{for } 0 \le x \le l.$$

The boundary condition $u_t(x,0) = \psi(x)$ gives

$$-cF'(x) + cG'(x) = \psi(x) \qquad \text{for } 0 \le x \le l.$$

Integrating the last equation over the interval $[0,x]$ gives

$$-F(x) + G(x) = \frac{1}{c}\int_0^x \psi(s)\,ds - F(0) + G(0)$$

Let us recall from equation (4.15) that solving for F and G, we get

$$F(\xi) = \frac{1}{2}\varphi(\xi) - \frac{1}{2c}\int_0^\xi \psi(s)\,ds + \frac{F(0)-G(0)}{2} \qquad \text{for } 0 \le \xi \le l \tag{4.113}$$

$$G(\eta) = \frac{1}{2}\varphi(\eta) + \frac{1}{2c}\int_0^\eta \psi(s)\,ds - \frac{F(0)-G(0)}{2} \qquad \text{for } 0 \le \eta \le l \tag{4.114}$$

Since the constant terms in (4.113) cancel each other in the expression for $u(x,t)$, we drop them from now onwards, and we write

$$F(\xi) = \frac{1}{2}\varphi(\xi) - \frac{1}{2c}\int_0^\xi \psi(s)\,ds \qquad \text{for } 0 \le \xi \le l \tag{4.115}$$

$$G(\eta) = \frac{1}{2}\varphi(\eta) + \frac{1}{2c}\int_0^\eta \psi(s)\,ds \qquad \text{for } 0 \le \eta \le l \tag{4.116}$$

Substituting these values in the general solution, we get

$$u(x,t) = \frac{\varphi(x-ct)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds \tag{4.117}$$

for (x,t) such that $0 \le x-ct \le l$ and $0 \le x+ct \le l$. Thus the solution is completely determined in the triangular region

$$\big\{(x,t) \in [0,l] \times [0,\infty) : 0 \le x-ct \le l,\ 0 \le x+ct \le l\big\}, \tag{4.118}$$

which is the triangular region denoted by $0,0$ in Figure 4.2.

Note that the boundary conditions (4.111d) and (4.111e) do not play any role in determining (and thus does not influence) the solution within the region $0,0$.

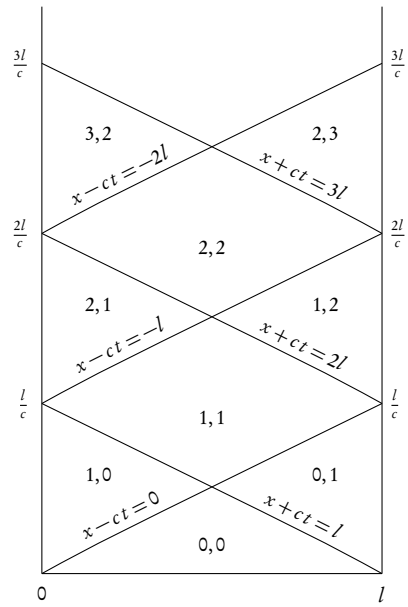

Figure 4.2. *Diamond picture.*

Step 2: Solution in the region 1,0

Note that in Step 1, we could obtain the solution in the region denoted by 0,0. In order to find the solution at other points of the domain $[0,l]\times[0,\infty)$, we need to extend the functions F,G outside the interval $[0,l]$ in which they were already determined by (4.115). The boundary conditions (4.111d) and (4.111e) impose conditions on such extensions by intertwining the values of the extended functions (which we still denote using the same F,G) as seen below.

Using the boundary condition (4.111d), in the general form of the solution (4.112), we get

$$F(-ct)+G(ct)=0 \text{ for } t\geq 0, \tag{4.119}$$

Denoting $\zeta=-ct$, the equation (4.119) gives a way to extend the function $F(\zeta)$ for negative arguments ζ, namely, by setting

$$F(\zeta)=-G(-\zeta). \tag{4.120}$$

For the moment, the formula (4.120) is meaningful for $-l\leq\zeta<0$. This is because G is defined only on the interval $[0,l]$, and whenever $-l\leq\zeta<0$ holds, the $-\zeta$ satisfies $0<-\zeta\leq l$, and thus $G(-\zeta)$ is defined.

$$F(\zeta)=-G(-\zeta)=-\frac{1}{2}\varphi(-\zeta)-\frac{1}{2c}\int_0^{-\zeta}\psi(s)\,ds \qquad \text{for } -l\leq\zeta<0. \tag{4.121}$$

Thus, in the triangular region 1,0 in Figure 4.2 which is described by

$$\big\{(x,t)\in[0,l]\times[0,\infty):-l\leq x-ct\leq 0,\ 0\leq x+ct\leq l\big\}, \tag{4.122}$$

we have the following expression for the solution $u(x,t)$ of the IBVP

$$u(x,t) = F(x-ct) + G(x+ct) = -G(ct-x) + G(x+ct),$$

which takes the following form in terms of the given data:

$$u(x,t) = \frac{-\varphi(ct-x)+\varphi(x+ct)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,ds. \tag{4.123}$$

