Lecture notes on Ordinary Differential Equations

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Part I

Ordinary Differential Equations

Chapter 1

Initial Value Problems

In this chapter we introduce the notion of an initial value problem (IVP) for first order systems of ODE, and discuss questions of existence, uniqueness of solutions to IVP. We also discuss well-posedness of IVPs and maximal interval of existence for a given solution to the IVP. We complement the theory with examples from the class of first order scalar equations.

The main hypotheses in the studies of IVPs is **Hypothesis** ($\mathbf{H_{IVPS}}$), which will be in force throughout our discussion.

> **Hypothesis** ($\mathbf{H_{IVPS}}$)
>
> Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\mathbb{I} \subseteq \mathbb{R}$ be an open interval. Let $\boldsymbol{f} : \mathbb{I} \times \Omega \to \mathbb{R}^n$ be a continuous function defined by $(x, \boldsymbol{y}) \mapsto \boldsymbol{f}(x, \boldsymbol{y})$ where $\boldsymbol{y} = (y_1, \dots y_n)$. Let $(x_0, \boldsymbol{y}_0) \in \mathbb{I} \times \Omega$ be an arbitrary point.

Definition 1.1 (IVP). *Assume Hypothesis* ($\mathrm{H_{IVPS}}$) *on* $\boldsymbol{f}$. *An IVP for a first order system of* n *ordinary differential equations is given by*

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}), \tag{1.1a}$$

$$\boldsymbol{y}(x_0) = \boldsymbol{y}_0. \tag{1.1b}$$

Definition 1.2 (Solution of an IVP). *An* n*-tuple of functions* $\boldsymbol{u} = (u_1, \dots u_n) \in \mathbf{C}^1(\mathbb{I}_0)$ *where* $\mathbb{I}_0 \subseteq \mathbb{I}$ *is an open interval containing the point* $x_0 \in \mathbb{I}$ *is called a solution of IVP (1.1) if for every* $x \in \mathbb{I}_0$*, the* $(n+1)$*-tuple* $(x, u_1(x), u_2(x), \cdots, u_n(x)) \in \mathbb{I} \times \Omega$,

$$\boldsymbol{u}'(x) = \boldsymbol{f}(x, \boldsymbol{u}(x)) \;\; \forall x \in \mathbb{I}_0, \tag{1.2a}$$

$$\boldsymbol{u}(x_0) = \boldsymbol{y}_0. \tag{1.2b}$$

If $\mathbb{I}_0$ *is not an open interval, the left hand side of the equation (1.2a) is interpreted as the appropriate one-sided limit at the end point(s).*

Remark 1.3.

1. The **boldface** notation is used to denote vector quantities and we drop boldface for scalar quantities. We denote this solution by $\boldsymbol{u} = \boldsymbol{u}(x; \boldsymbol{f}, x_0, \boldsymbol{y}_0)$ to remind us that the solution depends on $\boldsymbol{f}, \boldsymbol{y}_0$ and $\boldsymbol{u}(x_0) = \boldsymbol{y}_0$. This notation will be

used to distinguish solutions of IVPs having different initial data, with the understanding that these solutions are defined on a common interval.

2. The IVP (1.1) involves an interval $\mathbb{I}$, a domain Ω, a continuous function $\boldsymbol{f}$ on $\mathbb{I}\times\Omega$, $x_0 \in \mathbb{I}$, $\boldsymbol{y}_0 \in \Omega$. Given $\mathbb{I}$, $x_0 \in \mathbb{I}$ and Ω, we may pose many IVPs by varying the data $(\boldsymbol{f}, \boldsymbol{y}_0)$ belonging to the set $\mathbf{C}(\mathbb{I}\times\Omega)\times\Omega$.
3. Solutions which are defined on the entire interval $\mathbb{I}$ are called *global solutions*. Note that a solution may be defined only on a subinterval of $\mathbb{I}$ according to Definition 1.2. Such solutions which are not defined on the entire interval $\mathbb{I}$ are called *local solutions* to the initial value problem. See Exercise **??** for an example of an IVP that has only a local solution.
4. In Definition 1.2, we may replace the condition $\boldsymbol{u} = (u_1, \dots u_n) \in \mathbf{C}^1(\mathbb{I}_0)$ by requiring that the function $\boldsymbol{u}$ be differentiable. However, since the right hand side of (1.2a) is continuous, both these conditions are equivalent. ■

Basic questions associated to initial value problems

Basic questions associated to initial value problems are

Existence Given $(\boldsymbol{f}, \boldsymbol{y}_0) \in \mathbf{C}(\mathbb{I}\times\Omega)\times\Omega$, does the IVP (1.1) admit at least one solution? This question is of great importance as an IVP without a solution is not of any practical use. When an IVP is modeled on a real-life situation, it is expected that the problem has a solution. In case an IVP does not have a solution, we have to reassess the modeling ODE and/or the notion of the solution. Existence of solutions will be discussed in Section 1.1.

Uniqueness Note that if $\boldsymbol{u} \in \mathbf{C}^1(\mathbb{I}_0)$ is a solution to the IVP (1.1), then restricting $\boldsymbol{u}$ to any subinterval of $\mathbb{I}_0$ will also give a solution. Thus once a solution exists, there are infinitely many solutions. But we do not wish to treat them as different solutions. Thus the interest is to know the total number of 'distinct solutions', and whether that number is one. This is the question of uniqueness. It will be discussed in Section 1.2.

Continuous dependence Assuming that there exists an interval $\mathbb{I}_0$ containing x_0 such that the IVP (1.1) admits a unique solution $\boldsymbol{u}(x; \boldsymbol{f}, x_0, \boldsymbol{y}_0)$ for each $(\boldsymbol{f}, \boldsymbol{y}_0) \in \mathbf{C}(\mathbb{I}\times\Omega)\times\Omega$, it is natural to ask the nature of the function

$$\mathscr{S} : \mathbf{C}(\mathbb{I}\times\Omega)\times\Omega \longrightarrow \mathbf{C}^1(\mathbb{I}_0) \tag{1.3}$$

defined by

$$(\boldsymbol{f}, \boldsymbol{y}_0) \longmapsto \boldsymbol{u}(x; \boldsymbol{f}, x_0, \boldsymbol{y}_0). \tag{1.4}$$

The question is if the above function is continuous. Unfortunately we are unable to prove this result. However we can prove the continuity of the function $\mathscr{S}$ where the co-domain is replaced by $\mathbf{C}(\mathbb{I}_0)$. It will be discussed in Section 1.3.

Continuation of solutions Note that the notion of a solution to an IVP requires a solution to be defined on some interval containing the point x_0 at which initial condition is prescribed. Thus the interest is in finding out if a given solution to an IVP is the restriction of another solution that is defined on a bigger interval. In other words, the question is whether a given solution to an IVP can be extended to a solution on a bigger interval, and how big that interval can be. This is the question of continuation of solutions, and it will be discussed in Section 1.4. In particular, we would like to know if a given solution can be extended to a solution on the interval $\mathbb{I}$, and the possible obstructions that prevent such an extension.

1.1 ▪ Existence of local solutions

There are two important results concerning existence of solutions to IVP (1.1). One of them is proved for any function $\boldsymbol{f}$ satisfying Hypothesis ($\mathrm{H_{IVPS}}$) and the second assumes *Lipschitz continuity* of the function $\boldsymbol{f}$ in addition. This extra assumption on $\boldsymbol{f}$ not only gives rise to another proof of existence but also makes sure that the IVP has a unique solution, as we shall see in Section 1.2.

1.1.1 ▪ Preliminaries

Equivalent integral equation formulation

Proof of existence of solutions is based on the equivalence of IVP (1.1) and an integral equation, which is stated in the following result.

Lemma 1.4. *Let $\boldsymbol{u}$ be a continuous function defined on an interval $\mathbb{I}_0$ containing the point x_0. Then the following statements are equivalent.*

1. *The function $\boldsymbol{u}$ is a solution of IVP (1.1).*
2. *The function $\boldsymbol{u}$ satisfies the integral equation*

$$\boldsymbol{y}(x) = \boldsymbol{y}_0 + \int_{x_0}^{x} \boldsymbol{f}(s, \boldsymbol{y}(s))\, ds \quad \forall\, x \in \mathbb{I}_0, \tag{1.5}$$

where the integral of a vector quantity is understood as the vector consisting of integrals of its components.

Proof. **(Proof of 1 $\Longrightarrow$ 2:)** If $\boldsymbol{u}$ is a solution of IVP (1.1), then by definition of solution we have

$$\boldsymbol{u}'(x) = \boldsymbol{f}(x, \boldsymbol{u}(x)). \tag{1.6}$$

Integrating the above equation from x_0 to x yields the integral equation (1.5).

(Proof of 2 $\Longrightarrow$ 1:) Let $\boldsymbol{u}$ be a solution of integral equation (1.5). Observe that, due to continuity of the function $t \to \boldsymbol{u}(t)$, the function $t \to \boldsymbol{f}(t, \boldsymbol{u}(t))$ is continuous on $\mathbb{I}_0$. Thus RHS of (1.5) is a differentiable function w.r.t. x by fundamental theorem of integral calculus and its derivative is given by the function $x \to \boldsymbol{f}(x, \boldsymbol{u}(x))$ which is a continuous function. Thus $\boldsymbol{u}$, being equal to a continuously differentiable function via equation (1.5), is also continuously differentiable. The function $\boldsymbol{u}$ is a solution of ODE (1.1) follows by differentiating the equation (1.5). Evaluating (1.5) at $x = x_0$ gives the initial condition $\boldsymbol{u}(x_0) = \boldsymbol{y}_0$. □

Remark 1.5. In view of Lemma 1.4, existence of a solution to the IVP (1.1) follows from the existence of a solution to the integral equation (1.5). This is the strategy followed in proving both the existence theorems that we are going to discuss. ∎

Join of two solutions is a solution

The graph of any solution to the ordinary differential equation (1.1a) is called a solution curve, and it is a subset of $\mathbb{I}\times\Omega$. The space $\mathbb{I}\times\Omega$ is called *extended phase space*. If we join (concatenate) two solution curves, the resulting curve will also be a solution curve. This is the content of the next result.

Lemma 1.6 (Concatenation of two solutions). *Assume Hypothesis* ($\mathrm{H_{IVPS}}$). *Let* $[a,b]$ *and* $[b,c]$ *be two subintervals of* $\mathbb{I}$. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{w}$ *defined on intervals* $[a,b]$ *and* $[b,c]$ *respectively be solutions of IVP with initial data* (a,ξ) *and* $(b,\boldsymbol{u}(b))$ *respectively. Then the concatenated function* $\boldsymbol{z}$ *defined on the interval* $[a,c]$ *by*

$$\boldsymbol{z}(x)=\begin{cases}\boldsymbol{u}(x) & \text{if} \quad x\in[a,b].\\ \boldsymbol{w}(x) & \text{if} \quad x\in(b,c].\end{cases} \tag{1.7}$$

is a solution of IVP with initial data (a,ξ).

