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Hence $0 = 0x$. By closure axioms $0 \in W$. If $x \in W$ then $-x = (-1)x$ is in W by closure axioms. □

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   More generally, the set of solutions of a homogeneous system of linear equations in n variables forms a subspace of $\mathbb{K}^n$. In other words, if $A \in M_{m,n}(\mathbb{K})$, then the set $\{\mathbf{x} \in \mathbb{K}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{K}^n$. It is called the null space of A.

Linear Span of a set in a Vector Space

Definition

Let S be a subset of a vector space V. The linear span of S is the subset
$L(S) = \left\{\sum_{i=1}^{n} c_i x_i : x_1, \ldots, x_n \in S \text{ and } c_1, \ldots, c_n \text{ are scalars}\right\}$.
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Remark

(i) *Different sets may span the same subspace. For example*

$$L(\{\hat{i},\hat{j}\}) = L(\{\hat{i},\hat{j},\hat{i}+\hat{j}\}) = \mathbb{R}^2.$$

More generally, the standard basis vectors $\mathbf{e}_1,\ldots,\mathbf{e}_n$ *span* $\mathbb{R}^n$ *and so does any set* $S \subset \mathbb{R}^n$ *containing* $\mathbf{e}_1,\ldots,\mathbf{e}_n$

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(ii) *The vector space* $\mathbb{R}[t]$ *is spanned by* $\{1, t, t^2, \ldots, t^n, \ldots\}$ *and also by* $\{1, (1+t), \ldots, (1+t)^n, \ldots\}$.

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(iv) *While talking about the linear span or any other vector space notion, the underlying field of scalars is understood. If we change this we get different objects and relations. For instance, the real linear span of* $1 \in \mathbb{C}$ *is* $\mathbb{R}$ *where as the complex linear span is the whole of* $\mathbb{C}$.

Linear Dependence

Definition

Let V be a vector space. A subset S of V is called linearly dependent (L.D.) if there exist distinct elements $\mathbf{v}_1, \ldots, \mathbf{v}_n \in S$ and $\alpha_i \in \mathbb{K}$, not all zero, such that

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(iv) If E_{ij} denotes the $m \times n$ matrix with 1 in $(i, j)^{\text{th}}$ position and 0 elsewhere, then the set $\{E_{ij} : i = 1, \dots, m,\ j = 1, \dots, n\}$ is linearly independent in the vector space $M_{m,n}(\mathbb{K})$.

A useful Lemma

Lemma

Let T be a linearly independent subset of a vector space V. If $\mathbf{v} \in V \setminus L(T)$, then $T \cup \{\mathbf{v}\}$ is linearly independent.

Proof: Suppose there is a linear dependence relation in the elements of $T \cup \{\mathbf{v}\}$ of the form

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Definition

A subset S of a vector space V is called a basis of V if

(i) *$V = L(S)$, and*

(ii) *S is linearly independent.*

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Let S be a finite subset of a vector space V such that $V = L(S)$. Suppose S_1 is a subset of S which is linearly independent. Then there exists a basis S_2 of V such that $S_1 \subset S_2 \subset S$.

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Remark

By the above theorem, it follows that every finite dimensional vector space has a finite basis.

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By the above theorem, it follows that every finite dimensional vector space has a finite basis. To see this, choose a finite set S such that $L(S) = V$ and apply the theorem with $S_1 = \emptyset$.

Theorem

If a vector space V contains a finite subset $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ such that $V = L(S)$, then every subset of V with $n + 1$ (or more) elements is linearly dependent.

Proof:
Let $\mathbf{u}_1, \ldots \mathbf{u}_{n+1}$ be any $n + 1$ elements of V. Since $V = L(S)$, we can write

$$\mathbf{u}_i = \sum_{j=1}^{n} a_{ij}\mathbf{v}_j \quad \text{for some } a_{ij} \in \mathbb{K} \text{ and for } i = 1, \ldots, n+1$$

Consider the $(n + 1) \times n$ matrix $A = (a_{ij})$. Let A' be a REF of A. Then A' can have at most n pivots, and so the last row of A' must be full of zeros. On the other hand, $A' = RA$ for some $(n + 1) \times (n + 1)$ invertible matrix R (which is a product of elementary matrices). Now if $(c_1, \ldots, c_{n+1})$ denotes the last row of R, then not all c_i's are zero since R is invertible. Also