The formula (4.123) takes the d'Alembert form, in terms of suitably extended initial data in the following sense:

Let φ_o and ψ_o denote the extensions of the functions φ and ψ respectively to the interval $[-l,0)$ as odd functions w.r.t. 0. That is,

$$\varphi_0(x) = -\varphi(-x) \text{ for } -l \le x < 0, \tag{4.124a}$$
$$\psi_0(x) = -\psi(-x) \text{ for } -l \le x < 0. \tag{4.124b}$$

In terms of the extended functions φ_o and ψ_o, the formula (4.121) takes the form

$$F(\zeta) = \frac{1}{2}\varphi_0(\zeta) - \frac{1}{2c}\int_0^{\zeta} \psi_0(s)\,ds \qquad \text{for } -l \le \zeta \le 0, \tag{4.125}$$

and thus the solution takes the following d'Alembert form

$$u(x,t) = \frac{\varphi_o(x-ct)+\varphi_o(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_o(s)\,ds. \tag{4.126}$$

Step 3: Solution in the region 0,1, region 1,1 and region 1,2

Let us use the second boundary condition (4.111e), in the general form of the solution (4.112), to get

$$F(l-ct) + G(l+ct) = 0 \text{ for } t \ge 0. \tag{4.127}$$

Denoting $\zeta = l+ct$, the equation (4.127) gives a way to extend the function $G(\zeta)$ for $\zeta > l$, namely, by setting

$$G(\zeta) = -F(2l-\zeta). \tag{4.128}$$

For the moment, the formula (4.128) is meaningful for $l < \zeta \le 3l$. This is because F is known only on the interval $[-l,l]$, and whenever $l < \zeta < 3l$ holds, the $2l-\zeta$ satisfies $-l \le 2l-\zeta \le l$, and thus $F(2l-\zeta)$ is defined.

$$G(\zeta) = -F(2l-\zeta) = \begin{cases} -\frac{1}{2}\varphi(2l-\zeta) + \frac{1}{2c}\int_0^{2l-\zeta} \psi(s)\,ds & \text{for } l < \zeta \le 2l. \\ G(\zeta-2l) \qquad \text{for } 2l \le \zeta \le 3l. \end{cases} \tag{4.129}$$

Thus the function G is now known on the interval $[0,3l]$, and F is known on the interval $[-l,l]$. Using this information, the solution of the IBVP will be determined in three regions.

Solution in the region 0,1

The region $0,1$ is given by

$$\left\{(x,t) \in [0,l]\times[0,\infty) : 0 \le x-ct \le l,\ l \le x+ct \le 2l\right\}, \tag{4.130}$$

and the solution in this region is given by

$$u(x,t) = F(x-ct) + G(x+ct) = F(x-ct) - F(2l-x-ct),$$

which in terms of the initial data take the following form

$$u(x,t) = \frac{\varphi(x-ct) - \varphi(2l-x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{2l-x-ct} \psi(s)\,ds. \tag{4.131}$$

As before, the formula (4.131) may easily seen to be in d'Alembert form (4.126), provided we extend the functions φ_0 and ψ_0 to whole of the real line as $2l$-periodic functions.

Solution in the region 1,1

The region $1,1$ is given by

$$\big\{(x,t) \in [0,l] \times [0,\infty) : -l \le x-ct \le 0,\ l \le x+ct \le 2l\big\}, \tag{4.132}$$

and the solution in this region is given by

$$u(x,t) = F(x-ct) + G(x+ct) = F(x-ct) - F(2l-x-ct),$$

which in terms of the initial data take the following form

$$u(x,t) = \frac{-\varphi(ct-x) - \varphi(2l-x-ct)}{2} + \frac{1}{2c}\int_{ct-x}^{2l-x-ct} \psi(s)\,ds. \tag{4.133}$$

Solution in the region 1,2

The region $1,2$ is given by

$$\big\{(x,t) \in [0,l] \times [0,\infty) : -l \le x-ct \le 0,\ 2l \le x+ct \le 3l\big\}, \tag{4.134}$$

and the solution in this region is given by

$$u(x,t) = F(x-ct) + G(x+ct) = F(x-ct) - F(2l-x-ct),$$

which in terms of the initial data take the following form

$$u(x,t) = \frac{-\varphi(ct-x) + \varphi(x+ct-2l)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct-2l} \psi(s)\,ds. \tag{4.135}$$

Step 4: Solution in other regions By continuing the above procedure, we can determine the solution in the region $[0,l] \times [0,\infty)$. As we saw in the previous steps, the main idea is to incrementally extend the functions F, G to have domains equal to $\mathbb{R}$, which were initially determined on $[0,l]$, using the boundary conditions via equations (4.120) and (4.128).