Proof. It is easy to see that the function $\boldsymbol{z}$ is continuous on $[a,c]$. Therefore, by Lemma 1.4, it is enough to show that $\boldsymbol{z}$ satisfies the integral equation

$$\boldsymbol{z}(x)=\xi+\int_a^x f(s,\boldsymbol{z}(s))\,ds \quad \forall\, x\in[a,c]. \tag{1.8}$$

Clearly the equation (1.8) is satisfied for $x\in[a,b]$, once again, by Lemma 1.4 since $\boldsymbol{u}$ solves IVP with initial data (a,ξ) and $\boldsymbol{z}(x)=\boldsymbol{u}(x)$ for $x\in[a,b]$. Thus it remains to prove (1.8) for $x\in(b,c]$.

For $x\in(b,c]$, once again by Lemma 1.4, we get

$$\boldsymbol{z}(x)=\boldsymbol{w}(x)=\boldsymbol{u}(b)+\int_b^x f(s,\boldsymbol{w}(s))\,ds=\boldsymbol{u}(b)+\int_b^x f(s,\boldsymbol{z}(s))\,ds. \tag{1.9}$$

Since

$$\boldsymbol{u}(b)=\xi+\int_a^b f(s,\boldsymbol{u}(s))\,ds=\xi+\int_a^b f(s,\boldsymbol{z}(s))\,ds, \tag{1.10}$$

substituting for $\boldsymbol{u}(b)$ in (1.9) finishes the proof of lemma. □

Remark 1.7 (Importance of concatenation lemma). We list below where the concatenation lemma is used in the sequel.

(i) Peano's theorem asserts the existence of a solution to the IVP (1.1) in a bilateral interval containing the point x_0 at which initial condition (1.1b) is prescribed. However the proof is divided into two parts: existence of a solution is proved on an interval of the type $[x_0,x_0+\delta]$ (called a *left solution*), and then on an interval of the type $[x_0-\delta,x_0]$ (called a *right solution*). Concatenation lemma is then used to assert that a solution exists on the bilateral interval $[x_0-\delta,x_0+\delta]$.

(ii) In view of (i) above, all the discussion of solutions on a bilateral interval containing x_0 can be reduced to those of left solutions and right solutions.

(iii) In the discussion of continuation of a given solution to the IVP (1.1), if the given solution is defined on the interval $[x_0,x_1]$, and if we find a solution on $[x_1,x_2)$ such that both functions agree at x_1, then concatenation lemma asserts that the concatenated function is a solution on $[x_0,x_2)$. ∎

Rectangles

The proofs of existence theorems rely on an approximation procedure. This necessitates introduction of measurement of distances between functions, for which dealing with functions on open sets is problematic. For example a continuous function defined on an open interval need not be bounded, while its restriction to any compact interval is bounded.

A good class of compact subsets of the open set $\mathbb{I} \times \Omega$ is that of 'rectangles', which are defined below.

Definition 1.8 (Rectangle). *Let $a > 0, b > 0$. A rectangle $R \subseteq \mathbb{I} \times \Omega$ centred at (x_0, y_0) is defined by*

$$R = \{x : |x - x_0| \leq a\} \times \{y : ||y - y_0|| \leq b\} \tag{1.11}$$

Note that rectangle R in (1.11) is symmetric in the x space as well. However a solution may be defined on an interval that is not symmetric about x_0. Thus it looks restrictive to consider R as defined above. It is definitely the case when x_0 is very close to one of the end points of the interval $\mathbb{I}$. Indeed in results addressing the existence of solutions for IVP (left solutions), separately on intervals left and right to x_0 (right solutions) consider rectangles $R^\dagger \subseteq \mathbb{I} \times \Omega$ of the forms

$$R_\dagger = [x_0 - a, x_0] \times \{y : ||y - y_0|| \leq b\}, \tag{1.12a}$$

$$R^\dagger = [x_0, x_0 + a] \times \{y : ||y - y_0|| \leq b\}. \tag{1.12b}$$

respectively.

Remark 1.9. Since $\mathbb{I} \times \Omega$ is an open set, we can find such an R (for some positive real numbers a, b) for each point $(x_0, y_0) \in \mathbb{I} \times \Omega$. ∎

1.1.2 ▪ Peano's existence theorem

Theorem 1.10 (Peano). *Assume Hypothesis* ($\mathrm{H}_{\mathrm{IVPS}}$). *Let R be a rectangle contained in $\mathbb{I} \times \Omega$ given by*

$$R = \{x : |x - x_0| \leq a\} \times \{y : ||y - y_0|| \leq b\}$$

for some $a > 0$, $b > 0$. Then the IVP (1.1) has at least one solution on the interval $|x - x_0| \leq \delta$ where $\delta = \min\{a, \frac{b}{M}\}$, where $M = \sup_R ||f(x, y||$.

The plan of action for proving Peano's theorem is as follows:

1. In view of concatenation lemma (Lemma 1.6), it is enough to show the existence of a left solution and a right solution to IVP. We are going to prove the existence of a right solution (*i.e.*, a solution defined on an interval to the right of x_0) and similar arguments yield a left solution (*i.e.*, a solution defined on an interval to the left of x_0).

2. A special case of Peano's theorem is proved, namely when f is a bounded function defined on the domain $\mathbb{I} \times \mathbb{R}^n$.

3. Peano's theorem is then deduced from its special case.

Special case of Peano's theorem

Theorem 1.11. *Assume in additon to Hypothesis* $(\mathrm{H}_{\mathrm{IVPS}})$ *that* $\Omega = \mathbb{R}^n$. *Let* f *be bounded on* $\mathbb{I} \times \mathbb{R}^n$, *and* $M := \sup_{\mathbb{I}\times\mathbb{R}^n} \|f(x,y)\|$. *Let* R *be a rectangle contained in* $\mathbb{I} \times \mathbb{R}^n$ *given by*

$$R = \{x : |x - x_0| \leq a\} \times \{y : \|y - y_0\| \leq b\}$$

for some $a > 0$, $b > 0$. *Then the IVP (1.1) has at least one solution on the interval* $[x_0,\, x_0 + \delta]$ *for every* $\delta > 0$ *such that* $[x_0,\, x_0 + \delta] \subset \mathbb{I}$.

Before we prove the special case of Peano's theorem, consider an auxiliary problem under the same hypotheses as that of Theorem 1.11, which resembles the integral equation (1.5) corresponding to IVP (1.1) but for the *delay* factor $\lambda > 0$ in the argument of the integrand in (1.13), given by

$$y_\lambda(x) = \begin{cases} y_0 & \text{if} \quad x \in [x_0 - \lambda,\, x_0]. \\ y_0 + \displaystyle\int_{x_0}^{x} f(s, y_\lambda(s-\lambda))\, ds & \text{if} \quad x \in [x_0,\, x_0 + a]. \end{cases} \tag{1.13}$$

We have the following result concerning solutions of the auxiliary problem (1.13).

Lemma 1.12. *Let* $\delta > 0$ *be such that* $[x_0,\, x_0 + \delta] \subset \mathbb{I}$. *Then*

(i) *for each* $\lambda > 0$, *the problem (1.13) has a unique solution defined on the interval* $[x_0,\, x_0 + \delta]$.
(ii) *the sequence* (y_λ) *is equicontinuous.*
(iii) *the sequence* (y_λ) *is uniformly bounded.*

Proof.
Step 1: Proof of (i)
Note that the problem (1.13) has a solution, which is determined first on the interval $[x_0, x_0+\lambda]$, followed by the intervals $[x_0+\lambda, x_0+2\lambda]$,$[x_0+2\lambda, x_0+3\lambda]$, $\cdots$. Proceeding this way, the interval $[x_0,\, x_0+\delta]$ will be covered in a finite number of steps, and thus a solution to (1.13) would be determined on the interval $[x_0,\, x_0 + \delta]$. Proving the uniqueness of solutions to (1.13) is left as an exercise.
Step 2: Proof of (ii)
Recall that on the interval $[x_0,\, x_0 + \delta]$, each of the functions y_λ satisfy the equation

$$y_\lambda(x) = y_0 + \int_{x_0}^{x} f(s, y_\lambda(s-\lambda))\, ds. \tag{1.14}$$

For x_1, x_2 belonging to the interval $[x_0,\, x_0 + \delta]$, using (1.14), we get

$$y_\lambda(x_1) - y_\lambda(x_2) = \int_{x_2}^{x_1} f(s, y_\lambda(s-\lambda))\, ds. \tag{1.15}$$

Taking the euclidean norm on both sides of the equation (1.15) yields

$$\|y_\lambda(x_1) - y_\lambda(x_2)\| \leq M|x_1 - x_2|, \tag{1.16}$$

where $M := \sup_{\mathbb{I}\times\mathbb{R}^n} \|f(x,y)\|$. Since M is independent of λ, equicontinuity of the sequence (y_λ) follows from the inequality (1.16).

Step 3: Proof of (iii)
Taking the euclidean norm on both sides of the equation (1.14), we get

$$\|y_\lambda(x)\| \le M|x-x_0| \le M\delta. \tag{1.17}$$

This proves that the sequence (y_λ) is uniformly bounded. □

Proof. **(of Theorem 1.11)**
In view of Lemma 1.12, we can apply Ascoli-Arzela theorem and conclude that there exists a subsequence of (y_λ), and a continuous function $z : [x_0, x_0+\delta] \to \mathbb{R}^n$ such that

$$y_{\lambda_k} \to z \quad \text{uniformly on the interval } [x_0, x_0+\delta].$$

Passing to the limit as $k \to \infty$, and as $\lambda_k \to 0$ in the equation (1.14), which is justified because of the uniform convergence, yields

$$z(x) = y_0 + \int_{x_0}^{x} f(s, z(s))\, ds. \tag{1.18}$$

By the equivalence of integral equation (1.18) with the initial value problem, we conclude that z is a solution of the initial value problem. This completes the proof of the special case of Peano's theorem (Theorem 1.11). □

Proof of Peano's theorem

Proof. **(of Theorem 1.10)** Recall that according to Hypothesis ($\mathrm{H_{IVPS}}$), the function f is defined on $\mathbb{I}\times\Omega$ and it may not be equal to $\mathbb{I}\times\mathbb{R}^n$. We define a function g defined on $\mathbb{I}\times\mathbb{R}^n$ using f and ρ defined in (1.20) by

$$g(x,y) = f(x,\rho(y)) \quad \forall (x,y) \in \mathbb{I}\times\mathbb{R}^n, \tag{1.19}$$

where the function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(y) = \begin{cases} y & \text{if} \quad y \in B[y_0, b]. \\ y_0 + b\,\dfrac{y-y_0}{\|y-y_0\|} & \text{if} \quad y \in \mathbb{R}^n \setminus B[y_0, b]. \end{cases} \tag{1.20}$$

Here $B[y_0, b]$ denotes the closed ball of radius $b > 0$ centred at y_0. Note that g is a bounded and continuous function defined on $\mathbb{I}\times\mathbb{R}^n$. Therefore by the special case of Peano's theorem, namely, Theorem 1.11, there exists a solution for the IVP defined by g on the interval $[x_0, x_0+\delta]$ for every $\delta > 0$ such that $[x_0, x_0+\delta] \subset \mathbb{I}$.

Note that the solution curve starts inside $B[y_0, b]$ (due to initial condition) and thus will remain inside $B[y_0, b]$ due to continuity of a solution, *i.e.,* there exists an $\alpha > 0$ such that solution restricted to the interval $[x_0, x_0+\alpha]$ takes its values in $B[y_0, b]$. But by construction of g, we know that

$$g(x,y) = f(x,y) \quad \forall (x,y) \in \mathbb{I}\times B[y_0, b] \tag{1.21}$$

in view of the definition of the function ρ. Thus we have found a solution of IVP defined by f on an interval $[x_0, x_0+\alpha]$. Now we would like to find a good α if not the best.

