Lectures on
Two Scale Convergence and Homogenization

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1.1 Motivation

Consider the following problem on an open and bounded domain $\Omega \subset \mathbb{R}^N$:

$$-\text{div}\left(A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f \text{ in } \Omega \tag{1.1}$$

$$u_\epsilon = 0 \text{ on } \partial\Omega \tag{1.2}$$

where the matrix $A(x, y)$ is defined on $\Omega \times Y$ such that A is smooth and Y-periodic in y. We assume that there exist constants $0 < \alpha \leq \beta$ such that for all $\xi \in \mathbb{R}^N$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{N} A_{ij}(x, y)\xi_i\xi_j \leq \beta|\xi|^2.$$

By Lax-Milgram lemma, we know that there exists a unique solution $u_\epsilon \in H_0^1(\Omega)$ of (1.1)-(1.2). We are interested in the behaviour of u_ϵ as $\epsilon \to 0$ and this is called the homogenization problem associated to (1.1)-(1.2).
In this context, there are at least three motivating reasons to consider the concept of **Two Scale Convergence**. They are described below.

(i) As a first step in analyzing the problem (1.1)-(1.2), we introduced a **formal two-scale asymptotic expansion** for the solution u_ϵ of (1.1)-(1.2) of the form

$$u_\epsilon(x) = \sum_{i=0}^{\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right)$$

The above expansion was proposed, expecting that the solution u_ϵ will exhibit a two scale variation, namely x and $y = \dfrac{x}{\epsilon}$. This is due to the presence of two scales in the problem. We then substituted this expansion in the equation (1.1) and derived the homogenized equation. The formal two-scale asymptotic expansion method is not rigorous. The homogenization process is then justified using Tartar's method of **oscillating test functions**. This method involves construction of test functions which are also oscillating. Two Scale convergence method incorporates the lessons learnt from the two-scale asymptotic expansion and gives rise to another rigurous justification of the homogenization process.

(ii) In the studies of Homogenization, the primary difficulty is to the passage to limit in integrals of the following type:

$$\int_\Omega f^\epsilon(x)g(x)h\left(\frac{x}{\epsilon}\right)\, dx,$$

where f^ϵ converges only weakly, $g \in C_c^\infty(\Omega)$ or $g \in L^2(\Omega)$, and h is a Y- periodic function. Thus the integrand is a product of two weakly converging sequences. We get integrals of this type in the weak formulation of the system (1.1)-(1.2).

(iii) Another point of view is that weak limit of a given oscillating sequence does not adequately describe the oscillations, see Example 1.8. Thus we need to generalize the concept of weak convergence so that we gain extra knowledge and capture the oscillations present in the sequence through its limit, see Example 1.11.

1.2 Some Important Function Spaces

Notations

- $Y = [0, 1]^N$ denotes the unit cube in $\mathbb{R}^N$ which is the fundamental period of the periodic structures under study.

- $C^\infty(\Omega)$, $C_c^\infty(\Omega)$ (also denoted by $\mathcal{D}(\Omega)$), $L^p(\Omega)$, $H^1(\Omega)$ are the standard spaces.
- The symbol $\#$, generally represents the periodicity in Y. For example,
  $\star$ The space $C_\#^\infty(Y) = \{f \in C^\infty(\mathbb{R}^N) : f$ is $Y-$ periodic $\}$.
  $\star$ $L^2_\#(Y)$ is the completion of $C_\#^\infty(Y)$ with respect to $L^2(Y)$ norm.
  $\star$ $H^1_\#(Y)$ is the completion of $C_\#^\infty(Y)$ with respect to $H^1(Y)$ norm.
- Let X be a function space. Then $\mathcal{D}(\Omega; X)$ denotes the space of $\mathcal{D}(\Omega)$ functions with values in X.
- Let $1 \le p < \infty$ and X be a normed linear space. Then $L^p(\Omega; X)$ denotes the space of measurable functions $f : \Omega \longrightarrow X$ such that $\int_\Omega \|f(x)\|^p_{\|X\|} < \infty$.

Some function spaces and their properties

Theorem 1.1 *[5] A function f belongs to $L^2(\Omega; C_\#(Y))$ if and only if there exists a subset E of measure zero in Ω such that*
(a) *For any $x \in \Omega \setminus E$, the function $y \longmapsto f(x,y)$ is continuous and Y-periodic.*
(b) *For any $y \in Y$, the function $x \longmapsto f(x,y)$ is measurable.*
(c) *The function $x \longmapsto \sup_{y\in Y} |f(x,y|$ has finite $L^2(\Omega)$ norm.*

Remark 1.2 (i) *Let f be a function defined on $\Omega \times Y$ such that f is continuous in either of the variables and measurable in the remaining variable. Let f be a Y-periodic function for every fixed $x \in \Omega$. Then it is well-known that f is a* **Caratheodory function**. *As a consequence the function $x \longmapsto f\left(x, \frac{x}{\epsilon}\right)$ defined on Ω is measurable [1, 4].*

(ii) *Let $1 \le p < \infty$ and $f \in L^p(\Omega \times Y)$. We use the notation $f(x,y)$ where $x \in \Omega$ and $y \in Y$. Then the space $C_c^\infty(\Omega) \bigotimes C_\#^\infty(Y)$ consisting of all finite sums of functions of the form $g(x)h(y)$ is dense in $L^p(\Omega \times Y)$. We do not expect such a result for $L^\infty(\Omega \times Y)$. This is analogous to the fact that $C[0,1]$ is closed w.r.t. the uniform convergence and hence cannot be a dense subspace of $L^\infty[0,1]$. [3]*