Theorem 4.18 (Existence and uniqueness theorem). *The IBVP (4.111) has a unique classical solution when the initial data $\varphi \in C^2(0,l)$, $\psi \in C^1(0,l)$, and satisfy the following compatibility conditions:*

$$\varphi(0) = \varphi(l) = \psi(0) = \psi(l) = 0, \tag{4.136a}$$

$$\varphi'_+(0),\ \varphi'_-(l),\ \varphi''_+(0),\ \varphi''_-(l),\ \psi'_+(0),\ \psi'_-(l) \text{ exist}, \tag{4.136b}$$

$$\varphi''_+(0) = \varphi''_-(l) = 0. \tag{4.136c}$$

Proof. Note that in the region $0,0$ the expression for u given by the d'Alembert formula (4.117), and thus it is the unique classical solution to the IBVP in this region due to the smoothness assumptions on φ, ψ.

The solution in the region $\mathbf{1},0$ is given by the formula (4.126), which is once again in the d'Alembert form in terms of the extended functions φ_o and ψ_o. Thus if $\varphi_o \in C^2(-l,l)$ and $\psi_o \in C^1(-l,l)$ then the formula (4.126) gives the unique classical solution in the region $\mathbf{1},0$. Let us find out the necessary and sufficient conditions to have $\varphi_o \in C^2(-l,l)$. They are

(i) φ_o is a continuous function on $(-l,l)$ if and only if $\varphi(0)=0$.

(ii) The function φ_o is differentiable at all points of the interval $(-l,l)$ except possibly at zero. Since we assumed that

$$\varphi'_+(0) = \lim_{x\to 0+} \frac{\varphi(x)-\varphi(0)}{x} \quad \text{exists,}$$

we have

$$\varphi'_-(0) = \lim_{s\to 0-} \frac{\varphi(s)-\varphi(0)}{s} = \lim_{x\to 0+} \frac{\varphi(x)-\varphi(0)}{x},$$

where the last equality holds by a change of variable $s=-x$ and since φ is an odd function about $x=0$. Thus the function φ_o is differentiable on the interval $(-l,l)$, and the derivative is also continuous.

(iii) By a similar analysis, using the remaining compatibility conditions, it also follows that φ_o has a continous second derivative on the interval $(-l,l)$.

(iv) Finally it follows that φ_o when extended to $\mathbb{R}$ by $2l$-periodicity, belongs to $C^2(\mathbb{R})$ by using compatibility conditions at $x=l$. By a similar reasoning, $\psi_o \in C^1(\mathbb{R})$.

Since $\varphi_o \in C^2(\mathbb{R})$, and $\psi_o \in C^1(\mathbb{R})$, it follows by induction that the solution u as constructed earlier (in various regions) is indeed a classical solution. The proof will be complete if we verify that u is twice continuously differentiable at all points on the boundaries of different regions, whose proof is left to the reader. □

Example 4.19. Let us find the value of $u(\frac{1}{2},\frac{3}{2})$ where u solves the following initial-boundary value problem (IBVP)

$$u_{tt} - u_{xx} = 0 \text{ for } 0<x<1,\ t>0, \tag{4.137a}$$
$$u(x,0) = 0 \quad \text{for } 0\le x\le 1, \tag{4.137b}$$
$$\frac{\partial u}{\partial t}(x,0) = x(1-x) \quad \text{for } 0\le x\le 1, \tag{4.137c}$$
$$u(0,t) = 0 \quad \text{for } t\ge 0, \tag{4.137d}$$
$$u(1,t) = 0 \quad \text{for } t\ge 0. \tag{4.137e}$$

From (4.120), (4.128), and (4.115), we get

$$u\left(\frac{1}{2},\frac{3}{2}\right) = F(-1)+G(2) = -G(1)-F(0) = -\frac{1}{2}\int_0^1 s(1-s)\,ds - 0 = -\frac{1}{6}. \tag{4.138}$$

∎

Remark 4.20 (interpretation of the solution using Reflections of waves). Have a look at the formulae (4.118), (4.122), (4.130), (4.132), and (4.134). Each of them has the form

$$u(x,t) = \frac{(-1)^p \varphi(L) + (-1)^q \varphi(R)}{2} + \frac{1}{2c}\int_L^R \psi(s)\,ds, \tag{4.139}$$

Let us start at the point (x,t) and follow the left characteristic till we hit the left boundary, and then follow the right characteristic till we hit right boundary, and once again follow the left characteristic through that point, and continue this procedure till we hit a point L (lying in the interval $[0,l]$) on the x-axis. Let p denote the number of turns that we took till reaching L. Similarly by following the right characteristic reach a point R, and let q denote the number of turns that we took till reaching a point R (lying in the interval $[0,l]$ on the x-axis.