Note that α was chosen such that solution remains in $B[y_0, b]$ using continuity of a solution. Using the integral equation corresponding to this IVP, we find the best α. Since the interval $[x_0, x_0+\delta]$ should lie inside $[x_0, x_0+a]$, which can be guaranteed provided $\delta \leq a$. Since $||z(x)-y_0|| \leq M\delta$, $M\delta \leq b$ can be guaranteed if $\delta \leq \frac{b}{M}$. Thus $\delta = \min\{a, \frac{b}{M}\}$ guarantees both, and thus theorem is proved. □

1.1.3 ▪ Cauchy-Lipschitz-Picard existence theorem

Lipschitz continuity

From real analysis, we know that continuity of a function at a point is a local concept (as it involves values of the function in a neighbourhood of the point at which continuity of function is in question). We talk about uniform continuity of a function with respect to a domain. Similarly we can define Lipschitz continuity at a point and on a domain of a function defined on subsets of $\mathbb{R}^n$.

Definition 1.13. *Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function.*

(i) *The function f is said to be Lipschitz continuous if there exists a $K > 0$ such that*

$$||f(y_1)-f(y_2)|| \leq K||y_1-y_2|| \quad \forall y_1, y_2 \in \mathbb{R}^m,$$

where $||\cdot||$ denotes any norm.

(ii) *The function f is said to be locally Lipschitz continuous if for every point $y_0 \in R^m$, there exists a closed and bounded ball $B[y_0, r]$, and a $K > 0$ such that*

$$||f(y_1)-f(y_2)|| \leq K||y_1-y_2|| \quad \forall y_1, y_2 \in B[y_0, r].$$

Example 1.14. (i) The norm function $||\cdot|| : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz. This is a consequence of triangle inequality.

(ii) Any continuously differentiable function is locally Lipschitz on its domain. ■

The following result helps in identifying functions that are not Lipschitz. The result can be proved from the definition of Lipschitz continuity, and definition of derivative at a point, and proving the same is left to the reader as an exercise.

Lemma 1.15. *Let $\mathbb{I}$ be an open interval, and $f : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Let f be differentiable on $\mathbb{I}$. Then there exists a $C > 0$ such that for every $x \in \mathbb{I}$, the inequality $|f'(x)| \leq C$ holds.* ■

Remark 1.16. The function $f(x) = \sqrt{x}$ is not Lipschitz on any interval of the types $[0, b]$ and $(0, b)$ for any $b > 0$. Since the derivative of f is $\frac{1}{2\sqrt{x}}$ which is not bounded on intervals of the type described earlier. However f is Lipschitz on intervals of the type $[\epsilon, b]$, (ϵ, b) for every $\epsilon > 0$. ■

For ODE purposes we need functions of $(n+1)$ variables and Lipschitz continuity w.r.t. the last n variables. Thus we define concept of Lipschitz continuity used for such functions.

Let $R \subseteq \mathbb{I} \times \Omega$ be a rectangle centred at (x_0, y_0) defined by two positive real numbers a, b (see equation (1.11)).

Definition 1.17 (Lipschitz continuity). *A function f is said to be Lipschitz continuous on a rectangle R with respect to the variable y if there exists a $K > 0$ such that*

$$||f(x, y_1) - f(x, y_1)|| \le K||y_1 - y_2|| \quad \forall (x, y_1), (x, y_2) \in R. \tag{1.22}$$

Cauchy-Lipschitz-Picard theorem

We now state the existence theorem and the method of proof is different from that of Peano's theorem and yields a bilateral interval containing x_0 on which existence of a solution is asserted.

Theorem 1.18 (Cauchy-Lipschitz-Picard).
Assume Hypothesis ($\mathrm{H_{IVPS}}$). *Let R be a rectangle contained in $\mathbb{I} \times \mathbb{R}^n$ given by*

$$R = \{x : |x - x_0| \le a\} \times \{y : ||y - y_0|| \le b\}$$

for some $a > 0$, $b > 0$. Let f be Lipschitz continuous with respect to the variable y on R. Then the IVP (1.1) has at least one solution on the interval $\mathbb{J} : |x - x_0| \le \delta$ where $\delta = \min\{a, \frac{b}{M}\}$, where $M = \sup\limits_R ||f(x, y||$.

Proof.
Step 1: Equivalent integral equation By Lemma 1.4, the given IVP is equivalent to the following integral equation.

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,ds \quad \forall\, x \in \mathbb{I}. \tag{1.23}$$

By the equivalence of above integral equation with the IVP, it is enough to prove that the integral equation has a solution. This proof is accomplished by constructing, what are known as Picard approximations, a sequence of functions that converges to a solution of the integral equation (1.23).

Step 2: Construction of Picard approximations
Define the first function $y_0(x)$, for $x \in \mathbb{I}$, by

$$y_0(x) := y_0 \tag{1.24}$$

Define $y_1(x)$, for $x \in \mathbb{I}$, by

$$y_1(x) := y_0 + \int_{x_0}^{x} f(s, y_0(s))\,ds. \tag{1.25}$$

Note that the function $y_1(x)$ is well-defined for $x \in \mathbb{I}$. However, when we try to define the next member of the sequence, $y_2(x)$, for $x \in \mathbb{I}$, by

$$y_2(x) := y_0 + \int_{x_0}^{x} f(s, y_1(s))\,ds, \tag{1.26}$$

caution needs to exercised. This is because, we do not know about the values that the function $y_1(x)$ assumes for $x \in \mathbb{I}$, there is no reason that those values are inside Ω. However, it happens that for $x \in \mathbb{J}$, where the interval $\mathbb{J}$ is as in the statement of the theorem, the expression on RHS of (1.26) which defined function $y_2(x)$ is meaningful, and hence the function $y_2(x)$ is well-defined for $x \in \mathbb{J}$. By restricting to the interval $\mathbb{J}$, we can prove that the following sequence of functions is well-defined: Define for $k \geq 1$, for $x \in \mathbb{J}$,

$$y_k(x) = y_0 + \int_{x_0}^{x} f(s, y_{k-1}(s))\, ds. \tag{1.27}$$

Proving the well-definedness of Picard approximations is left as an exercise, by induction. In fact, the graphs of each Picard approximant lies inside the rectangle R (see statement of our theorem). That is,

$$||y_k(x) - y_0|| \leq b, \quad \forall\, x \in \mathbb{J} := [x_0 - \delta,\, x_0 + \delta]. \tag{1.28}$$

The proof is immediate from

$$y_k(x) - y_0 = \int_{x_0}^{x} f(s, y_{k-1}(s))\, ds, \quad \forall\, x \in \mathbb{J}. \tag{1.29}$$

$$\text{Therefore,} \quad ||y_k(x) - y_0|| \leq M|x - x_0|, \quad \forall\, x \in \mathbb{J}. \tag{1.30}$$

and for $x \in \mathbb{J}$, we have $|x - x_0| \leq \delta$.

Step 3: Convergence of Picard approximations

We prove the uniform convergence of sequence of Picard approximations y_k on the interval $\mathbb{J}$, by proving that this sequence corresponds to the partial sums of a uniformly convergent series of functions given by

$$y_0 + \sum_{l=0}^{\infty} \left[y_{l+1}(x) - y_l(x) \right]. \tag{1.31}$$

Note that the sequence y_{k+1} corresponds to partial sums of series (1.31). That is,

$$y_{k+1}(x) = y_0 + \sum_{l=0}^{k} \left[y_{l+1}(x) - y_l(x) \right]. \tag{1.32}$$

Step 3A: Unifrom convergence of series (1.31) on $x \in \mathbb{J}$

We are going to compare series (1.31) with a convergence series of real numbers, uniformly in $x \in \mathbb{J}$, and thereby proving uniform convergence of the series. From the expression

$$y_{l+1}(x) - y_l(x) = \int_{x_0}^{x} \left\{ f(s, y_l(s)) - f(s, y_{l-1}(s)) \right\} ds, \quad \forall\, x \in \mathbb{J}, \tag{1.33}$$

and using induction, the following estimate follows :

$$||y_{l+1}(x) - y_l(x)|| \leq M L^l \frac{|x - x_0|^{l+1}}{(l+1)!} \leq \frac{M}{L} L^{l+1} \frac{\delta^{l+1}}{(l+1)!} \tag{1.34}$$

We conclude that the series (1.31), and hence the sequence of Picard iterates, converge uniformly on $\mathbb{J}$. This is because the above estimate (1.34) says that general term of series (1.31) is uniformly smaller than that of a convergent series, namely, for the function $e^{\delta L}$ times a constant.

Let $y(x)$ denote the uniform limit of the sequence of Picard iterates $y_k(x)$ on $\mathbb{J}$.

Step 4: The limit function $y(x)$ solves integral equation (1.23)
We want to take limit as $k \to \infty$ in

$$y_k(x) = y_0 + \int_{x_0}^{x} f(s, y_{k-1}(s))\, ds. \tag{1.35}$$

Taking limit on LHS of (1.35) is trivial. Therefore, for $x \in \mathbb{J}$, if we prove that

$$\int_{x_0}^{x} f(s, y_{k-1}(s))\, ds \longrightarrow \int_{x_0}^{x} f(s, y(s))\, ds, \tag{1.36}$$

then we would obtain, for $x \in \mathbb{J}$,

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds, \tag{1.37}$$

and this finishes the proof. Therefore, it remains to prove (1.36). Let us estimate, for $x \in \mathbb{J}$, the quantity

$$\int_{x_0}^{x} f(s, y_{k-1}(s))\, ds - \int_{x_0}^{x} f(s, y(s))\, ds = \int_{x_0}^{x} \left\{ f(s, y_{k-1}(s)) - f(s, y(s)) \right\} ds \tag{1.38}$$

Since the graphs of y_k lie inside rectangle R, so does the graph of y. This is because rectangle R is closed. Now we use that the vector field f is Lipschitz continuous (with Lipschitz constant L) in the variable y on R, we get

$$\begin{aligned}
\left\| \int_{x_0}^{x} \left\{ f(s, y_{k-1}(s)) - f(s, y(s)) \right\} ds \right\| &\leq \left| \int_{x_0}^{x} \left\| \left\{ f(s, y_{k-1}(s)) - f(s, y(s)) \right\} \right\| ds \right| \\
&\leq L \left| \int_{x_0}^{x} \left\| y_{k-1}(s) - y(s) \right\| ds \right| \\
&\leq L|x - x_0| \sup_{\mathbb{J}} \left\| y_{k-1}(x) - y(x) \right\| \\
&\leq L\delta \sup_{\mathbb{J}} \left\| y_{k-1}(x) - y(x) \right\|.
\end{aligned} \tag{1.39}$$

$$\begin{aligned}
\left\| \int_{x_0}^{x} \left\{ f(s, y_{k-1}(s)) - f(s, y(s)) \right\} ds \right\| &\leq \left| \int_{x_0}^{x} \left\| \left\{ f(s, y_{k-1}(s)) - f(s, y(s)) \right\} \right\| ds \right| \\
\leq L \left| \int_{x_0}^{x} \left\| y_{k-1}(s) - y(s) \right\| ds \right| &\leq L|x - x_0| \sup_{\mathbb{J}} \left\| y_{k-1}(x) - y(x) \right\| \\
&\leq L\delta \sup_{\mathbb{J}} \left\| y_{k-1}(x) - y(x) \right\|.
\end{aligned} \tag{1.40}$$

The estimate (1.39) finishes the proof of (1.36), since $y(x)$ is the uniform limit of the sequence of Picard iterates $y_k(x)$ on $\mathbb{J}$, and hence for sufficiently large k, the quantity $\sup_{\mathbb{J}} \left\| y_{k-1}(x) - y(x) \right\|$ can be made arbitrarily small.