Theorem 1.3 *[5] Let $f \in L^2(\Omega; C_\#(Y))$. Then the function $x \longmapsto f\left(x, \frac{x}{\epsilon}\right)$ defined on Ω is measurable and*

$$\left\| f\left(x, \frac{x}{\epsilon}\right) \right\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega; C_\#(Y))} := \left(\int_\Omega \|f(x,.)\|^2_{C_\#(Y)}\, dx \right)^{\frac{1}{2}}, \tag{1.3}$$

$$\lim_{\epsilon \to 0} \int_\Omega \left| f\left(x, \frac{x}{\epsilon}\right) \right|^2 dx = \int_\Omega \int_Y |f(x,y)|^2\, dy\, dx. \tag{1.4}$$

Theorem 1.4 *[5] Let $B_p(\Omega, Y)$, $1 \le p < \infty$ denote any of the spaces $L^p(\Omega; C_\#(Y))$, $L^p_{\text{loc}}(Y; C(\overline{\Omega})$, $C(\overline{\Omega}; C_\#(Y))$. Then $B_p(\Omega, Y)$ has the following properties:*
(a) *$B_p(\Omega, Y)$ is a separable Banach space.*
(b) *$B_p(\Omega, Y)$ is dense in $L^p(\Omega \times Y)$.*
(c) *For every $f \in B_p(\Omega, Y)$ The function $x \longmapsto f\left(x, \frac{x}{\epsilon}\right)$ defined on Ω is measurable and*

$$\left\| f\left(x, \frac{x}{\epsilon}\right) \right\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega; C_\#(Y))} := \left(\int_\Omega \|f(x,.)\|^p_{C_\#(Y)}\, dx \right)^{\frac{1}{2}}. \tag{1.5}$$

(d) *For every $f \in B_p(\Omega, Y)$, we have*

$$\lim_{\epsilon \to 0} \int_\Omega \left| f\left(x, \frac{x}{\epsilon}\right) \right|^p dx = \int_\Omega \int_Y |f(x,y)|^p\, dy\, dx. \tag{1.6}$$

1.3 Weak Convergence

Let us recall the definition of weak convergence in L^p spaces.

Definition 1.5 *A sequence $\{u^\epsilon\}$ in $L^p(\Omega)$ is said to converge weakly to $u \in L^p(\Omega)$ if*

$$\lim_{\epsilon\to 0}\int_\Omega u^\epsilon(x)v(x)\,dx = .\int_\Omega u(x)v(x)\,dx,\ \forall v\in L^q(\Omega)$$

where q is such that $\frac{1}{p}+\frac{1}{q}=1$ if $1<p<\infty$. If $p=1$, then $q=\infty$. If $p=\infty$, we take $q=1$ and the convergence is called weak- convergence in $L^\infty(\Omega)$.*

The functions $v \in L^q(\Omega)$ in the above definition are called **test functions for the weak convergence**.

Exercise 1.6 *Let $\{u^\epsilon\}$ be a sequence in $L^p(\Omega)$ such that $u^\epsilon \rightharpoonup u$ in $L^p(\Omega)$-weak and $u^\epsilon \rightharpoonup v$ in $L^p(\Omega)$-weak. Then $u = v$.*

Exercise 1.7 *Let $\{u^\epsilon\}$ be a bounded sequence in $L^p(\Omega)$. Then $u^\epsilon \rightharpoonup u$ in $L^p(\Omega)$-weak if and only if*

$$\lim_{\epsilon\to 0}\int_\Omega u^\epsilon(x)\phi(x)\,dx = .\int_\Omega u(x)\phi(x)\,dx,\ \forall\phi\in C_0^\infty(\Omega).\blacksquare$$

Exercise 1.8 *Let $g:\mathbb{R}^N\to\mathbb{R}$ be a Y-periodic function. Let $\omega\subseteq\mathbb{R}^N$ be an open set and $g\in L^p_{\text{loc}}(\mathbb{R}^N)$. Define $g^\epsilon(x)=g\left(\frac{x}{\epsilon}\right)$. Then $g^\epsilon \rightharpoonup \frac{1}{|Y|}\int_Y g(y)\,dy$ in $L^p(\omega)$-weak for $1\le p<\infty$ and in $L^\infty(\omega)$ weak-$\star$ for $p=\infty$.*

It follows from Exercise 1.8 that weak limit of g^ϵ depends only on the average of the periodic function g. In particular the weak limits do not see the shape of the oscillations present in the sequence g^ϵ. This fact is attributed to and is interpreted as the inability of the class of test functions used in the definition of weak convergence. The class of test functions should be modified in order to pick-up the oscillations via some other type of weak limits. Roughly speaking, the test functions that we use to define the two scale convergence help in describing the oscillations via the 2-scale limits.

1.4 Two Scale Convergence

Definition 1.9 *A sequence of functions $\{u_\epsilon\}$ in $L^2(\Omega)$ is said to* **two-scale converge** *to $u_0\in L^2(\Omega\times Y)$ if for every $\psi\in L^2(\Omega;C_\#(Y))$ we have*

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\psi\left(x,\frac{x}{\epsilon}\right)\,dx = \int_\Omega\int_Y u_0(x,y)\psi(x,y)\,dy\,dx. \tag{1.7}$$

Notation: *We shall write $u_\epsilon \overset{2-s}{\rightharpoonup} u_0$.*

Exercise 1.10 *Let $\{u_\epsilon\}$ be a sequence in $L^2(\Omega)$ and $u,v\in L^2(\Omega\times Y)$ be such that $u_\epsilon\overset{2-s}{\rightharpoonup}u$ and $u_\epsilon\overset{2-s}{\rightharpoonup}v$. Then $u=v$. (Hint: use the density of $L^2(\Omega;C_\#(Y))$ in $L^2(\Omega\times Y)$).*

Thanks to Theorem 1.3, we have a large class of Two-scale convergent sequences.