Method of separation of variables

We were considering the following IBVP at the beginning of this section, and we discussed how to solve the same starting from first principles.

$$u_{tt} - c^2 u_{xx} = 0 \text{ for } 0 < x < l,\ t > 0, \tag{4.140a}$$
$$u(x,0) = \varphi(x) \qquad \text{for } 0 \le x \le l, \tag{4.140b}$$
$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \qquad \text{for } 0 \le x \le l, \tag{4.140c}$$
$$u(0,t) = 0 \qquad \text{for } t \ge 0, \tag{4.140d}$$
$$u(l,t) = 0 \qquad \text{for } t \ge 0. \tag{4.140e}$$

In this subsection, we use a general method called *separation of variables method* to solve the IBVP (4.140). Solution of the IBVP (4.140) can be found by finding solutions u_1 (with $\varphi \equiv 0$), and u_2 (with $\psi \equiv 0$) of the IBVP (4.140). Since the equation (4.140a) is linear, by superposition principle $u = u_1 + u_2$ solves the given IBVP (4.140). Thus we will assume that $\psi \equiv 0$ for the rest of this discussion.

Method of separation of variables assumes a solution of (4.140a) is of the form

$$u(x,t) = X(x)T(t) \text{ for } x \in (0,l),\ t > 0. \tag{4.141}$$

Substituting the expression for u from (4.141) in the equation (4.140a) gives

$$X(x)T''(t) - c^2 X''(x)T(t) = 0. \tag{4.142}$$

On dividing both sides of the equation (4.142) with $X(x)T(t)$ and re-arranging terms yields

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}. \tag{4.143}$$

Note that the LHS of the equation (4.143) is a function of t only, while the RHS is a function of x only. Thus both of them must be equal to a constant, and let us denote it by λ. Thus we have

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = \lambda. \tag{4.144}$$

Thus we get two ODEs from (4.144) which are given by

$$X'' - \frac{\lambda}{c^2} X = 0, \;\; T'' - \lambda T = 0. \tag{4.145}$$

Using the boundary condition (4.140d), we get

$$u(0,t) = X(0)T(t) = 0 \quad \text{for all } t > 0. \tag{4.146}$$

Since we do not want to find a trivial solution $u \equiv 0$, we cannot admit $T(t) \equiv= 0$. Thus we get $X(0) = 0$.

Similarly using the boundary condition (4.140e), we get

$$u(l,t) = X(l)T(t) = 0 \quad \text{for all } t > 0. \tag{4.147}$$

Since we do not want to find a trivial solution $u \equiv 0$, we cannot admit $T(t) \equiv= 0$. Thus we get $X(l) = 0$.

Thus we are led to the boundary value problem for X given by

$$X'' - \frac{\lambda}{c^2} X = 0, \tag{4.148a}$$

$$X(0) = X(l) = 0. \tag{4.148b}$$

We are interested in finding non-trivial solutions to the boundary value problem (4.148), which depends on the value of λ. The λs for which the BVP admits a non-trivial solution is called an eigenvalue and corresponding non-trivial solutions are called eigenfunctions. Let us find the eigenvalues, and corresponding eigenfunctions now.

(i) ($\lambda = 0$) General solution of the ODE (4.148a) is given by $X(x) = ax + b$. Applying the boundary conditions (4.148b), we get $a = b = 0$. Thus $\lambda = 0$ is not an eigenvalue.

(ii) ($\lambda > 0$) When $\lambda > 0$, we may write $\lambda = \mu^2$ where $\mu > 0$. The ODE (4.148a) then becomes

$$X'' - \frac{\mu^2}{c^2} X = 0,$$

whose general solution is given by $X(x) = ae^{\frac{\mu}{c}x} + be^{-\frac{\mu}{c}x}$. Applying the boundary conditions (4.148b), we get $a = b = 0$. Thus $\lambda > 0$ is not an eigenvalue.

(iii) ($\lambda < 0$) When $\lambda < 0$, we may write $\lambda = -\mu^2$ where $\mu > 0$. The ODE (4.148a) then becomes

$$X'' + \frac{\mu^2}{c^2} X = 0,$$

whose general solution is given by $X(x) = a\cos\left(\frac{\mu}{c}x\right) + b\sin\left(\frac{\mu}{c}x\right)$. Applying the boundary conditions (4.148b), we get

$$a = 0, \tag{4.149a}$$

$$a\cos\left(\frac{\mu}{c}l\right) + b\sin\left(\frac{\mu}{c}l\right) = 0. \tag{4.149b}$$

Since we are interested in non-trivial solutions to the BVP (4.148), at least one of the constants a, b should be non-zero. However, we already have $a = 0$ from (4.149). Thus in order to have $b \neq 0$, we must have

$$\sin\left(\frac{\mu}{c} l\right) = 0. \tag{4.150}$$

Solving the equation (4.150), we get $\mu_n = \frac{cn\pi}{l}$ for each $n \in \mathbb{N}$. Thus we have the following eigenvalue and corresponding eigenfunction, indexed by $n \in \mathbb{N}$:

$$\lambda_n = -\frac{c^2 n^2 \pi^2}{l^2}, \quad X_n(x) = \sin\left(\frac{n\pi}{l} x\right). \tag{4.151}$$