□

Some comments on existence theorems

Remark 1.19. (i) Note that the interval of existence depends only on the dimensions of the rectangle a, b, and on the bound M of the function f on the rectangle, and not on any other property of the function f.

(ii) If g is any Lipschitz continuous function (with respect to the variable y on R) in an α neighbourhood of f and ζ is any vector in a β neighbourhood of y_0, then solution to IVP with data (g, ζ) exists on the interval $\delta = \min\{a, \frac{b-\beta}{M+\alpha}\}$. Note that this interval depends on data (g, ζ) only in terms of its distance to (f, y_0). ■

1.2 ▪ Uniqueness

Recalling the definition of a solution, we note that if u solves IVP (1.1) on an interval $\mathbb{I}_0$ then $w \stackrel{\text{def}}{=} u|_{\mathbb{I}_1}$ is also a solution to the same IVP where $\mathbb{I}_1$ is any subinterval of $\mathbb{I}_0$ containing the point x_0. In principle we do not want to consider the latter as a different solution. Thus we are led to define a concept of equivalence of solutions of an IVP that does not distinguish w from u near the point x_0. Roughly speaking, two solutions of IVP are said to be equivalent if they agree on some interval containing x_0. This interval itself may depend on the given two solutions.

Definition 1.20 (local uniqueness). *An IVP is said to have local uniqueness property if for each $(x_0, y_0) \in \mathbb{I} \times \Omega$ and for any two solutions y_1 and y_2 of IVP (1.1) defined on intervals $\mathbb{I}_1$ and $\mathbb{I}_2$ respectively, there exists an open interval $\mathbb{I}' \subseteq \mathbb{I}_1 \cap \mathbb{I}_2$ such that the point $x_0 \in \mathbb{I}'$ and $y_1(x) = y_2(x)$ for all $x \in \mathbb{I}'$.*

Definition 1.21 (global uniqueness). *An IVP is said to have global uniqueness property if for each $(x_0, y_0) \in \mathbb{I} \times \Omega$ and for any two solutions y_1 and y_2 of IVP (1.1) defined on intervals $\mathbb{I}_1$ and $\mathbb{I}_2$ respectively, the equality $y_1(x) = y_2(x)$ holds for all $x \in \mathbb{I}_1 \cap \mathbb{I}_2$.*

Remark 1.22. 1. It is easy to understand the presence of adjectives local and global in Definition 1.20 and Definition 1.21 respectively.

2. There is no loss of generality in assuming that the interval appearing in Definition 1.20, namely $\mathbb{I}_{\delta l r}$, is of the form $\mathbb{J}_{\delta l r} := [x_0 - \delta_l, x_0 + \delta_r]$. This is because in any open interval containing a point x_0, there is a closed interval containing the same point x_0 and vice versa.

Though it may appear that local and global uniqueness properties are quite different from each other, indeed they are the same. This is the content of the next result.

Lemma 1.23. *The following are equivalent.*

1. *An IVP has local uniqueness property.*
2. *An IVP has global uniqueness property.*

Proof. From definitions, clearly $2 \Longrightarrow 1$. We turn to the proof of $1 \Longrightarrow 2$.

Let y_1 and y_2 be solutions of IVP (1.1) defined on intervals $\mathbb{I}_1$ and $\mathbb{I}_2$ respectively. We prove that $y_1 = y_2$ on the interval $\mathbb{I}_1 \cap \mathbb{I}_2$. If the interval $\mathbb{I}_1 \cap \mathbb{I}_2$ is not bounded below or above, we consider its infimum as $-\infty$ and its supremum as ∞.

We split its proof into two parts. Note that $\mathbb{I}_1 \cap \mathbb{I}_2$ is intersection of two intervals containing x_0, and hence is an interval. We first prove the equality to the right of x_0, *i.e.,* on the interval $[x_0, \sup(\mathbb{I}_1 \cap \mathbb{I}_2))$ and proving the equality to the left of x_0 (*i.e.,* on the interval $(\inf(\mathbb{I}_1 \cap \mathbb{I}_2), x_0]$) follows a canonically modified argument. Let us consider the following set

$$\mathscr{K}_r = \{t \in \mathbb{I}_1 \cap \mathbb{I}_2 : \boldsymbol{y}_1(x) = \boldsymbol{y}_2(x) \quad \forall x \in [x_0, t]\}. \tag{1.41}$$

The set $\mathscr{K}_r$ has the following properties:

(i) The set $\mathscr{K}_r$ is non-empty. This follows by applying local uniqueness property of the IVP with initial data $(x_0, \boldsymbol{y}_0)$.

(ii) Without loss of generality we may assume that $\mathscr{K}_r$ is bounded. We claim that the equality $\sup \mathscr{K}_r = \sup(\mathbb{I}_1 \cap \mathbb{I}_2)$ holds.

Proof: Note that infimum and supremum of an open interval equals the left and right end points of its closure (the closed interval) whenever they are finite. Observe that $\sup \mathscr{K}_r \leq \sup(\mathbb{I}_1 \cap \mathbb{I}_2)$ since $\mathscr{K}_r \subseteq \mathbb{I}_1 \cap \mathbb{I}_2$. Thus it is enough to prove that strict inequality can not hold. On the contrary, let us assume that $a_r := \sup \mathscr{K}_r < \sup(\mathbb{I}_1 \cap \mathbb{I}_2)$. This means that $a_r \in \mathbb{I}_1 \cap \mathbb{I}_2$ and hence $\boldsymbol{y}_1(a_r), \boldsymbol{y}_2(a_r)$ are defined. Since a_r is the supremum of the set $\mathscr{K}_r$ on which $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ coincide, it follows that $\boldsymbol{y}_1(a_r) = \boldsymbol{y}_2(a_r)$ holds. If this were not true, then $\boldsymbol{y}_1(a_r) \neq \boldsymbol{y}_2(a_r)$. By continuity of the functions $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, it follows that $\boldsymbol{y}_1(x) \neq \boldsymbol{y}_2(x)$ for $x \in (a_r - \delta, a_r + \delta)$, which contradicts the fact that a_r is the supremum of the set $\mathscr{K}_r$.

Thus we find that the functions $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ are still solutions of ODE on the interval $\mathbb{I}_1 \cap \mathbb{I}_2$, and also that $\boldsymbol{y}_1(a_r) = \boldsymbol{y}_2(a_r)$. Thus applying once again local uniquness property of IVP but with initial data $(a_r, \boldsymbol{y}_1(a_r))$, we conclude that $\boldsymbol{y}_1 = \boldsymbol{y}_2$ on an interval $\mathbb{J}_{\delta l r} := [a_r - \delta_l, a_r + \delta_r]$ (see Remark 1.22). Thus combining with arguments of previous paragraph we obtain the equality of functions $\boldsymbol{y}_1 = \boldsymbol{y}_2$ on the interval $[x_0, a_r + \delta_r]$. This means that $a_r + \delta_r \in \mathscr{K}_r$ and thus a_r is not an upper bound for $\mathscr{K}_r$. This contradiction to the definition of a_r finishes the proof of 2.

As mentioned at the beginning of the proof, similar statements to the left of x_0 follow. This finishes the proof of lemma. ∎

Remark 1.24. (i) By Lemma 1.23 we can use either of the two definitions Definition 1.20 and Definition 1.20 while dealing with questions of uniqueness. Henceforth we use the word *uniqueness* instead of using adjectives local or global since both of them are equivalent.

(ii) One may wonder then, why there is a need to define both local and global uniqueness properties. The reason is that it is easier to prove local uniqueness compared to proving global uniqueness, and at the same time retaining what we intuitively feel about *uniqueness*. ∎

Example 1.25 (Peano). The initial value problem

$$y' = 3y^{2/3}, \quad y(0) = 0. \tag{1.42}$$

has infinitely many solutions. ∎

However there are sufficient conditions on $\boldsymbol{f}$ so that the corresponding IVP has a unique solution. One such condition is that of Lipschitz continuity w.r.t. variable $\boldsymbol{y}$.

Lemma 1.26. *Assume Hypothesis* ($\mathrm{H_{IVPS}}$). *If f is Lipschitz continuous on every rectangle R inside $\mathbb{I}\times\Omega$, then we have global uniqueness.*

Proof. Let y_1 and y_2 be solutions of IVP (1.1) defined on intervals $\mathbb{I}_1$ and $\mathbb{I}_2$ respectively. By Lemma 1.4, we have

$$y_i(x) = y_0 + \int_{x_0}^{x} f(s, y_i(s))\,ds, \quad \forall x \in \mathbb{I}_i \quad \text{and} \quad i = 1,2. \tag{1.43}$$

Subtracting one equation from another we get

$$y_1(x) - y_2(x) = \int_{x_0}^{x} \big(f(s, y_1(s)) - f(s, y_2(s))\big)\,ds, \quad \forall x \in \mathbb{I}_1 \cap \mathbb{I}_2. \tag{1.44}$$

Applying norm on both sides yields, for $x \in \mathbb{I}_1 \cap \mathbb{I}_2$

$$\|y_1(x) - y_2(x)\| = \left\| \int_{x_0}^{x} \big(f(s, y_1(s)) - f(s, y_2(s))\big)\,ds \right\| \tag{1.45}$$

$$\leq \int_{x_0}^{x} \big\|f(s, y_1(s)) - f(s, y_2(s))\big\|\,ds \tag{1.46}$$

Choose δ such that for each $s \in [x_0 - \delta, x_0 + \delta]$, we have $\|y_1(s) - y_0\| \leq b$ and $\|y_2(s) - y_0\| \leq b$, since we know that f is locally Lipschitz, it will be Lipschitz on the rectangle R with Lipschitz constant $L > 0$. As a consequence we get

$$\|y_1(x) - y_2(x)\| \leq L \sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\|\,|x - x_0| \tag{1.47}$$

$$\leq L\delta \sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\| \tag{1.48}$$

It is possible to arrange δ such that $L\delta < 1$. From here, if $\sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\| \neq 0$, we conclude that

$$\sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\| < \sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\| \tag{1.49}$$

Thus we conclude $\sup_{|x-x_0|\leq\delta} \|y_1(x) - y_2(x)\| = 0$. This establishes local uniqueness and hence global uniqueness follows from Lemma 1.23. ∎

1.3 ▪ Continuous dependence

In situations where a physical process is described (modelled) by an initial value problem for a system of ODEs, it is desirable that any errors made in the measurement of either initial data or the vector field, do not influence the solution very much. In mathematical terms, this is known as continuous dependence of solution of an IVP, on the data present in the problem. An honest effort to formulate this idea in mathematical terms would make us realize the difficulty and subtleties involved in it. In fact, many introductory books on ODEs do not highlight the subtle points (which will be highlighted here), and rather give a weak version of it. See, however, the books of Wolfgang [34], Piccinini *et. al.* [23].