Example 1.11 *Let $f\in L^2(\Omega;C_\#(Y))$. Then the sequence $\{u_\epsilon\}$ defined by $u_\epsilon(x):=f\left(x,\frac{x}{\epsilon}\right)$ two-scale converges to f. In particular the sequence $\sin\left(\frac{x}{\epsilon}\right)\overset{2-s}{\rightharpoonup}\sin(y)$ and thus retaining the information on the shape of oscillations present in the sequence. Note that the sequence oscillates at the same frequencies as the test functions considered in the two-scale convergence.*

Example 1.12 *Let $f \in L^2(\Omega; C_\#(Y))$ and $g \in L^2(\Omega; C_\#(Y))$. Then the sequence u_ϵ defined by $u_\epsilon(x) := f\left(x, \frac{x}{\epsilon}\right) + \epsilon\, g\left(x, \frac{x}{\epsilon}\right)$ two-scale converges to f.*

Roughly speaking, this example says that any two-scale asymptotic expansion

$$u_\epsilon(x) = \sum_{i=0}^{\infty} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right)$$

two-scale converges to $u_0(x,y)$, under smoothness assumptions on u_is.

Example 1.13 *Two-scale convergence does not see the oscillations which are not in resonance with those of the test functions. If $u(x,y)$ is a smooth function Y-periodic in Y, and if $v_\epsilon(x) = u\left(x, \frac{x}{\epsilon^2}\right)$, then*

$$v_\epsilon \overset{2-s}{\rightharpoonup} \int_Y u(x,y)dy. \quad \blacksquare$$

Remark 1.14 *The Example 1.13 shows that the test functions $\psi\left(x, \frac{x}{\epsilon}\right)$ will only capture the oscillations which are in resonance with $\psi\left(x, \frac{x}{\epsilon}\right)$. To capture higher order oscillations, one requires to work with test functions having such oscillations. This leads to the study of multi-scale convergence [2].* ■

1.5 Compactness Theorem

Theorem 1.15 *If $\{u_\epsilon\}$ is a bounded sequence in $L^2(\Omega)$, then there exists a subsequence of $\{u_\epsilon\}$ (subsequence is still denoted by ϵ) and an $u_0 \in L^2(\Omega \times Y)$ such that $\{u_\epsilon\}$ two-scale converges to u_0.*

Proof: The proof is not very difficult. The main idea is to use **Banach-Alaouglu theorem** which says that every bounded sequence in $X^\star$ has a weak-$\star$ convergent subsequence whenever X is a separable Banach space. The X which is a natural choice for us is the space $L^2(\Omega; C_\#(Y))$ which is a separable Banach space and $\{u_\epsilon\}$ defines a sequence in its dual space naturally.

Step 1: Define a sequence of bounded linear functionals $\{l_\epsilon\}$ on $L^2(\Omega; C_\#(Y))$ using the given sequence $\{u_\epsilon\}$ as follows:

$$l_\epsilon(\psi) = \int_\Omega u_\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)\, dx$$

Indeed, since $||u_\epsilon||_{L^2(\Omega)} \leq C$ and for any $\psi \in L^2(\Omega; C_\#(Y))$ that $\psi(x, x/\epsilon) \in L^2(\Omega)$, thanks to Hölder inequality we have

$$(1.8) \quad |l_\epsilon(\psi)| \leq \left| \int_\Omega u_\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)\, dx \right| \leq C||\psi(x, x/\epsilon)||_{L^2(\Omega)} \leq C||\psi(x,y)||_{L^2(\Omega; C_\#(Y))}.$$

The last two inequalities follows from Theorem 1.3. Thanks to the inequalities (1.8), the sequence $\{l_\epsilon\}$ is bounded in $\left(L^2(\Omega; C_\#(Y))\right)^\star$.

Step 2: Apply Banach-Alaouglu theorem to the sequence $\{l_\epsilon\}$ and conclude the existence of a subsequence of $\{l_\epsilon\}$ and an $l \in \left(L^2(\Omega; C_\#(Y))\right)^\star$ such that $l_\epsilon \rightharpoonup l$ in the weak-$\star$ topology in $l \in \left(L^2(\Omega; C_\#(Y))\right)^\star$. We can apply Banach-Alaouglu theorem since the space $L^2(\Omega; C_\#(Y))$ is separable. Thus we have (for a subsequence)

$$l_\epsilon(\psi) \to l(\psi) \quad \forall\, \psi \in L^2(\Omega; C_\#(Y))$$

Step 3: Passing to the limit, as $\epsilon \to 0$ in (1.8), we see that

$$|l(\psi)| \leq C||\psi(x,y)||_{L^2(\Omega\times Y)} \quad \forall\, \psi \in L^2(\Omega; C_\#(Y)). \tag{1.9}$$

This follows easily from (1.4).