We now solve the ODE for T with $\lambda = \lambda_n$ for each $n \in \mathbb{N}$. The initial condition (4.140c) gives rise to the condition $T'(0) = 0$. Let us solve the ODE $T'' - \lambda_n T = 0$ with the condition $T'(0) = 0$. We get the following solutions T_n indexed by $N \in \mathbb{N}$:

$$T_n(t) = b \cos\left(\frac{n\pi c}{l} t\right).$$

Since the equation is linear, and each of the functions $X_n(x)T_n(t)$ is a solution, we propose a formal solution using 'superposition principle' by the formula

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right) \cos\left(\frac{n\pi c}{l} t\right) \tag{4.152}$$

We use the initial condition (4.140b) to determine the coefficients b_n. Indeed the function u defined by (4.152) satisfies the initial condition (4.140b) if and only if

$$\varphi(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right). \tag{4.153}$$

Thus b_n are the fourier sine coefficients of the function φ. Extend the function φ to the interval $[-l, l]$ as an odd function around $x = 0$, and let the extended function be still denoted by φ. Then the fourier series of φ takes the form

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right), \tag{4.154}$$

where b_n is given by

$$b_n = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi}{l} x\right) dx. \tag{4.155}$$

Thus the formal solution of the IBVP (4.140) is given by

$$u(x,t) \approx \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{n\pi}{l} x\right) dx \right) \sin\left(\frac{n\pi}{l} x\right) \cos\left(\frac{n\pi c}{l} t\right). \tag{4.156}$$

Theorem 4.21. *Let $\varphi \in C^4[0,l]$ such that $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0$. Then the function defined by (4.156) is a solution of the IBVP (4.140).*

Proof. In order to establish that the formal solution given by (4.156) is indeed a solution to the IBVP (4.140), we need to show that

(i) the series in (4.156) is twice continuously differentiable w.r.t. both the variables x and t, thus making sure that u_{tt} and u_{xx} are meaningful. Then we need to show that equation (4.140a) is satisfied,

(ii) the function $u(x,t)$ given by (4.156) is continuous upto the boundary of $(0,\pi)\times(0,T)$ and that the initial-boundary conditions (4.140b) - (4.140e) are satisfied.

Step 1: Proof of (i) By formal differentiation of the equation (4.156), using the notation introduced in (4.155), we get

$$u_t(x,t) = -\frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \sin\Big(\frac{n\pi}{l}x\Big)\sin\Big(\frac{n\pi c}{l}t\Big) \tag{4.157}$$

$$u_{tt}(x,t) = -\Big(\frac{\pi c}{l}\Big)^2\sum_{n=1}^{\infty} n^2 b_n \sin\Big(\frac{n\pi}{l}x\Big)\cos\Big(\frac{n\pi c}{l}t\Big) \tag{4.158}$$

$$u_x(x,t) = \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n \cos\Big(\frac{n\pi}{l}x\Big)\cos\Big(\frac{n\pi c}{l}t\Big) \tag{4.159}$$

$$u_{xx}(x,t) = -\Big(\frac{\pi c}{l}\Big)^2\sum_{n=1}^{\infty} n^2 b_n \sin\Big(\frac{n\pi}{l}x\Big)\cos\Big(\frac{n\pi c}{l}t\Big). \tag{4.160}$$

From (4.157), it is clear that $u_{tt} = u_{xx}$. Thus it remains to justify that the function defined by (4.156) is twice differentiable w.r.t. x and t variables. By a theorem of real analysis Theorem D.1, it is enough to show that the series in (4.157) are uniformly convergent, and the series in (4.156) converges at some point. Clearly all the series in (4.157) are uniformly convergent (by comparison test) if the following condition is satisfied: there exists a constant $C > 0$ such that for each $n \in \mathbb{N}$ the following inequalities hold.

$$n^2|b_n| \le \frac{1}{n^2} \tag{4.161}$$

Let us show that the conditions (4.161) are satisfied.

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l \varphi(x)\sin\Big(\frac{n\pi}{l}x\Big)\,dx \\
&= -\frac{2}{n\pi}\int_0^l \varphi(x)\frac{d}{dx}\Big(\cos\Big(\frac{n\pi}{l}x\Big)\Big)\,dx \\
&= \frac{2}{n\pi}\int_0^l \varphi'(x)\cos\Big(\frac{n\pi}{l}x\Big)\,dx - \varphi(x)\cos\Big(\frac{n\pi}{l}x\Big)\Big|_0^l \\
&= \frac{2}{n\pi}\int_0^l \varphi'(x)\cos\Big(\frac{n\pi}{l}x\Big)\,dx \\
&= \frac{2l}{n^2\pi^2}\int_0^l \varphi'(x)\frac{d}{dx}\Big(\sin\Big(\frac{n\pi}{l}x\Big)\Big)\,dx \\
&= -\frac{2l}{n^2\pi^2}\int_0^l \varphi''(x)\sin\Big(\frac{n\pi}{l}x\Big)\,dx + \varphi'(x)\sin\Big(\frac{n\pi}{l}x\Big)\Big|_0^l \\
&= -\frac{2l}{n^2\pi^2}\int_0^l \varphi''(x)\sin\Big(\frac{n\pi}{l}x\Big)\,dx.
\end{aligned}$$