Proof Contd.

since the the last row of $A' = RA$ is $\mathbf{0}$, we see that

$$\sum_{i=1}^{n+1} c_i a_{ij} = 0 \quad \text{for each } j = 1, \ldots, n.$$

Multiplying by v_j and summing over j, we obtain

$$0 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n+1} c_i a_{ij} \right) \mathbf{v}_j = \sum_{i=1}^{n+1} c_i \left(\sum_{j=1}^{n} a_{ij} v_j \right) = \sum_{i=1}^{n+1} c_i \mathbf{u}_i.$$

This proves that $\mathbf{u}_1, \ldots \mathbf{u}_{n+1}$ are linearly dependent. □

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For a finite dimensional vector space, the number of elements in any two bases are the same.

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Corollary

For a finite dimensional vector space, the number of elements in any two bases are the same.

Definition

Given a finite dimensional vector space V, the dimension of V is defined to be the number of elements in any basis for V.

Exercises

(i) Show that in any vector space of dimension n any subset S such that $L(S) = V$ has at least n elements.

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(iv) Let V and W be vector subspaces of another vector space U. Show that $L(V \cup W) = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}$. This subspace is denoted by $V + W$ and is called the sum of V and W.

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$$\dim(V + W) = \dim V + \dim W - \dim(V \cap W).$$

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4. Linear Transformations

Definition

Let V, W be any two vector spaces over $\mathbb{K}$. By a linear map (or a linear transformation) $f : V \to W$ we mean a function f satisfying

$$f(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2) = \alpha_1 f(\mathbf{v}_1) + \alpha_2 f(\mathbf{v}_2)$$

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Thus, it follows that f is completely determined by its value on a basis of V, i.e., if f and g are two linear maps such that $f(v_i) = g(v_i)$ for all $i = 1, \ldots, k$, then $f = g$.

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(iii) *If* V *has a basis* $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *then for each ordered* k *tuple* $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ *of elements of* W *we obtain a unique linear transformation* $f : V \to W$ *by choosing* $f(\mathbf{v}_i) = \mathbf{w}_i$ *for all* i,

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Can you say $\mathcal{I} \circ D = Id$? Is D a one-one map?

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Exercise: On the vector space $\mathcal{P}[x]$ of all polynomials in one-variable, determine all linear maps $\phi : \mathcal{P}[x] \to \mathcal{P}[x]$ having the property $\phi(fg) = f\phi(g) + g\phi(f)$ and $\phi(x) = 1$.

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(c) Put (a) and (b) together. $\square$

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Remark

Because of the above theorem any vector space of dimension n is isomorphic to $\mathbb{K}^n$.

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- *It follows that the study of 'linear transformations on finite dimensional vector spaces' can also be converted into the 'study of matrices'.*

Exercises:

(1) Clearly a bijective linear transformation is invertible. Show that the inverse is also linear.

(2) Let V be a finite dimensional vector space and $f : V \to V$ be a linear map. Prove that the following are equivalent:

(i) f is an isomorphism.

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Proof: If f and g denote the corresponding linear maps then $f \circ g = Id : \mathbb{R}^n \to \mathbb{R}^n$.
From the exercise (2) above, f is an isomorphism and $f \circ g = g \circ f = Id$. Hence $AB = I_n = BA$ which means $A = B^{-1}$.

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(Rank and Nullity Theorem): The rank and nullity of a linear transformation $f : V \to W$ on a finite dimensional vector space V add up to the dimension of V :

$$rk(f) + n(f) = \dim V.$$

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Hence there are scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k = \beta_1 \mathbf{w}_1 + \beta_2 \mathbf{w}_2 + \ldots + \beta_{n-k} \mathbf{w}_{n-k}.$$

By linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-k}\}$ we conclude that $\beta_1 = \beta_2 = \ldots = \beta_{n-k} = 0$. Hence T is L. I. Therefore it is a basis of $\mathcal{R}(f)$. $\square$

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$$trace\,(A) = \sum_{i=1}^{n} a_{ii}$$