Step 4: Since $L^2(\Omega; C_\#(Y))$ is dense in $L^2(\Omega\times Y)$, we can extend l as a bounded linear functional on $L^2(\Omega\times Y)$ and denote the extension by $\tilde{l}$. We then have

$$|l(\psi)| \leq C||\psi(x,y)||_{L^2(\Omega\times Y)} \quad \forall\, \psi \in L^2(\Omega\times Y). \tag{1.10}$$

Since $L^2(\Omega\times Y)$ is a Hilbert space, by Riesz representation theorem, the bounded linear functional l can be identified with an element $u_0 \in L^2(\Omega\times Y)$. We then have

$$\lim_{\epsilon\to 0}\int_\Omega u_\epsilon(x)\psi\left(x,\frac{x}{\epsilon}\right)\,dx = \lim_{\epsilon\to 0} l_\epsilon(\psi) = l(\psi) = \int_\Omega\int_Y u_0(x,y)\psi(x,y)dy$$

for every $\psi \in L^2(\Omega; C_\#(Y))$. This completes the proof. ■

1.6 Relations between Strong, 2-scale and weak convergences

For a sequence $\{u_\epsilon\}$ in the space $L^2(\Omega)$, we have three notions of convergence: Strong, Two-scale and weak. In this section we will explore the relation between them. We have the following picture

Strong Convergence $\Longrightarrow$ **Two-scale Convergence** $\Longrightarrow$ **Weak convergence**
Weak Convergence $\not\Rightarrow$ **Two-scale Convergence** $\not\Rightarrow$ **Strong convergence**

Theorem 1.16 (Strong and Two-scale limits) *If $\{u_\epsilon\}$ converges strongly to u_0 in $L^2(\Omega)$, then $u_\epsilon(x) \overset{2-s}{\rightharpoonup} u_0(x)$.*

Proof: We already know that if a sequence converges strongly, then it converges weakly and the limit is equal to the strong limit. The present result is a two-scale version of the same.

Let $\psi \in L^2(\Omega; C_\#(Y))$. Then

$$\begin{aligned}
&\left|\int_\Omega u_\epsilon(x)\psi\left(x,\frac{x}{\epsilon}\right)\,dx - \int_\Omega\int_Y u_0(x)\psi(x,y)\,dy\,dx\right| \\
&\qquad\qquad \leq \quad \|u_\epsilon - u_0\|_{L^2(\Omega)}\left\|\psi\left(x,\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} + \\
&\qquad \left|\int_\Omega u_0(x)\psi\left(x,\frac{x}{\epsilon}\right)\,dx - \int_\Omega\int_Y u_0(x)\psi(x,y)\,dy\,dx\right| \\
&\leq \quad C\|u_\epsilon - u_0\|_{L^2(\Omega)} + \left|\int_\Omega u_0(x)\left(\psi\left(x,\frac{x}{\epsilon}\right) - \int_Y \psi(x,y)\,dy\right)\,dx\right|.
\end{aligned}$$

In the last inequality, the first term on RHS goes to zero due to the strong convergence of $\{u_\epsilon\}$ while the second term also goes to zero since $u_0(x)\psi(x,y) \in L^1(\Omega; C_\#(Y))$ and (1.6). ■

Theorem 1.17 (Weak and Two-scale limits) *If $\{u_\epsilon\}$ is a sequence in $L^2(\Omega)$ and $u_\epsilon \overset{2-s}{\rightharpoonup} u_0(x,y)$, then $u_\epsilon \rightharpoonup u$ weakly in $L^2(\Omega)$ where*

$$u(x) = \int_Y u_0(x,y)dy,$$

and $\{u_\epsilon\}$ is bounded.

Proof: The proof of weak convergence follows by taking the test functions $\psi \in L^2(\Omega; C_\#(Y))$ which are constant w.r.t. the variable $y \in Y$. Every weakly convergent sequence is bounded is a well-known fact. ■

Example 1.18 (A two-scale convergent sequence that does not converge strongly) *This is the first example one studies while dealing with oscillating functions. Consider the sequence $\{u_\epsilon\}$ defined by $u_\epsilon = \sin(\frac{x}{\epsilon})$. We saw already that this sequence has a two-scale limit, namely $\sin(y)$ but the sequence does not converge strongly in $L^2(\Omega)$.* ■

Example 1.19 (A weakly convergent sequence that does not two-scale converge) *Consider the sequence $\{u_n\}$ defined by $u_n(x) = \sin(nx)$ if n is odd and $u_n(x) = \cos(nx)$ if n is even. This sequence converges weakly to the zero function in $L^2(\Omega)$. This sequence does not have a two-scale limit (Why?). Note however that every weakly convergent sequence has a subsequence that two-scale converges (to some limit).* ■

Remark 1.20 (An important Remark:) *Usually in the literature, the definition of two scale convergence was given by taking the test function space $\mathcal{D}(\Omega; C^\infty_\#(Y))$ instead of $L^2(\Omega; C_\#(Y))$ as in our case.*

Such a definition will not yield Theorem 1.17.

Example 1.21 *Let $\Omega = (0,1)$ and define*

$$u_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{if} \quad 0 < x < \epsilon \\ 0 & \text{if} \quad \epsilon < x < 1. \end{cases}$$

Then

$$\int_\Omega u_\epsilon(x)\psi(x, x/\epsilon)dx \longrightarrow \int_\Omega \int_Y u_0(x,y)\psi(x,y)dydx = 0,$$

for all $\psi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$.

But $\{u_\epsilon\}$ is neither bounded nor converges to 0 weakly in $L^2(\Omega)$ because by taking $g \equiv 1$, we see that

$$\int_\Omega u_\epsilon g = 1. \quad ■$$

Even, one cannot replace $\mathcal{D}(\Omega; C^\infty_\#(Y))$ by a bigger space $C(\Omega; C^\infty_\#(Y))$.