Continuing the above computations two more times, we get

$$b_n = \frac{2l^3}{n^4\pi^4}\int_0^l \varphi^{(iv)}(x)\sin\left(\frac{n\pi}{l}x\right)dx.$$

The last equation gives the following estimate:

$$|b_n| \leq M\frac{2l^4}{n^4\pi^4},$$

where $M = \max_{x\in[0,l]}|\varphi^{(iv)}(x)|$, from which the estimate (4.161) follows.

Step 2: Proof of (ii) Due to assumptions on φ, the fourier sine series of φ converges to φ, and hence the initial conditions (4.140b) are satisfied. Other conditions (4.140c) - (4.140e) are checked by substituting appropriately the values of x and t in (4.156).
□

Remark 4.22. (i) Note that the conditions on the initial data φ in the above theorem are very restrictive. Luckily they are only sufficient conditions, and one can find weaker conditions on φ under which the above theorem is valid. Note that theorem cannot be applied in the case of an initially plucked string, which corresponds to φ that is piecewise linear (graph of φ is of triangle shape).

(ii) When φ is only piecewise continuous, the formal solution may be interepreted as a weak solution. For more details on weak solutions to wave equation, see Section 5.7 (in Chapter 5). ■

Exercises

General

4.1. Any solution of $\xi_t - c\xi_x = 0$ is a function of $x + ct$. Any solution of $\xi_t + c\xi_x = 0$ is a function of $x - ct$.

4.2. Let u be a twice continuously differentiable function that satisfies the wave equation for $x \in \mathbb{R}$ and $t \in \mathbb{R}$. That is, $u_{tt} - c^2 u_{xx} = 0$. Prove that u is constant on a member of the family of characteristics $x - ct =$ constant if and only if it is constant along each member of this family.

4.3. Let u be a classical solution to Wave equation $u_{tt} - u_{xx} = 0$ for $(x,t) \in \mathbb{R} \times (0,\infty)$. Show that

(i) for each fixed $y \in \mathbb{R}$, the function $w(x,t) := u(x-y,t)$ is also a solution.

(ii) for each $k \in \mathbb{N}$, the function $w(x,t) := \frac{\partial^k u}{\partial x^k}(x,t)$ is also a solution.

(iii) for each $a > 0$, the function $w(x,t) = u(ax, at)$ is also a solution.

Cauchy problem in full space

4.4. Show that the Cauchy problem for wave equation for $x \in \mathbb{R}$ and $t > 0$ is ill-posed, unlike the corresponding problem posed for $x \in \mathbb{R}$ and $0 \le t \le T$ for a fixed $T > 0$.

4.5. Let $T > 0$. Show that function u given b d'Alembert formula is also a solution for $x \in \mathbb{R}$ and $T < t \le 0$ and show that the Cauchy problem is well-posed for $x \in \mathbb{R}$ and $T \le t \le 0$.

4.6. If the Cauchy data φ, ψ are even (resp. odd or periodic) functions, then the solution of Cauchy problem for the homogeneous wave equation is also an even (resp. odd or periodic) function.

4.7. Isn't it a surprise that we proved the existence of solutions to Cauchy problem for $d = 1$ followed by $d = 3$, and then for $d = 2$ via method of descent? Where does the method of spherical means hits the roadblock when $d = 2$?

4.8. Consider the Cauchy problem

$$u_{tt} - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = f(x) = \begin{cases} 0 & \text{if } -\infty < x < -1, \\ x+1 & \text{if } -1 \le x \le 0, \\ 1-x & \text{if } 0 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty. \end{cases},$$

$$u_t(x,0) = g(x) = \begin{cases} 0 & \text{if } -\infty < x < -1, \\ 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{if } 1 < x < \infty. \end{cases}.$$

(i) Evaluate u at the point $(1, \frac{1}{2})$.

(ii) On what domain does d'Alembert's formula gives a classical solution to the above Cauchy problem? Discuss the smoothness of the function given by d'Alembert's formula.

4.9. [32] Let u be a solution of the Cauchy problem

$$u_{tt} - 9u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,$$

$$u(x,0)=f(x)=\begin{cases} 1 & \text{if } |x|\le 2,\\ 0 & \text{if } |x|>2. \end{cases},$$
$$u_t(x,0)=g(x)=\begin{cases} 1 & \text{if } |x|\le 2,\\ 0 & \text{if } |x|>2. \end{cases}.$$

(i) Evaluate u at the point $(0,\frac{1}{6})$.

