Department of Mathematics, Indian Institute of Technology Bombay
MA522: Fourier Analysis and Applications
Assignment 7

Date: **08-04-2011** Due Date: **None**
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(1) Let $\psi \in \mathcal{E}(\mathbb{R}^n)$. Then the following statements about ψ are equivalent.

(i) For every pair α, β of multi-indices, $\lim_{|x|\to\infty} \left|x^\alpha \partial^\beta \psi(x)\right| = 0$.

(ii) For every pair α, β of multi-indices, $\sup_{x\in\mathbb{R}^n} \left|\partial^\alpha \left(x^\beta \psi(x)\right)\right| < \infty$.

(iii) For every polynomial $P(x)$ and for every partial differential operator L with constant coefficients, the function $P(x)\, L\psi(x)$ is bounded in $\mathbb{R}^n$.

(iv) For every integer $k \geq 0$, and every multi-index α, the function $\left(1+|x|^2\right)^k \partial^\alpha \psi(x)$ is bounded in $\mathbb{R}^n$.

(v) For every natural number N, and every multi-index α, there exists a positive constant $C_{\alpha,N}$ such that

$$|\left(\partial^\alpha \psi\right)(x)| \leq \frac{C_{\alpha,N}}{\left(1+|x|\right)^N}$$

(2) Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Prove that there exists a sequence (ϕ_k) in $\mathcal{D}(\mathbb{R}^n)$ such that $\phi_k \to \psi$ in $\mathcal{S}(\mathbb{R}^n)$.

(3) Prove that $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and the inclusion map is continuous. Also prove that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ and the inclusion map is continuous.

(4) For what $l > 0$ is the function $(1+|x|)^{-l}$ is integrable on $\mathbb{R}^n$? For what $l > 0$ is the function $\left(1+|x|^2\right)^{-l}$ is integrable on $\mathbb{R}^n$? Prove your assertion.

(5) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Compute the Fourier transform of $\left(\frac{d^2}{dx^2} - x^2\right) f$ in terms of $\hat{f}$.

(6) Let $f \in \mathcal{S}(\mathbb{R})$, and define $g(x) = f(|x|)$ for $x \in \mathbb{R}^n$. Show that if f is an even function, then $g \in \mathcal{S}(\mathbb{R}^n)$. What can be said if f is an odd function?

(7) Let $f \in \mathcal{S}(\mathbb{R}^2)$. Define ϕ by $\phi(x) = f(x,0)$. What is the relationship between $\hat{f}$ and $\hat{\phi}$?

(8) We know that Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. What can we say about $\lambda \in \mathbb{C}$ satisfying the eigenvalue equation $\hat{f} = \lambda f$ for some non-zero $f \in \mathcal{S}(\mathbb{R}^n)$?