Example 1.22 *Let $\Omega = (0,1)$ and $\tilde{u}_\epsilon$ be the periodic extension of u_ϵ above and define*

$$u_\epsilon(x) = \begin{cases} \tilde{u}_\epsilon(\frac{x}{\epsilon}) & \text{if} \quad \frac{1}{4} < x < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_\Omega \bar{u}_\epsilon(x)\psi(x, x/\epsilon)dx \longrightarrow \int_\Omega \int_Y u(x,y)\psi(x,y)dydx,$$

for all $\psi \in C(\Omega; C^\infty_\#(Y))$, where

$$u(x,y) = \begin{cases} 1 & \text{if} \quad \frac{1}{4} < x < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Again note that $\{u_\epsilon\}$ is not bounded in $L^2(\Omega)$.

But under the boundedness assumption, we have the following theorem, whose proof, we leave it as an exercise.

Theorem 1.23 *Let $\{u_\epsilon\}$ be a bounded sequence in $L^2(\Omega)$ and the convergence in the definition of two scale convergence be satisfied only for the test functions $\psi \in \mathcal{D}(\Omega; C^\infty_\#(Y))$. Then u_ϵ two-scale converges to u.* ■

It is a well-known result in $L^2(\Omega)$ weak convergence theory that if a sequence $\{u_\epsilon\}$ converges weakly to u, then

$$\|u\|_{L^2(\Omega)} \leq \liminf_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)}.$$

A similar result holds for two-scale convergence which is the content of the following result.

Theorem 1.24 *Let $\{u_\epsilon\}$ be a sequence in $L^2(\Omega)$ and two-scale converges to $u \in L^2(\Omega\times Y)$. Then*

$$\liminf_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)} \geq ||u||_{L^2(\Omega\times Y)} \geq ||v||_{L^2(\Omega)}, \tag{1.11}$$

where $v = \int_Y u(x,y)\,dy$.

Proof: Step 1: Let $\psi_m \in L^2(\Omega; C_\#(Y))$ be a sequence that converges to u in $L^2(\Omega\times Y)$. We will make use of ψ_m which have some nice properties (1.3) and (1.4) which we will exploit in this proof. We have

$$2\int_\Omega u_\epsilon(x)\psi_m\left(x, \frac{x}{\epsilon}\right)\,dx \leq \int_\Omega |u_\epsilon(x)|^2\,dx + \int_\Omega \left|\psi_m\left(x, \frac{x}{\epsilon}\right)\right|^2\,dx \tag{1.12}$$

Let us take lim inf on both sides of the inequality now. In fact, limit exists for the quantity on LHS and also for the second term on the RHS. Thanks to the definition of two-scale convergence and (1.4), we get

$$2\int_\Omega\int_Y u(x,y)\psi_m(x,y)\,dy\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega |u_\epsilon(x)|^2\,dx + \int_\Omega\int_Y |\psi_m(x,y)|^2\,dx\,dy. \tag{1.13}$$

Step 2: Let us now pass to the limit in (1.13) as $m\to\infty$. We need to use that if a sequence converges strongly in a normed linear space to a limit, then corresponding norms also converge to the norm of the limit. This passage to the limit yields

$$2\int_\Omega\int_Y |u(x,y)|^2\,dy\,dx \leq \liminf_{\epsilon\to 0}\int_\Omega |u_\epsilon(x)|^2\,dx + \int_\Omega |u(x,y)|^2\,dx. \tag{1.14}$$

From the last inequality, it follows that $\liminf_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)} \geq ||u||_{L^2(\Omega\times Y)}$.

Step 3: By Jensen's inequality, we have

$$\|v\|^2_{L^2(\Omega)} \leq \int_\Omega \left|\int_Y u(x,y)\,dy\right|^2\,dx \leq \int_\Omega\int_Y |u(x,y)|^2\,dy\,dx. \tag{1.15}$$

This finishes the proof of the theorem. ■

Remark 1.25 (Interpretation) *When we considered purely oscillating sequences, we observed that information about the shape of oscillations is fully present in its two-scale limit. If the weak and two-scale limits are different, we may say from the above theorem that there is more information in the two-scale limit than the weak limit about the oscillating sequence. One also says that two-scale convergence is intermediary between strong and weak convergences in the space $L^2(\Omega)$.* ■

Question: We know that if $u_\epsilon \rightharpoonup u$ in $L^2(\Omega)$-weak and $\|u_\epsilon\|_{L^2(\Omega)} \to \|u\|_{L^2(\Omega)}$, then $u_\epsilon \to u$ in $L^2(\Omega)$. We ask a similar question now for a two-scale convergent sequence. Suppose $u_\epsilon \overset{2-s}{\rightharpoonup} u$ and $\lim_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)} = ||u||_{L^2(\Omega\times Y)}$. Since the sequence $\{u_\epsilon\}$ is a function on Ω and the two-scale limit u is a function on $\Omega\times Y$, we can ask if the sequence strongly converges in the space $L^2(\Omega\times Y)$ after identifying the sequence $\{u_\epsilon\}$ in the space $L^2(\Omega\times Y)$ as a sequence which is constant w.r.t. the variable $y\in Y$. The answer is negative in general.