(ii) Discuss the large time behaviour of the solution, *i.e.*, on $\lim_{t\to\infty} u(x_0,t)$ for each fixed x_0.

(iii) What is the largest subdomain of $\mathbb{R}\times(0,\infty)$ on which u is a classical solution to the Cauchy problem?

4.10. [32] Solve the following Cauchy problem for a nonhomogeneous wave equation

$$\begin{aligned} u_{tt}-u_{xx}&=t^7,\ x\in\mathbb{R},\ t>0,\\ u(x,0)&=f(x)=2x+\sin x \text{ for } x\in\mathbb{R},\\ u_t(x,0)&=g(x)=0 \text{ for } x\in\mathbb{R}. \end{aligned}$$

4.11. [23] Let $\varphi\in C^2(\mathbb{R})$ and $\psi\in C^1(\mathbb{R})$ be such that $\varphi\equiv\psi\equiv 0$ outside an interval $[a,b]$. Let $u(x,t)$ be the solution of the Cauchy problem (4.4).

(i) Show that for each fixed $t>0$, the function $x\mapsto u(x,t)$ is identically zero outside an interval.

(ii) Show that the functions F,G in the decomposition (4.14) for u can be of compact support only when $\int_{\mathbb{R}}\psi(s)\,ds=0$.

4.12. [39] Find all the spherical solutions of the three-dimensional wave equation.

4.13. [39] Solve the three-dimensional wave equation in $\mathbb{R}^3\setminus\{0\}\times(0,\infty)$ with zero initial conditions and with the limiting condition

$$\lim_{r\to 0} 4\pi r^2 u_r(r,t)=g(t).$$

Assume that $g(0)=g'(0)=g''(0)=0$.

Wave equation on proper subdomains

4.14. [23] Let $\varphi,\psi,h\in C^2([0,\infty))$. Solve the following initial boundary value problem (IBVP).

$$\begin{aligned} u_{tt}-u_{xx}&=0,\ 0<x<\infty,\ t>0,\\ u(x,0)&=\varphi(x), & 0\le x<\infty,\\ u_t(x,0)&=\psi(x), & 0\le x<\infty,\\ u(0,t)&=h(t), & t>0. \end{aligned}$$

from the first principles. Derive the compatibility condition on φ,ψ,h so that the solution derived above is indeed a classical solution. State and prove a relevant well-posedness result for the IBVP.

4.15. Let $\varphi,\psi,h\in C^2([0,\infty))$. Solve the following initial boundary value problem IBVP.

$$u_{tt}-u_{xx}=0,\ 0<x<\infty,\ t>0,$$

$$
\begin{aligned}
u(x,0) &= \varphi(x), \quad 0 \le x < \infty,\\
u_t(x,0) &= \psi(x), \quad 0 \le x < \infty,\\
\frac{\partial u}{\partial x}(0,t) &= h(t), \quad t > 0
\end{aligned}
$$

from the first principles. Derive the compatibility condition on φ, ψ, h so that the solution derived above is indeed a classical solution. State and prove a relevant well-posedness result for the IBVP.

4.16. [23] Let $\alpha \in \mathbb{R}$ be such that $\alpha + c \neq 0$. Let u be the solution of the homogeneous wave equation posed in the domain $(x,t) \in (0,\infty) \times (0,\infty)$ such that

$$
\begin{aligned}
u(x,0) = \varphi(x), u_t(x,0) = \psi(x) \quad &\text{for} \quad x \ge 0\\
u_t(0,t) = \alpha u_x(0,t) \quad &\text{for} \quad t \ge 0,
\end{aligned}
$$

where φ and ψ belong to the class $C^2(0,\infty) \cap C[0,\infty)$ and $\varphi(0) = \psi(0) = 0$. Show that generally no solution exists when $\alpha + c = 0$. (**Hint:** Use the decomposition (4.14)).

4.17. Let u be the solution to the IBVP

$$
\begin{aligned}
u_{tt} - 4u_{xx} &= 0, 0 < x < \infty, t > 0,\\
u(x,0) &= x^2, \quad 0 \le x < \infty,\\
u_t(x,0) &= 6x, \quad 0 \le x < \infty.\\
u(0,t) &= t^2, \quad t > 0,
\end{aligned}
$$

Evaluate $u(3,2)$ and $u(2,3)$.

4.18. Let $\varphi, \psi \in C^2([0,l])$. Solve the following initial boundary value problem IBVP.

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0, 0 < x < l, t > 0,\\
u(x,0) &= \varphi(x), \quad 0 \le x \le l,\\
u_t(x,0) &= \psi(x), \quad 0 \le x \le l,\\
\frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0,\\
\frac{\partial u}{\partial x}(l,t) &= 0, \quad t > 0
\end{aligned}
$$

from the first principles. Derive the compatibility condition on φ, ψ so that the solution derived above is indeed a classical solution. State and prove a relevant well-posedness result for the IBVP.

