Department of Mathematics, Indian Institute of Technology Bombay
MA522: Fourier Analysis and Applications
Assignment 3

Date: 28-01-2011 — Due Date: 04-02-2011
Instructor: S. Sivaji-Ganesh — MA522/04

Instructions:

(a) Submit solutions to all problems from **Section B**.

(b) Add the coverpage after duly filling-in.

(c) Avoid writing incomplete sentences. Each sentence should be math-grammatically correct.

Section A

(1) Do problems **3.1-3.6** from [**Duistermaat-Kolk**].

(2) Let $k \in \mathbb{N} \cup \{0\}$ and $X \subseteq \mathbb{R}^n$ be an open set. In the space $C_0^k(X)$ define convergence of a sequence $\phi_n \to \phi$ by requiring that there exists a compact subset K of X that contains supports of all the functions in the sequence and ϕ, and the sequence along with its derivatives of all orders upto k converge to ϕ and its corresponding derivatives uniformly on K. (note that this definition is similar to that of convergence in $\mathcal{D}(X)$). Now prove that $\mathcal{D}(X)$ is a dense subset of $C_0^k(X)$. Also show that the inclusion map $i : \mathcal{D}(X) \hookrightarrow C_0^k(X)$ is continuous.

(3) Find the order of the distribution defined by

$$\phi \in \mathcal{D}(\mathbb{R}) \longmapsto \int_{\mathbb{R}} \phi(x)\, dx.$$

(4) Let (f_k) be a sequence from the space $C(\mathbb{R}^n)$ and let $f \in C(\mathbb{R}^n)$. Assume that $f_k \to f$ uniformly on every compact subset of $\mathbb{R}^n$. Let $a \in \mathbb{R}^n$ be fixed. Prove that $f_k(a) \to f(a)$.

Section B

(1) Let T be a distribution on $\mathbb{R}^n$ having the property

$$\phi \in \mathcal{D}(\mathbb{R}^n) \text{ and } \phi \geq 0 \quad \Longrightarrow \quad \langle T, \phi \rangle \geq 0.$$

Show that T is order 0 on any compact subset of $\mathbb{R}^n$. Hence T is order 0 on $\mathbb{R}^n$.

(2) Find the order of the distribution defined by

$$\phi \in \mathcal{D}(\mathbb{R}) \longmapsto \phi''(0).$$

(3) Let $X \subseteq \mathbb{R}^n$ be an open set and $K \subset X$ be a compact set. For each $m \in \mathbb{N}$, let $\psi_m \in C_0^\infty(K)$ be such that $\|\psi_m\|_{C^m(K)} < \frac{1}{m}$. Prove that $\psi_m \to 0$ in $\mathcal{D}(X)$.