So, we ask a different question: We know that if we consider a product of two functions where one of them strongly converges and the other weakly converges, then we can pass to the limit in the product in some sense. More precisely,

Lemma 1.26 *If $u^\epsilon \to u$ in $L^2(\Omega)$ and $v^\epsilon \rightharpoonup v$ in $L^2(\Omega)$-weak, then $u^\epsilon v^\epsilon \rightharpoonup uv$ in $\mathcal{D}'(\Omega)$.*

Is there such a result concerning functions that converge in the sense of two-scales? First we need to define what should be the concept of a strong two-scale convergence! ■

Theorem 1.27 *[1, 5] Let $u_\epsilon \overset{2-s}{\rightharpoonup} u_0$. Assume that*

$$\lim_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)} = ||u_0||_{L^2(\Omega\times Y)}.$$

Then if $v_\epsilon \overset{2-s}{\rightharpoonup} v_0$, we have

$$u_\epsilon(x)v_\epsilon(x) \rightharpoonup \int_Y u_0(x,y)v_0(x,y)dy \text{ in } \mathcal{D}'(\Omega).$$

Further, if $u_0 \in L^2(\Omega; C_\#(Y))$, then

$$\lim_{\epsilon\to 0} \left\| u_\epsilon(x) - u_0\left(x, \frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} = 0.$$

■

Remark 1.28

(1) *Recall the inequalities* (1.11). *In the hypothesis of Theorem 1.27, we have a weaker assumption than the assumption that both inequalities are replaced with equalities which amounts to the assumption that the sequence $\{u_\epsilon\}$ converges strongly in $L^2(\Omega)$.*

(2) *Theorem 1.27 helps in obtaining a* **Corrector result.**

Theorem 1.27 results in the following natural definition of a strong two-scale convergence.

Definition 1.29 *A sequence $\{u_\epsilon\}$ in $L^2(\Omega)$ is said to* **strongly two-scale converge** *to $u_0 \in L^2(\Omega\times Y)$ if*

$$u_\epsilon \overset{2-s}{\rightharpoonup} u_0 \quad \textit{and} \quad \lim_{\epsilon\to 0} ||u_\epsilon||_{L^2(\Omega)} = ||u_0||_{L^2(\Omega\times Y)}.$$

Notation *We use the notation $u_\epsilon \overset{2-s}{\longrightarrow} u_0$.*

In view of the above definition, the theorem can be rewritten as: if

$$u_\epsilon \overset{2-s}{\longrightarrow} u_0 \quad \text{and} \quad v_\epsilon \overset{2-s}{\rightharpoonup} v_0,$$

then

$$u_\epsilon(x)v_\epsilon(x) \rightharpoonup \int_Y u_0(x,y)v_0(x,y)dy \text{ in } \mathcal{D}'(\Omega).$$

1.7 Bounded sequences in H^1 and their two-scale limits

We derive further results for bounded sequences in $H^1(\Omega)$ keeping homogenization problems in mind. Since the result that we are going to prove holds only for a subsequence, there is no loss of generality in assuming that the sequence also weakly converges.

Theorem 1.30 *Let $\{u_\epsilon\}$ be a sequence in $H^1(\Omega)$ such that $u_\epsilon \rightharpoonup u$ weakly in $H^1(\Omega)$. Then*

(1) *The sequence $\{u_\epsilon\}$ two-scale converges to u.*

(2) *There exists a subsequence of ϵ and $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$. such that*

$$\nabla u_\epsilon \overset{2-s}{\rightharpoonup} \nabla_x u(x) + \nabla_y u_1(x,y). \tag{1.16}$$

Proof:
Step 1: As $\{u_\epsilon\}$ converges weakly in $H^1(\Omega)$, it is a bounded sequence. As a consequence $\{u_\epsilon\}$ and $\{\nabla u_\epsilon\}$ are bounded sequences in $L^2(\Omega)$ and $\left(L^2(\Omega)\right)^N$ respectively. By the compactness theorem for two-scale convergence (Theorem 1.15), there exists a subsequence (still denoted by ϵ), $u_0 \in L^2(\Omega \times Y)$ and $\xi_0 \in L^2(\Omega \times Y)^N$ such that

$$u_\epsilon \overset{2-s}{\rightharpoonup} u_0 \quad \text{and} \quad \nabla u_\epsilon \overset{2-s}{\rightharpoonup} \xi_0.$$

Step 2: By Theorem 1.17, we know that $u(x) = \int_Y u_0(x,y)\,dy$. If we prove that u_0 does not depend on $y \in Y$, we get that the whole sequence two-scale converges to u. This follows from the uniqueness of the weak limit.
We want to do some integrations by parts, so let $\Psi \in [\mathcal{D}(\Omega; C^\infty_\#(Y)]^N$. Then

$$\int_\Omega \nabla u_\epsilon(x) \cdot \Psi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega u_\epsilon(x) \left[\operatorname{div}_x \Psi\left(x, \frac{x}{\epsilon}\right)\right] dx + \frac{1}{\epsilon}\int_\Omega u_\epsilon(x) \operatorname{div}_y \Psi\left(x, \frac{x}{\epsilon}\right) dx. \tag{1.17}$$

Multiply the equation (1.17) by ϵ and then pass to the limit on both sides as $\epsilon \to 0$ to get

$$0 = \int_\Omega \int_Y u_0(x,y) \operatorname{div}_y \Psi(x,y) dy dx.$$

This shows that u_0 does not depend on y and finishes the proof of (1).