1.3.1 ▪ Sandwich theorem for IVPs / Comparison theorem

Theorem 1.27. *Let $\mathbb{I}, \Omega$ be subintervals of $\mathbb{R}$. Let $f : \mathbb{I} \times \Omega \to \mathbb{R}$ and $g : \mathbb{I} \times \Omega \to \mathbb{R}$ be continuous functions such that for each $(x, y) \in \mathbb{I} \times \Omega$, the inequalify $f(x, y) < g(x, y)$ holds. Let $\zeta \in \Omega$ and $x_0 \in \mathbb{I}$. Consider the initial value problems*

$$y' = f(x, y), \quad y(x_0) = \zeta; \tag{1.50a}$$

$$z' = g(x, z), \quad z(x_0) = \zeta \tag{1.50b}$$

Let $(a, b) \subseteq \mathbb{I}$ be an interval containing x_0, and $\varphi : (a, b) \to \mathbb{R}$ and $\psi : (a, b) \to \mathbb{R}$ be solutions of the initial value problems (1.50a) and (1.50b) respectively. Then the following inequalities hold:

$$\psi(x) < \varphi(x) \ \text{for } x \in (a, x_0), \tag{1.51a}$$

$$\varphi(x) < \psi(x) \ \text{for } x \in (x_0, b). \tag{1.51b}$$

Proof. We will prove (1.51b), and (1.51a) can be proved by similar arguments.

Consider the following set

$$S = \{\alpha \in (a, b) : \forall x \in (x_0, \alpha),\ \varphi(x) < \psi(x) \text{ holds.}\}.$$

Step 1: The set S is non-empty Note that

$$\varphi'(x_0) = f(x_0, \varphi(x_0)) = f(x_0, \zeta) < g(x_0, \zeta) = g(x_0, \psi(x_0)) = \psi'(x_0).$$

Since φ', ψ' are continuous functions on (a, b), there exists a $\delta > 0$ such that $\varphi'(x) < \psi'(x)$ holds for $x \in [x_0, x_0 + \delta)$. Thus for $x \in [x_0, x_0 + \delta)$, using mean value theorem on the interval $[x_0, x]$, we get for some $\xi \in (x_0, x)$

$$(\varphi - \psi)(x) - (\varphi - \psi)(x_0) = (\varphi - \psi)'(\xi)(x - x_0) < 0.$$

Thus the set S is non-empty.

Step 2: The set S is either bounded above or is unbounded. In case the set S is unbounded, the assertion (1.51b) is obvious. Thus we may assume that S is bounded above. By l.u.b. property of real numbers, there exists a real number γ such that $\gamma = \sup S$. Once again, if $\gamma = b$, there is nothing to prove. Hence we may assume that $\gamma < b$. By continuity of the functions φ and ψ, we have $\varphi(\gamma) \le \psi(\gamma)$. If $\varphi(\gamma) < \psi(\gamma)$ holds, then continuity of φ and ψ assures that $\varphi(x) < \psi(x)$ for $x \in (\gamma, \gamma + \delta_1)$ for some $\delta_1 > 0$, which contradicts the definition of γ. Thus $\varphi(\gamma) = \psi(\gamma)$. From the definition of S, we have for small enough $h > 0$

$$\frac{\varphi(\gamma) - \varphi(\gamma - h)}{h} > \frac{\psi(\gamma) - \psi(\gamma - h)}{h},$$

and passing to the limit as $h \to 0$ yields $\varphi'(\gamma) \ge \psi'(\gamma)$. On the other hand, by the arguments presented in Step 1, we get $\varphi'(\gamma) < \psi'(\gamma)$. This contradiction means that $\gamma < b$ is not possible, and thus $\gamma = b$. This completes the proof of (1.51b).

□

Remark 1.28 (On sandwich theorem for IVPs). When f depends only on y, we have an easier physical interpretation of the sandwich theorem. Imagine that two persons Hari and Ravi start running at the same time instant, and from the same location. If at every location Hari's speed is higher than that of Ravi, then at every time instant Hari is ahead of Ravi. Note that at some time instants, Ravi's speed might be higher. ∎

1.3.2 ▪ Theorem on Continuous dependence

We now state a result on continuous dependence, following Wolfgang [34]. In fact, the following result asserts that solution to an IVP has not only continuous dependence on initial data but also on the vector field $\boldsymbol{f}$.

Theorem 1.29 (Continuous dependence). *Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $\mathbb{I} \subseteq \mathbb{R}$ be an interval containing the point x_0. Let $\mathbb{J}$ be a closed and bounded subinterval of $\mathbb{I}$, such that $x_0 \in \mathbb{J}$. Let $\boldsymbol{f} : \mathbb{I} \times \Omega \to \mathbb{R}^n$ be a continuous function. Let $\boldsymbol{y}(x; x_0, \boldsymbol{y}_0)$ be a solution on $\mathbb{J}$ of the initial value problem*

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}), \quad \boldsymbol{y}(x_0) = \boldsymbol{y}_0. \tag{1.52}$$

Let S_α denote the α-neighbourhood of graph of $\boldsymbol{y}$, i.e.,

$$S_\alpha := \left\{(x, \boldsymbol{y}) \, : \, ||\boldsymbol{y} - \boldsymbol{y}(x; x_0, \boldsymbol{y}_0)|| \le \alpha, \, x \in \mathbb{J}\right\}. \tag{1.53}$$

Suppose that there exists an $\alpha > 0$ such that $\boldsymbol{f}$ satisfies Lipschitz condition w.r.t. variable $\boldsymbol{y}$ on S_α. Then the solution $\boldsymbol{y}(x; x_0, \boldsymbol{y}_0)$ depends continuously on the initial values and on the vector field $\boldsymbol{f}$.

That is: Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $\boldsymbol{g}$ is continuous on S_α and the inequalities

$$||\boldsymbol{g}(x, \boldsymbol{y}) - \boldsymbol{f}(x, \boldsymbol{y})|| \le \delta \text{ on } S_\alpha, \quad ||\zeta - \boldsymbol{y}_0|| \le \delta \tag{1.54}$$

are satisfied, then every solution $\boldsymbol{z}(x; x_0, \zeta)$ of the IVP

$$\boldsymbol{z}' = \boldsymbol{g}(x, \boldsymbol{z}), \quad \boldsymbol{z}(x_0) = \zeta. \tag{1.55}$$

exists on all of $\mathbb{J}$, and satisfies the inequality

$$||\boldsymbol{z}(x; x_0, \zeta) - \boldsymbol{y}(x; x_0, \boldsymbol{y}_0)|| \le \epsilon, \quad x \in \mathbb{J}. \quad \blacksquare \tag{1.56}$$

Remark 1.30. (i) In words, the above theorem says : Any solution corresponding to an IVP where the vector field $\boldsymbol{g}$ near a Lipschitz continuous vector field $\boldsymbol{f}$ and initial data (x_0, ζ) near-by $(x_0, \boldsymbol{y}_0)$, stays near the unique solution of IVP with vector field $\boldsymbol{f}$ and initial data $(x_0, \boldsymbol{y}_0)$.

(ii) Note that, under the hypothesis of the theorem, any IVP with a vector field $\boldsymbol{g}$ which is only continuous, also has a solution defined on $\mathbb{J}$.

(iii) Note that the above theorem does not answer the third question we posed at the beginning of this chapter. The above theorem does not say anything about the function in (1.3). ■

Gronwall's lemma

Lemma 1.31. *Let h be a real-valued continuous function on an interval $[a, b]$, be such that*

$$h(x) \le \alpha + \beta \int_a^x h(s)\, ds, \quad \text{on } [a, b] \text{ with } \beta > 0. \tag{1.57}$$

Then

$$h(x) \le \alpha\, e^{\beta(x-a)}, \quad \text{on } [a, b]. \tag{1.58}$$

Proof. Denote the RHS of inequality (1.57) by $\psi(x)$. Note that $\psi'(x) = \beta h(x)$. Since $h(x) \leq \psi(x)$, we get $\psi'(x) \leq \beta\psi(x)$. That is, $\left(e^{-\beta x}\psi(x)\right)' \leq 0$.

Hence $e^{-\beta x}\psi(x)$ is decreasing, and therefore

$$e^{-\beta x}\psi(x) \leq e^{-\beta a}\psi(a) = \alpha e^{-\beta a}. \tag{1.59}$$

From the last inequality, we get

$$h(x) \leq \psi(x) \leq \alpha e^{\beta(x-a)}. \tag{1.60}$$

□

Well-posed problems

A mathematical problem is said to be well-posed (or, properly posed) if it has the **EUC property**.

1. **E**xistence: The problem should have at least one solution.
2. **U**niqueness: The problem has at most one solution.
3. **C**ontinuous dependence: The solution depends continuously on the data that are present in the problem.

Theorem 1.32. *Initial value problem for an ODE $y' = f(x, y)$, where f is Lipschitz continuous on a rectangle containining the initial data (x, y_0), is well-posed.* ■

1.4 ▪ Continuation

We answer the following two questions in this section.

1. When can a given solution be continued?

2. If a solution can not be continued to a bigger interval, what are the obstructions?

The existential results of Section 1.1 provide us with an interval, containing the point at which initial condition is prescribed (x_0), on which solution for IVP exists. The length of the interval depends on the data of the problem as can be seen from its expression in the statement of Theorem 1.10. However this does not rule out solutions of IVP defined on bigger intervals. In this section we address the issue of extending a given solution to a bigger interval and the difficulties that arise in extending.

Intuitive idea for extension Take a local solution $\boldsymbol{u}$ defined on an interval $\mathbb{I}_0 = (x_1\, x_2)$ containing the point x_0. An intuitive idea to extend $\boldsymbol{u}$ is to take the value of $\boldsymbol{u}$ at the point $x = x_2$ and consider a new IVP for the same ODE by posing the initial condition prescribed at the point $x = x_2$ to be equal to $\boldsymbol{u}(x_2)$. Consider this IVP on a rectangle containing the point $(x_2, \boldsymbol{u}(x_2))$. Now apply any of the existence theorems of Section 1.1 and conclude the existence of solution on a bigger interval. Repeat the previous steps and obtain a solution on the entire interval $\mathbb{I}$. Reader is instructed to make an analogy with a relay running race.

However note that this intuitive idea may fail for any one of the following reasons. They are (i). limit of function $\boldsymbol{u}$ does not exist as $x \to x_2$ (ii). limit in (i) may exist but there may not be any rectangle around the point $(x_2, \boldsymbol{u}(x_2))$ as required by existence theorems.

In fact these are the principal difficulties in extending a solution to a bigger interval. In any case we show that for any solution there is a biggest interval beyond which it can not be extended as a solution of IVP and is called *maximal interval of existence* via Zorn's lemma.

We start with a few definitions.