4.19. Let $\varphi, \psi \in C^2([0,l])$. Solve the following initial boundary value problem IBVP.

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0, 0 < x < l, t > 0,\\
u(x,0) &= \varphi(x), \quad 0 \le x \le l,\\
u_t(x,0) &= \psi(x), \quad 0 \le x \le l,\\
\frac{\partial u}{\partial x}(0,t) &= 0, \quad t > 0,\\
u(l,t) &= 0, \quad t > 0
\end{aligned}
$$

from the first principles. Derive the compatibility condition on φ, ψ so that the solution derived above is indeed a classical solution. State and prove a relevant well-posedness result for the IBVP.

4.20. [41] Solve the following initial boundary value problem IBVP.

$$\begin{aligned} u_{tt} - u_{xx} &= 0,\ 0 < x < \frac{\pi}{2},\ t > 0, \\ u(x,0) &= \sin x, \quad 0 \le x \le \frac{\pi}{2}, \\ u_t(x,0) &= 0, \quad 0 \le x \le \frac{\pi}{2}, \\ u(0,t) &= 0, \quad t > 0, \\ \frac{\partial u}{\partial x}\left(\frac{\pi}{2}, t\right) &= 0, \quad t > 0. \end{aligned}$$

4.21. In the context of the IBVP (4.111), What is the domain of dependence for a point $(x_0, t_0) \in (0,l) \times (0,\infty)$? What is the range of influence of a subinterval $[a,b]$ of $(0,l)$?

4.22. Using separation of variables method solve the following IBVP for the damped wave equation:

$$\begin{aligned} u_{xx} + 2u_t - u_{tt} &= 0,\ 0 < x < \pi,\ t > 0, \\ u(x,0) &= \varphi(x), \quad 0 \le x \le \pi, \\ u_t(x,0) &= 0, \quad 0 \le x \le \pi, \\ u(0,t) &= 0, \quad t > 0, \\ u(\pi,t) &= 0, \quad t > 0. \end{aligned}$$

4.23. Using separation of variables method solve the following IBVP for the damped wave equation:

$$u_{tt} + u_t = u_{xx}, \quad 0 < x < \pi,\ t > 0, \tag{4.164a}$$

$$u(0,t) = u(\pi,t) = 0, \quad 0 \le t \tag{4.164b}$$

$$u(x,0) = \sin(x), \quad 0 \le x \le \pi \tag{4.164c}$$

$$u_t(x,0) = 0, \quad 0 \le x \le \pi \tag{4.164d}$$

(Ans: $u(x,t) = e^{-t/2}\left(\cos(\frac{\sqrt{3}}{2}t) + \frac{1}{\sqrt{3}}\sin(\frac{\sqrt{3}}{2}t)\right)\sin x$)

4.24. [41] Let $c > 0$ be such that $c^2 < 1$, and $\varphi \in C^2([0,l])$. Solve the following initial boundary value problem IBVP.

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0,\ 0 < x < 1,\ t > x, \\ u(x,x) &= \varphi(x), \quad 0 \le x \le 1, \\ u_t(x,x) &= 0, \quad 0 \le x \le 1, \\ u(0,t) &= 0, \quad t > 0, \\ u(l,t) &= 0, \quad t > 0. \end{aligned}$$

4.25. For each $C \in \mathbb{R}$, verify that

$$u_C(x,t) = C \sin \pi x \sin \pi t$$

is a solution to the following boundary value problem for the wave equation:

$$\begin{aligned} u_{tt} &= u_{xx}, \quad 0 < x < 1, 0 < t < 1, \\ u(x,0) &= 0,\ u(x,1) = 0 \quad \text{for} \quad 0 \le x \le 1, \\ u(0,t) &= u(1,t) = 0 \quad \text{for} \quad 0 \le t \le 1. \end{aligned}$$

What are the maximum and minimum values of the solution u_C in the domain $[0,1]\times[0,1]$? Where are they attained? What is your conclusion regarding a maximum principle for Wave equation?

4.26. [19, 32]

(i) Solve the **Darboux problem:**

$$u_{tt} - u_{xx} = 0 \quad , \; t > \max\{x, -x\}, \; t \geq 0, \tag{4.165a}$$

$$u(x,t) = \begin{cases} \phi(t) & \text{if} \quad x = t, \; t \geq 0, \\ \psi(t) & \text{if} \quad x = -t, \; t \geq 0, \end{cases} \tag{4.165b}$$

where $\phi, \psi \in C^2([0,\infty))$ satisfies $\phi(0) = \psi(0)$.

(ii) Find the domain of dependence for any point (x_0, t_0) with $t_0 > |x_0|$.

(iii) Explain what is meant by saying that above problem is well-posed. Is the above problem well-posed? On what domain is it well-posed? State and prove a precise statement concerning this.