Step 3: We turn to the proof of (1.16).
In the equation (1.17), further assume that Ψ also satisfies $\operatorname{div}_y \Psi(x,y) = 0$. Then we have

$$\int_\Omega \nabla u_\epsilon(x) \cdot \Psi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega u_\epsilon(x) \left[\operatorname{div}_x \Psi\left(x, \frac{x}{\epsilon}\right)\right] dx. \tag{1.18}$$

Passing to the limit as $\epsilon \to 0$ on both sides, we get

$$\int_\Omega \int_Y \xi_0(x,y) \Psi(x,y)\,dy\,dx = \int_\Omega \int_Y u(x) \operatorname{div}_x \Psi(x,y)\,dy\,dx.$$

Hence

$$\int_\Omega \int_Y [\xi_0(x,y) - \nabla_x u(x)] \Psi(x,y) dy dx = 0,$$

for all divergence free Ψ.
It is well known that a vector field orthogonal to divergence free vector fields must be a gradient. Thus there exists $u_1(x,y)$ such that

$$\xi_0(x,y) - \nabla u(x) = \nabla_y u_1(x,y).$$

This finishes the proof of (1.16). ∎
The following exercises are very easy and idea for solving them is already there in the last proof.

Exercise 1.31 *Let $\{u_\epsilon\}$ and $\{\epsilon\nabla u_\epsilon\}$ be bounded in $L^2(\Omega)$. Then there exists $u_0 \in L^2(\Omega; H^1_\#(Y))$ such that*

$$\left\{\begin{array}{l} u_\epsilon(x) \overset{2-s}{\rightharpoonup} u_0(x,y), \\ \\ \epsilon\nabla u_\epsilon(x) \overset{2-s}{\rightharpoonup} \nabla_y u_0(x,y). \end{array}\right. \tag{1.19}$$

Exercise 1.32 *Let $\{u_\epsilon\}$ be divergence free and bounded in $(L^2(\Omega))^N$ and let $u_\epsilon \overset{2-s}{\rightharpoonup} u_0(x,y) \in L^2(\Omega \times Y)^N$. Then*

$$\left\{\begin{array}{l} \mathrm{div}_y u_0(x,y) = 0, \\ \\ \displaystyle\int_Y \mathrm{div}_x u_0(x,y)dy = 0. \end{array}\right. \tag{1.20}$$

1.8 Proof of Homogenization theorem using two-scale convergence

We considered the following homogenization problem:

$$-\mathrm{div}\left(A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right) = f \text{ in } \Omega \tag{1.21}$$

$$u_\epsilon = 0 \text{ on } \partial\Omega \tag{1.22}$$

• We have seen in the earlier lectures that $\{u_\epsilon\}$ is bounded in $H^1_0(\Omega)$.
• By the previous lectures, it follows that there exist functions $u \in H^1_0(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ such that along a subsequence

$$u_\epsilon \rightharpoonup u \text{ weakly in } H^1_0(\Omega)$$
$$\nabla u_\epsilon \overset{2-s}{\rightharpoonup} \nabla u(x) + \nabla_y u_1(x,y).$$

Theorem 1.33 *The pair of functions (u, u_1) obtained above is the unique solution of the* **two scale homogenized** *system:*

$$\left\{\begin{array}{l} -\mathrm{div}_x\left[\int_Y (A(x,y)(\nabla u(x) + \nabla_y u_1(x,y))dy\right] = f \text{ in } \Omega \\ \\ -\mathrm{div}_y\left[(A(x,y)(\nabla u(x) + \nabla_y u_1(x,y))dy\right] = 0 \text{ in } \Omega \times Y \\ \\ u = 0 \text{ on } \Omega \\ \\ u_1(x,y) \text{ is } Y-\text{periodic in } y. \end{array}\right. \tag{1.23}$$

Remark 1.34 *By the uniqueness of solutions to the above system, the entire sequence $\{u_\epsilon\}, \{\nabla u_\epsilon\}$ two-scale converges.* ■

Proof of Theorem 1.33:

Step 1: Once again the key idea is to choose suitable test functions involving both variables. Multiplying the PDE (1.21) by the test function $\phi(x) + \epsilon\phi_1(x, x/\epsilon)$, where $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega; C^\infty_\#(Y))$, we get

$$\int_\Omega \nabla u_\epsilon(x) \cdot \Psi(x, x/\epsilon) = \int_\Omega f(x)[\phi(x) + \epsilon\phi_1(x, x/\epsilon)]dx. \tag{1.24}$$

where

$$\Psi(x,x/\epsilon) = A(x,x/\epsilon)\cdot[\nabla\phi(x) + \nabla_y\phi_1(x,x/\epsilon) + \epsilon\nabla_x\phi_1(x,x/\epsilon)]\,dx.$$

• Consider Ψ as a test function (due to smoothness of A) and passing to the limit in (1.24), we get

$$(1.25)\quad \int_\Omega\int_Y A(x,y)[\nabla u(x) + \nabla_y u_1(x,y)] \quad [\nabla\phi(x) + \nabla_y\phi_1(x,y)]dydx = \int_\Omega f(x)\phi(x)dx.$$

The equation (1.25) is the weak formulation of (1.23). We leave it as an exercise the derivation of the two equations in (1.23) from (1.25).

Step 2: Existence and uniqueness of solutions to the system(1.25).

This follows as an application of Lax-Milgram lemma applied to the space $X = H_0^1(\Omega)\times L^2(\Omega; H^1_\#(Y)/\mathbb{R})$.