Definition 1.33 (Continuation, Saturated solutions).
Let the function $\boldsymbol{u}$ defined on an interval $\mathbb{I}_0$ be a solution of IVP (1.1). Then

1. *The solution $\boldsymbol{u}$ is called continuable at the right if there exists a solution of IVP $\boldsymbol{w}$ defined on interval $\mathbb{J}_0$ satisfying $\sup \mathbb{I}_0 < \sup \mathbb{J}_0$ and the equality $\boldsymbol{u}(x) = \boldsymbol{w}(x)$ holds for $x \in \mathbb{I}_0 \cap \mathbb{J}_0$. Any such $\boldsymbol{w}$ is called an extension of $\boldsymbol{u}$. The solution $\boldsymbol{u}$ is called saturated at the right if it is not continuable at the right.*

2. *The solution $\boldsymbol{u}$ is called continuable at the left if there exists a solution of IVP $\boldsymbol{z}$ defined on interval $\mathbb{K}_0$ satisfying $\inf \mathbb{K}_0 < \inf \mathbb{I}_0$ and the equality $\boldsymbol{u}(x) = \boldsymbol{z}(x)$ holds for $x \in \mathbb{I}_0 \cap \mathbb{K}_0$. The solution $\boldsymbol{u}$ is called saturated at the left if it is not continuable at the left.*

3. *The solution $\boldsymbol{u}$ is called global at the right if $\sup \mathbb{I}_0 = \sup \mathbb{I}$. Similarly, $\boldsymbol{u}$ is called global at the left if $\inf \mathbb{I}_0 = \inf \mathbb{I}$.*

Remark 1.34. Let $\boldsymbol{w}$ be a solution of IVP (1.1) defined on $\mathbb{J}_0$, which is a right extension of a solution $\boldsymbol{u}$. By concatenating the solutions $\boldsymbol{u}$ and $\boldsymbol{w}$ we obtain a solution of IVP on a bigger interval $(\inf \mathbb{I}_0, \sup \mathbb{J}_0)$. A similar statement holds if $\boldsymbol{u}$ is continuable at the left. ∎

Let us define the notion of a right (left) solution to the IVP (1.1).

Definition 1.35. *A function $\boldsymbol{u} \in C^1[x_0, b)$ defined on an interval $[x_0, b)$ (respectively, on $(a, x_0]$) is said to be a right solution (respectively, a left solution) if $\boldsymbol{u}$ is a solution of ODE $\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y})$ on (x_0, b) (respectively, on (a, x_0)) and $\boldsymbol{u}(x_0) = \boldsymbol{y}_0$.*

Note that if $\boldsymbol{u}$ is a solution of IVP (1.1) defined on an interval (a, b), then $\boldsymbol{u}$ restricted to $[x_0, b)$ (respectively, to $(a, x_0]$) is a right solution (respectively, a left solution) to IVP.

For right and left solutions, the notions of continuation and saturated solutions become

Definition 1.36 (Continuation, Saturated solutions).
Let the function $\boldsymbol{u}$ defined on an interval $[x_0, b)$ be a right solution of IVP and let $\boldsymbol{v}$ defined on an interval $(a, x_0]$ be a left solution of IVP (1.1). Then

1. *The solution $\boldsymbol{u}$ is called continuable at the right if there exists a right solution of IVP $\boldsymbol{w}$ defined on interval $[x_0, d)$ satisfying $b < d$ and the equality $\boldsymbol{u}(x) = \boldsymbol{w}(x)$ holds for $x \in [x_0, b)$. Any such $\boldsymbol{w}$ is called a right extension of $\boldsymbol{u}$. The right solution $\boldsymbol{u}$ is called saturated at the right if it is not continuable at the right.*

2. *The solution $\boldsymbol{v}$ is called continuable at the left if there exists a solution of IVP $\boldsymbol{z}$ defined on interval $(c, x_0]$ satisfying $c < a$ and the equality $\boldsymbol{u}(x) = \boldsymbol{z}(x)$ holds for $x \in (a, x_0]$. The solution $\boldsymbol{u}$ is called saturated at the left if it is not continuable at the left.*

3. *The solution $\boldsymbol{u}$ is called global at the right if $b = \sup \mathbb{I}$. Similarly, $\boldsymbol{v}$ is called global at the left if $a = \inf \mathbb{I}$.*

Remark 1.37 (Important note). The rest of the discussion in this section is devoted to analysing "continuability at the right", "saturated at the right" as the analysis for the corresponding notions "at the left" is similar. We drop suffixing "at the right" from now on to save space to notions of continuability and saturation of a solution. ∎

1.4.1 ▪ Characterisation of continuable solutions

Lemma 1.38. *Assume Hypothesis* ($\mathrm{H}_{\mathrm{IVPS}}$). *Let* $\boldsymbol{u} : [x_0, d) \to \mathbb{R}^n$ *be a right solution of IVP (1.1). Then the following statements are equivalent.*

1. *The solution* $\boldsymbol{u}$ *is continuable.*
2. (i) $d < \sup \mathbb{I}$ *and there exists*
   (ii) $\boldsymbol{y}^* = \lim_{x \to d_-} \boldsymbol{y}(x)$ *and* $\boldsymbol{y}^* \in \Omega$.
3. *The graph of* $\boldsymbol{u}$ *i.e.,*
$$\operatorname{graph} \boldsymbol{u} = \{(x, \boldsymbol{u}(x)) : x \in [x_0, d)\} \tag{1.61}$$
   is contained in a compact subset of $\mathbb{I} \times \Omega$.

Proof. We prove $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 2 \Longrightarrow 1$. The implication $1 \Longrightarrow 2$ is obvious.

Proof of $2 \Longrightarrow 3$**:**
In view of 2, we can extend the function $\boldsymbol{u}$ to the interval $[x_0, d]$ and let us call this extended function $\tilde{\boldsymbol{u}}$. Note that the function $x \longmapsto (x, \tilde{\boldsymbol{u}}(x)$ is continuous on the interval $[x_0, d]$ and the image of $[x_0, d]$ under this map is graph of $\tilde{\boldsymbol{u}}$, denoted by graph $\tilde{\boldsymbol{u}}$, is compact. But graph $\boldsymbol{u} \subset$ graph $\tilde{\boldsymbol{u}} \subset \mathbb{I} \times \Omega$. Thus 3 is proved.

Proof of $3 \Longrightarrow 2$
Assume that graph $\boldsymbol{u}$ is contained in a compact subset of $\mathbb{I} \times \Omega$. As a consequence, owing to continuity of the function $\boldsymbol{f}$ on $\mathbb{I} \times \Omega$, there exists $M > 0$ such that $||\boldsymbol{f}(x, \boldsymbol{u}(x))|| < M$ for all $x \in [x_0, d)$. Also, since $\mathbb{I}$ is an open interval, necessarily $d < \sup \mathbb{I}$. We will now prove that the limit in (2)(ii) exists.
Since $\boldsymbol{u}$ is solution of IVP (1.1), by Lemma 1.4, we have

$$\boldsymbol{u}(x) = \boldsymbol{y}_0 + \int_{x_0}^{x} \boldsymbol{f}(s, \boldsymbol{u}(s))\, ds \quad \forall\, x \in [x_0, d). \tag{1.62}$$

Thus for $\xi, \eta \in [x_0, d)$, we get

$$||\boldsymbol{u}(\xi) - \boldsymbol{u}(\eta)|| \le \int_{\xi}^{\eta} ||\boldsymbol{f}(s, \boldsymbol{u}(s))||\, ds \le M|\xi - \eta|. \tag{1.63}$$

Thus $\boldsymbol{u}$ satisfies the hypothesis of Cauchy test on the existence of finite limit at d. Indeed, the inequality (1.63) says that $\boldsymbol{u} : [x_0, d) \to \mathbb{R}^n$ is uniformly continuous and hence limit of $\boldsymbol{u}(x)$ exists as $x \to d$. This follows from a property of uniformly continuous functions, namely they map Cauchy sequences to Cauchy sequences. Let us denote the limit by y^*. In principle being a limit, $y^* \in \overline{\Omega}$. To complete the proof we need to show that $y^* \in \Omega$. This is a consequence of the hypothesis that graph of $\boldsymbol{u}$ is contained in a compact subset K of $\mathbb{I} \times \Omega$, which implies that y^* belongs to the compact set. Since projection of the compact set K, so is its project to the second coordinate. Thus y^* belongs to a compact subset of Ω. Since Ω is an open set, $y^* \in \Omega$.

Proof of $2 \Longrightarrow 1$

As we shall see, the implication $2 \Longrightarrow 1$ is a consequence of existence theorem for IVP (Theorem 1.10) and concatenation lemma (Lemma 1.6).

Let $\boldsymbol{w}$ be a solution to IVP corresponding to the initial data $(d, y^*) \in \mathbb{I} \times \Omega$ defined on an interval $(e, f) \subseteq \mathbb{I}$ containing the point d, which exists by applying Peano's theorem as there exists a rectangle with center at (d, y^*) inside the domain $\mathbb{I} \times \Omega$. Let $\boldsymbol{w}|_{[d,f)}$ be the restriction of $\boldsymbol{w}$ to the interval $[d, f)$. Let $\tilde{\boldsymbol{u}}$ be defined as the continuous extension of $\boldsymbol{u}$ to the interval $[x_0, d])$ which makes sense due to the existence of the limit in (2)(ii). Concatenating $\tilde{\boldsymbol{u}}$ and $\boldsymbol{w}|_{[d,f)}$ yields a solution of the original IVP (1.1) that is defined on the interval $[x_0, f)$ and $d < f$. □

Remark 1.39. The important message of the above result is that a solution can be extended to a bigger interval provided the solution curve remains "well within" the domain $\mathbb{I} \times \Omega$ and its right end-point lies in Ω. ■

1.4.2 ▪ Existence and Classification of saturated solutions

The following result is concerning the existence of saturted solutions for an IVP. Once again we study "saturated at the right" and corresponding results for "saturated at the left" can be obtained by similar arguments. Thus for this discussion we always consider a soluton as defined on interval of the form $[x_0, d)$

Theorem 1.40 (Existence of saturated solutions). *If $\boldsymbol{u}$ defined on an interval $[x_0, d)$ is a right solution of IVP (1.1), then either $\boldsymbol{u}$ is saturated, or $\boldsymbol{u}$ can be continued up to a saturated one.*

Proof. If $\boldsymbol{u}$ is saturated, then there is nothing to prove. Therefore, we assume that $\boldsymbol{u}$ is not saturated. By definition of saturatedness of a solution, $\boldsymbol{u}$ is continuable. Thus the set $\mathscr{S}$, defined below, is non-empty.

$$\mathscr{S} = \text{Set of all solutions of IVP (1.1) which extend } \boldsymbol{u}. \tag{1.64}$$

We define a relation $\preceq$ on the set $\mathscr{S}$ as follows. For $\boldsymbol{w}, \boldsymbol{z} \in \mathscr{S}$ defined on intervals $[x_0, d_{\boldsymbol{w}})$ and $[x_0, d_{\boldsymbol{z}})$ respectively, we say that $\boldsymbol{w} \preceq \boldsymbol{z}$ if $\boldsymbol{z}$ is a continuation of $\boldsymbol{w}$. This defines a *partial order*.

Roughly speaking, if we take the largest (w.r.t. order $\preceq$) element of $\mathscr{S}$ then by it can not be further continued. To implement this idea, we need to apply *Zorn's lemma*. Zorn's lemma is equivalent to axiom of choice (see the book on Topology by JL Kelley for more) and helps in asserting existence of "maximal elements" provided the totally ordered subsets of $\mathscr{S}$ have an upper bound (upper bound for a subset $\mathscr{T} \subseteq \mathscr{S}$ is an element $\boldsymbol{h} \in \mathscr{S}$ such that $\boldsymbol{w} \preceq \boldsymbol{h}$ for all $\boldsymbol{w} \in \mathscr{S}$). Showing that any totally ordered subset of $\mathscr{S}$ has an upper bound in $\mathscr{S}$ is left as an exercise to the reader.