• Define a norm $\|.\|_X$ on X and a bilinear form $B(.,.)$ on $X\times X$ as follows:

$$\|(\phi,\phi_1)\|_X = ||\nabla\phi||_{L^2(\Omega)} + ||\nabla_y\phi_1||_{L^2(\Omega\times Y)}$$

and

$$B((\phi,\phi_1),(\psi,\psi_1)) = \int_\Omega\int_Y A(x,y)[\nabla\psi(x) + \nabla_y\psi_1(x,y)]\cdot[\nabla\phi(x) + \nabla_y\phi_1(x,y)].$$

• *Verify that the bilinear form B is coercive and conclude the existence and uniqueness.*

Remark 1.35 *One can replace the condition on the smoothness of A by that of* **admissibility (or strong two scale limit)** *of A, i.e. by*

$$\lim_{\epsilon\to 0}\int_\Omega a_{ij}(x,x/\epsilon)^2 dx = \int_\Omega\int_Y a_{ij}(x,y)^2 dydx$$

for all $1\le i,j\le N$, where $A = (a_{ij})$. In this case we appeal to the result we have derived in passing to the limit in product of two two-scale convergent sequences to get (1.25). ∎

Relation between the two-scaled homogenized system and the classical homogenized problem

Remark 1.36 *The two scale homogenized system (1.23) can be easily decoupled to get the familiar homogenized system. This is achieved by using the solutions of the so-called cell problem.*

• *Let $\{e_j\}$ be the unit normal basis of $\mathbb{R}^N$. Treating x as a parameter in the second equation in (1.23) and writing $\nabla u(x) = \sum_{j=1}^N \frac{\partial u}{\partial x_j}(x)e_j$, we may introduce the following cell problem: define $\chi^j(x,y)$ as the solution of*

$$(1.26)\quad \begin{cases} -\mathrm{div}_y(A(x,y)[\nabla_y\chi^j(x,y) - e_j]) = 0 \text{ in } Y \\ \chi^j \text{ is } Y-\text{periodic in } y. \end{cases}$$

• *Then one can immediately see that*

$$u_1(x,y) = -\sum_{i=1}^N \frac{\partial u}{\partial x_i}(x)\chi^i(x,y),$$

where u is the only unknown to be determined.

• *Substituting* u_1 *in the first equation in (1.23), we get the single equation for* u *as*

$$\begin{cases} -\mathrm{div}(A^*(x)\nabla u(x)) = f \ \text{ in } \ \Omega \\ \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $A^* = (q_{ij}(x))$ *with*

$$q_{ij}(x) = \int_Y A(x,y)[\nabla_y \chi^j - e_j] \cdot [\nabla_y \chi^i - e_i] dy.$$

■

Comments

♣ The two scale homogenized system (1.23) is a system of two equations with unknowns u and u_1 in which the variables x and y (macroscopic and microscopic, respectively) are mixed.
♣ Though apparently complicated, it is a well-posed system and is easily shown to have a unique solution via variational method.
♣ We could decouple the equations to recover the original homogenized equations obtained by the formal method or energy method. This was due to the simple nature of the second order elliptic equation we considered.
♣ For many other types of problems this decoupling may not be possible or may produce very complicated equations including integro-differential equations or non-explicit equations.
♣ The usual homogenized equation may not fall into a class with a nice theory of existence and uniqueness of solutions while the two scale form, though it has twice the number of unknowns and variables is, in most cases, of the same type as the original problem.
♣ Thus the presence of the microscopic variables in the 2-scale homogenized problem doubles the size of the problem but simplifies the structure, while eliminating them introduces strange effects, like memory or non-local effects. Both methods have their pros and cons. ■

1.9 Corrector result

We end the lectures on two scale convergence by establishing a **Corrector result**.

Theorem 1.37 *Assume that* $\nabla_y u_1(x,y)$ *is an admissible test function. Then* $[\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1(x, x/\epsilon)]$ *converges strongly to zero in* $(L^2(\Omega))^N$. *Thus, if* $u_1, \nabla u_1$ *and* $\nabla_y u$, *are all admissible, then*

$$u_\epsilon(x) - u(x) - \epsilon u_1(x, x/\epsilon) \to 0$$

strongly in $H^1(\Omega)$.

Proof : If A is smooth, say, $A \in C(\Omega; L^\infty_\#(Y))^{N^2}$, by standard regularity results of χ^j, the function $u_1(x, x/\epsilon)$ is in $L^2(\Omega)$ and can be seen as a test function for the two scale convergence.

Thus under these assumptions,

$$\int_\Omega A(x, x/\epsilon)[\nabla u_\epsilon - \nabla u(x) - \nabla_y u_1(x, x/\epsilon)]^2 dx$$

$$= \int_\Omega f(x) u_\epsilon(x) dx + \int_\Omega A(x, x/\epsilon)[\nabla u + \nabla_y u_1(x, x/\epsilon)]^2 dx \tag{1.27}$$

$$- \int_\Omega (A +^t A)(x, x/\epsilon) \nabla u_\epsilon(x) \cdot [\nabla u(x) + \nabla_y u_1(x, x/\epsilon)] dx.$$

Using the coercivity of A and on passing to the 2-scale limit, we get

$$\begin{aligned} &\alpha \limsup_{\epsilon\to 0} ||\nabla u_\epsilon(x) - \nabla u(x) - \nabla_y u_1(x, x/\epsilon)||^2_{L^2(\Omega)} \\ &\le \int_\Omega f(x)u(x)dx - \int_\Omega \int_Y A(x,y)[\nabla u(x) + \nabla_y u_1(x,y)]^2 dydx \end{aligned} \tag{1.28}$$

The term on right-hand side is zero (why!), thus completing the proof. ∎

1.10 Extensions of two-scale convergence

(i) Two-scale convergence in L^p spaces [5].

(ii) Multi-scale convergence and their application to Reiterated homogenization[2].

(iii) For a two-scale analogue of Young measures, see [8].

(iv) For a discussion on admissible test functions used in the definition of two-scale convergence, see [9]

(v) Alternative definitions for two-scale convergence have been proposed (see [6]) where the test functions used are less regular compared to those in the two-scale convergence of [1, 7].

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