By Zorn's lemma, there exists a maximal element $\boldsymbol{q}$ in $\mathscr{S}$. Note that this maximal solution is saturated in view of the definition of $\preceq$ and maximality $\boldsymbol{q}$.

□

Remark 1.41. Under the hypothesis of previous theorem, if a solution $\boldsymbol{u}$ of IVP (1.1) is continuable, then there may be more than one saturated solution extending $\boldsymbol{u}$. This possibility is due to non-uniqueness of solutions to IVP (1.1). Exercise 1.11 is concerned with this phenomenon. Further note that if solution $\boldsymbol{u}$ to IVP (1.1) is unique, then there will be a unique saturated solution extending it. ∎

Classification of saturated solutions

Theorem 1.42 (Classification of saturated solutions). *Let $\boldsymbol{u}$ be a saturated right solution of IVP (1.1), and its domain of definition be the interval $[x_0, d)$. Then one of the following alternatives holds.*

1. *The function $\boldsymbol{u}$ is unbounded on the interval $[x_0, d)$.*
2. *The function $\boldsymbol{u}$ is bounded on the interval $[x_0, d)$, and $\boldsymbol{u}$ is global i.e., $d = \sup \mathbb{I}$.*
3. *The function $\boldsymbol{u}$ is bounded on the interval $[x_0, d)$, and $\boldsymbol{u}$ is not global i.e., $d < \sup \mathbb{I}$ and each limit point of $\boldsymbol{u}$ as $x \to d\_$ lies on the boundary of Ω.*

Proof.

If (1) is not true, then definitely (2) or (3) will hold. Therefore we assume that both (1) and (2) do not hold. Thus we assume that $\boldsymbol{u}$ is bounded on the interval $[x_0, d)$ and $d < \sup \mathbb{I}$. We need to show that each limit point of $\boldsymbol{u}$ as $x \to d\_$ lies on the boundary of Ω.

Our proof is by method of contradiction. We assume that there exists a limit point $\boldsymbol{u}^*$ of $\boldsymbol{u}$ as $x \to d\_$ in Ω. We are going to prove that $\lim\limits_{x \to d\_} \boldsymbol{u}(x)$ exists. Note that, once the limit exists it must be equal to $\boldsymbol{u}^*$ which is one of its limit points. Now applying Lemma 1.38, we infer that the solution $\boldsymbol{u}$ is continuable and thus contradicting the hypothesis that $\boldsymbol{u}$ is a saturated solution.

Thus it remains to prove that $\lim\limits_{x \to d\_} \boldsymbol{u}(x) = \boldsymbol{u}^*$ *i.e.*, $||\boldsymbol{u}(x) - \boldsymbol{u}^*||$ can be made arbitrarily small for x near $x = d$.

Since Ω is an open set and $\boldsymbol{u}^* \in \Omega$, there exists $r > 0$ such that $B[\boldsymbol{u}^*, r] \subset \Omega$. As a consequence, $B[\boldsymbol{u}^*, \epsilon] \subset \Omega$ for every $\epsilon < r$. Thus on the rectangle $R \subset \mathbb{I} \times \Omega$ defined by

$$R = [x_0, d] \times \{\boldsymbol{y} : ||\boldsymbol{y} - \boldsymbol{u}^*|| \leq r\}, \tag{1.65}$$

$||\boldsymbol{f}(x, \boldsymbol{y})|| \leq M$ for some $M > 0$ since R is a compact set and $\boldsymbol{f}$ is continuous.

Since $\boldsymbol{u}^*$ is a limit point of $\boldsymbol{u}$ as $x \to d\_$, there exists a sequence (x_m) in $[x_0, d)$ such that $x_m \to d$ and $\boldsymbol{u}(x_m) \to \boldsymbol{u}^*$. As a consequence of definition of limit, we can find a $k \in \mathbb{N}$ such that

$$|x_k - d| < \min\left\{\frac{\epsilon}{2M}, \frac{\epsilon}{2}\right\} \quad \text{and} \quad ||\boldsymbol{u}(x_k) - \boldsymbol{u}^*|| < \min\left\{\frac{\epsilon}{2M}, \frac{\epsilon}{2}\right\}. \tag{1.66}$$

Claim: $\{(x, \boldsymbol{u}(x)) : x \in [x_k, d)\} \subset \mathbb{I} \times B[\boldsymbol{u}^*, \epsilon]$.

Proof of claim: If the inclusion in the claim were false, then there would exist a point on the graph of $\boldsymbol{u}$ (on the interval $[x_k, d)$) lying outside the set $\mathbb{I} \times B[\boldsymbol{u}^*, \epsilon]$. Owing to the continuity of $\boldsymbol{u}$, the graph must meet the boundary of $B[\boldsymbol{u}^*, \epsilon]$. Let $x^* > x_k$ be the first instance at which the trajectory touches the boundary of $B[\boldsymbol{u}^*, \epsilon]$. That is, $\epsilon = ||\boldsymbol{u}(x^*) - \boldsymbol{u}^*||$ and $||\boldsymbol{u}(x) - \boldsymbol{u}^*|| < \epsilon$ for $x_k \leq x < x^*$. Thus

$$\epsilon = ||\boldsymbol{u}(x^*) - \boldsymbol{u}^*|| \leq ||\boldsymbol{u}(x^*) - \boldsymbol{u}(x_k)|| + ||\boldsymbol{u}(x_k) - \boldsymbol{u}^*|| < \int_{x_k}^{x^*} ||\boldsymbol{f}(s, \boldsymbol{u}(s))||\, ds + \frac{\epsilon}{2} \tag{1.67}$$

$$< M(x^* - x_k) + \frac{\epsilon}{2} < M(d - x_k) + \frac{\epsilon}{2} < \epsilon \tag{1.68}$$

This contradiction finishes the proof of Claim.

Therefore, limit of $\boldsymbol{u}(x)$ as $x \to d$ exists. As noted at the beginning of this proof, it follows that $\boldsymbol{u}$ is continuable. This finishes the proof of the theorem. $\square$

1.5 ▪ Global Existence theorem

In this section we give some sufficient conditions under which every local solution of an IVP is global. One of them is the growth of $\boldsymbol{f}$ wrt $\boldsymbol{y}$. If the growth is at most linear, then we have a global solution.

Theorem 1.43. *Let $\boldsymbol{f} : \mathbb{I} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume that there exist two continuous functions h, $k : \mathbb{I} \to \mathbb{R}_+$ (non-negative real-valued) such that*

$$||\boldsymbol{f}(x, \boldsymbol{y})|| \leq k(x)||\boldsymbol{y}|| + h(x), \quad \forall (x, \boldsymbol{y}) \in \mathbb{I} \times \mathbb{R}^n. \tag{1.69}$$

Then for every initial data $(x_0, \boldsymbol{y}_0) \in \mathbb{I} \times \mathbb{R}^n$, IVP has at least one global solution.

Idea of the proof is to show that a local solution exists and it can be continued to whole of $\mathbb{I}$ or $\mathbb{R}$ whichever may the case.

Proof. Let $\boldsymbol{y} : [x_0, d) \to \mathbb{R}^n$ be a saturated right solution of the IVP. We are going to show that $\boldsymbol{y}$ is a global solution at right, by showing that $d = \sup \mathbb{I}$ (If $\mathbb{I}$ is not bounded above, then $\sup \mathbb{I}$ is taken to be ∞). If $d < \sup \mathbb{I}$, then ....Note that $\boldsymbol{y}$ satisfies the equivalent integral equation

$$\boldsymbol{y}(x) = \boldsymbol{y}_0 + \int_{x_0}^{x} \boldsymbol{f}(s, \boldsymbol{y}(s))\, ds. \tag{1.70}$$

Since h, k are continuous functions on $\mathbb{I}$, these functions will be bounded on the compact subinterval $[x_0, d]$. Let $M > 0$ be such that for each $x \in [x_0, d]$, the inequalities $h(x) \leq M$ and $k(x) \leq M$ hold. On taking norm on both sides of the equation (1.70), and using the assumption (1.69) we get for each $x \in [x_0, d)$

$$\begin{aligned} ||\boldsymbol{y}(x)|| &\leq ||\boldsymbol{y}_0|| + \int_{x_0}^{x} ||\boldsymbol{f}(s, \boldsymbol{y}(s))||\, ds \\ &\leq ||\boldsymbol{y}_0|| + \int_{x_0}^{x} \{k(s)||\boldsymbol{y}(s)|| + h(s)\}\, ds \\ &\leq ||\boldsymbol{y}_0|| + M \int_{x_0}^{x} ||\boldsymbol{y}(s)||\, ds + M(d - x_0) \end{aligned}$$

By Gronwall's lemma, we conclude

$$||\boldsymbol{y}(x)|| \leq (M(d - x_0) + ||\boldsymbol{y}_0||)\, e^{M(d-x_0)} \tag{1.71}$$

This implies that $||\boldsymbol{y}(x)||$ is bounded on the interval $[x_0, d)$, and hence $\boldsymbol{y}(x)$ has a limit point, denoted by $\boldsymbol{y}^*$, as $x \to d\_$. Since $\boldsymbol{y}$ is saturated at right, by Theorem 1.42, the limit point $\boldsymbol{y}^*$ must belong to boundary of $\mathbb{R}^n$ which is an empty set. Thus we conclude that

$b = \sup \mathbb{I}$. By a similar argument it follows that y is also a global solution at left. As a consequence y is a global solution.

□

Corollary 1.44. *Globally Lipschitz right hand side implies existence of global solution.*

Exercises

Existence of solutions

1.1. Let $B[y_0, r]$ denote the closed ball of radius $r > 0$ centred at y_0. Prove that the function $\rho : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\rho(y) = \begin{cases} y & \text{if} \quad y \in B[y_0, r]. \\ y_0 + r\, \dfrac{y - y_0}{||y - y_0||} & \text{if} \quad y \in \mathbb{R}^n \setminus B[y_0, r]. \end{cases}$$

is continuous. Draw the graph of ρ for $n = 1$. Show that the range of ρ equals $B[y_0, r]$.

1.2.
1. Let $n = 1$ and f be differentiable w.r.t. the variable y with a continuous derivative defined on $\mathbb{I} \times \Omega$. Show that f is Lipschitz continuous on any rectangle $R \subset \mathbb{I} \times \Omega$.
2. If f is Lipschitz continuous on every rectangle $R \subset \mathbb{I} \times \Omega$, is f differentiable w.r.t. the variable y?
3. Prove that the function h defined by $h(y) = y^{2/3}$ on $[0, \infty)$ is not Lipschitz continuous on any interval containing 0. Picard's iterates for IVP with $y(0) = 0$ converge, but IVP has infinitely many solutions.
4. Prove that the function $f(x, y) = y^2$ defined on doamin $\mathbb{R} \times \mathbb{R}$ is not Lipschitz continuous. (this gives yet another reason to define Lipschitz continuity on rectangles)

1.3. Prove that Picard's iterates need not converge if the vector field does not satisfy Lipschitz condition as in the existence theorem. Compute successive approximations for the IVP

$$y' = 2x - x\sqrt{y_+}, \quad y(0) = 0, \quad \text{with} \quad y_+ = \max\{y, 0\}, \tag{1.72}$$

and show that they do not converge. (Hint: $y_{2n} = 0$, $y_{2n+1} = x^2$, $n \in \mathbb{N}$).

1.4. [5] Solve the IVP

$$y' + 2y = f(x), \quad y(0) = 0, \quad \text{where}$$
$$f(x) = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1, \\ 0 & \text{if} \quad x > 1. \end{cases}$$

by finding a solution valid for $0 \le x \le 1$, and for $x > 1$, and then patching both of them. Note that such a solution will not be differentiable at $x = 1$. Explain why it does not contradict concatenation lemma Lemma 1.6. Such solutions are called "continuous solutions" by some, which causes confusion to a beginner. Perhaps calling it by a different name is better to avoid confusing the reader, for example we may call it a 'weak solution'.

1.5. Derive the estimate (1.34).

1.6. Using Ascoli-Arzela theorem, prove Cauchy-Lipascitz-Picard theorem. (Modify the given proof of C-L-P theorem).

Uniqueness

1.7. Show that the IVP

$$y' = \begin{cases} y \sin \frac{1}{y} & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0, \end{cases} \qquad y(0) = 0.$$

has unique solution, even though the RHS is not a Lipschitz continuous function w.r.t. variable y on any rectangle containing $(0,0)$.

Continuous dependence

1.8. For a fixed f, prove the continuous dependence theorem w.r.t. initial conditions. That is prove

$$||\boldsymbol{y}(x) - \boldsymbol{z}(x)|| \leq \left(||\boldsymbol{y}_0 - \boldsymbol{y}_1||\right) e^{K(x - x_0)}, \quad \forall\, x \in \mathbb{J}. \tag{1.73}$$

1.9. Using comparison theorem for initial value problems, show that if the solution to IVP $y' = x^2 + y^2$, $y(0) = 1$ blows up at some $\alpha \in (0,1)$, then $\frac{\pi}{4} \leq \alpha \leq 1$.

Continuation

1.10. Show that the relation $\preceq$ defines a partial order on the set $\mathscr{S}$. Prove that each totally ordered subset of $\mathscr{S}$ has an upper bound.

1.11. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x, y) = 3y^{2/3}$. Show that the solution $y : [-1, 0] \to \mathbb{R}$ defined by $y(x) = 0$ for all $x \in [-1, 0]$ of IVP satisfying the initial condition $y(-1) = 0$ has at least two saturated solutions extending it.

1.12. 1. Prove the following sharp result concerning saturated solutions.

(Classification of saturated solutions) Let $\boldsymbol{f} : \mathbb{I} \times \Omega \to \mathbb{R}^n$ be continuous on $\mathbb{I} \times \Omega$ and assume that it maps bounded subsets in $\mathbb{I} \times \Omega$ into bounded subsets in $\mathbb{R}^n$. Let $\boldsymbol{u} : [x_0, d) \to \mathbb{R}^n$ be a saturated right solution of IVP. Then one of the following alternatives holds.

(1)′ The function $\boldsymbol{u}$ is unbounded on the interval $[x_0, d)$. If $d < \infty$ there exists $\lim_{x \to d\_} ||\boldsymbol{u}(x)|| = \infty$.

(2) The function $\boldsymbol{u}$ is bounded on the interval $[x_0, d)$, and $\boldsymbol{u}$ is global *i.e.,* $d = \sup \mathbb{I}$.

(3)′ The function $\boldsymbol{u}$ is bounded on the interval $[x_0, d)$, and $\boldsymbol{u}$ is not global *i.e.,* $d < \sup \mathbb{I}$ and limit of $\boldsymbol{u}$ as $x \to d\_$ exists and lies on the boundary of Ω.

2. Let $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Let $\boldsymbol{u} : [x_0, d) \to \mathbb{R}^n$ be a saturated right solution of IVP. Use the previous exercise and conclude that one of the following two alternatives holds.

   1. The function $\boldsymbol{u}$ is global *i.e.,* $d = \infty$.
   2. The function $\boldsymbol{u}$ is not global *i.e.,* $d < \infty$ and $\lim_{x \to d\_} ||\boldsymbol{u}(x)|| = \infty$. This phenomenon is often referred to as $\boldsymbol{u}$ **blows up in finite time**.

1.13. **Basha theorem** If an IVP possesses two solutions, then it has infinitely many solutions.

Global existence

1.14. Using Global existence theorem, prove that any IVP corresponding to a first order linear system has a unique global solution.

1.15. Show that IVP $y' = (y-1)(y-3)$, $y(0) = 2$ has a global solution, without explicitly determining its solution.

from Ahmad-Ambrosetti

1.16. [1] Explain why $y' + \frac{\sin x}{1+e^x} y = 0$ cannot have a solution satisfying the conditions $y(1) = 1$ and $y(2) = -1$?

1.17. [1] Find $\zeta \geq 0$ such that the initial value problem $y' = |y|^{1/2}$, $y(0) = \zeta$ has a unique solution.

1.18. [1] Show that saturated solutions of $y' = \tan^{-1} y$ are defined for all $x \in \mathbb{R}$.

1.19. [1] Show that the initial value problem $y' = \max\{1, y\}$, $y(0) = 1$ has a unique solution which is defined for all $x \in \mathbb{R}$.

1.20. [1] Show that for every $\zeta \in (-1, 1)$, the solution φ of the initial value problem $y' = y^3 - y$, $y(0) = \zeta$ is defined for all $x \in \mathbb{R}$, and $\lim_{x\to\infty} \varphi(x) = 0$.

General

1.21. Discuss existence, uniqueness, maximal interval of existence of solutions for each of the following initial value problems.

(i)

$$y' = y^2, \quad y(1) = 1.$$

Good RHS, but only a local solution.

(ii)

$$y' = \begin{cases} 1 & \text{if} \quad y > 0 \\ 0 & \text{if} \quad y = 0 \\ -1 & \text{if} \quad y < 0, \end{cases} \qquad y(0) = 0.$$

Bad RHS with a global solution.

(iii)

$$y' = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0, \end{cases} \qquad y(0) = 0.$$

Bad RHS with a solution only on $[0, \infty)$.

(iv)

$$y' = y^{2/3}, \quad y(0) = 0.$$

Non-Lipschitz RHS, has infinitely many solutions.

(v)

$$y' = \begin{cases} y \sin \frac{1}{y} & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0, \end{cases} \qquad y(0) = 0.$$

has a unique solution, even though the RHS is not a Lipschitz continuous function w.r.t. variable y on any rectangle containing $(0, 0)$.

1.22. Show that IVP $y' = (y-1)(y-3)$, $y(0) = 2$ has a global solution, without explicitly determining its solution.

1.23. Using comparison theorem for initial value problems, show that if the solution to IVP $y' = x^2 + y^2$, $y(0) = 1$ blows up at some $\alpha \in (0,1)$, then $\frac{\pi}{4} \leq \alpha \leq 1$.

Bibliography

[1] S. Ahmad and A. Amborsetti, *A textbook on ordinary differential equations*, 2nd ed., (Springer, 2015). (Cited on p. 28)

[2] V.I. Arnold, *Ordinary differential equations*, translated by Silverman, (Printice-Hall of India, 1998). (Not cited)

[3] V.I. Arnold, *Geometrical methods in the theory of ordinary differential equations*, (Springer, 1988). (Not cited)

[4] G. Birkhoff and G.-C. Rota, *Ordinary differential equations*, (John wiley, 1978). (Not cited)

[5] W.E. Boyce and R.C. Diprima, *Elementary differential equations and boundary value problems*, (Johy wiley, 2003). (Cited on p. 26)

[6] F. Brauer and J.A. Nohel, *The qualitative theory of Ordinary differential equations: an introduction*, (Dover, 1989). (Not cited)

[7] M. Braun, *Differential equations and their applications : an introduction to applied mathematics*, (Springer-Verlag, 1993). (Not cited)

[8] L. Cesari, *Asymptotic behavior and stability problems in ordinary differential equations*, 2nd ed., (Academic Press, 1963). (Not cited)

[9] E.A. Coddington, *Introduction to ordinary differential equations*, (Prentice-Hall of India, 1974). (Not cited)

[10] E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, (McGraw-Hill, 1955). (Not cited)

[11] J. Cronin, *Differential equations: Introduction and qualitative theory*, (Marcel Dekker, 1980). (Not cited)

[12] S.G. Deo and V. Raghavendra, *Ordinary differential equations and stability theory*, (Tata McGraw-Hill, 1980). (Not cited)

[13] M.S.P. Eastham, *Theory of ordinary differential equations*, (Van Nostrand Reinhold, 1970). (Not cited)

[14] N.P. Erugin , *Linear systems of ordinary differential equations with periodic and quasi-periodic coefficients translation edited by Richard Bellman*, (Academic Press, 1966). (Not cited)

[15] M. Golomb and M. Shanks, *Elements of ordinary differential equations*, (McGraw-Hill, 1965). (Not cited)

[16] P. Hartman, *Ordinary differential equations*, (Wiley, 1964). (Not cited)

[17] M.W. Hirsh, S. Smale and R.L. Devaney, *Differential equations, dynamical systems & Chaos*, (Academic press, 2004). (Not cited)

[18] W. Hurewicz, *Lectures on ordinary differential equations*, (Cambridge : MIT Press, 1958). (Not cited)

[19] E.L. Ince, *Ordinary differential equations*, (Dover, 1956). (Not cited)

[20] D.W. Jordan and P. Smith, *Nonlinear ordinary differential equations : an introduction to dynamical systems*, (Oxford, 1987). (Not cited)

[21] T. Myint-U, *Ordinary differential equations*, (North-Holland, 1978). (Not cited)

[22] L. Perko, *Differential equations and dynamical systems* (Springer-Verlag, 1991). (Not cited)

[23] L.C. Piccinini, G. Stampacchia and G. Vidossich, *Ordinary differential equations in* $\mathbb{R}^N$*: problems and methods*, (Springer-Verlag, 1984). (Cited on p. 16)

[24] Y. Pinchover and J. Rubinstein, *An introduction to partial differential equations*, (Cambridge, 2005). (Not cited)

[25] L.S. Pontriagin , *Ordinary differential equations*, (Addison-Wesley, 1962). (Not cited)

[26] A.L. Rabenstein, *Introduction to ordinary differential equations*, (Academic Press, 1966). (Not cited)

[27] E.D. Rainville and P.E. Bedient, *A short course in differential equations*, (IBH, 1973; Macmillan, 1969). (Not cited)

[28] M.R.M. Rao, *Ordinary differential equations: theory and applications*, (Affiliated East-West, 1980). (Not cited)

[29] W.T. Reid, *Sturmian theory for ordinary differential equations*, (Springer-Verlag, 1980). (Not cited)

[30] S.L. Ross, *Introduction to ordinary differential equations*, (Waltham : Blaisdell Pub., 1966). (Not cited)

[31] D.A. Sanchez, *Ordinary differential equations and stability theory*, (Dover, 1979). (Not cited)

[32] D.A. Sanchez, *Ordinary differential equations: A brief eclectic tour*, (Mathematical association of America, 2002). (Not cited)

[33] I.I. Vrabie, *Differential equations*, (World scientific, 2004). (Not cited)

[34] W.Wolfgang, *Ordinary differential equations*, (Springer-Verlag, 1998). (Cited on pp. 16, 18)