Osculating Spaces of Varieties, Forms
Linear Network Codes

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Abstract

- We present a general theory to obtain linear network codes utilizing the osculating nature of algebraic varieties.

- Linear network coding transmits information in terms of a basis of a vector space and the information is received as a basis of a possible altered vector space.

- By way of example we present the osculating spaces of Veronese varieties and obtain families of vector spaces constituting linear network codes.

- The osculating spaces of Veronese varieties are equidistant in a certain metric.

- We present linear network codes derived from any set of normalized homogenous polynomials in $\mathbb{F}_q[X_0, \ldots, X_n]_e/ \sim$, generalising the above result.
Ordinary transmission: Independent data share the same resources and data are merely \textit{forwarded} and stay independent.

Linear network coding: Transmission is obtained by transmitting a number of packets into the network - a packet is a vector of length $N$ over a finite field $\mathbb{F}_q$. The packets travel the network through intermediate nodes, each \textit{forwarding} $\mathbb{F}_q$ - linear combinations of the packets it has available.

Figure: Linear network coding allows for linear combination of information - both receivers obtain complete information.
The problem of linear network code design

How to select which linear combinations to use?

- A simple algorithm is to select the combinations at random. However there is a certain probability that linearly dependent combinations are chosen!

- An alternative is to use deterministic algorithms to perform the design of linearly independent combinations.
Koetter and Kschischang (2008)

- Described a transmission model in terms of linear subspaces of $\mathbb{F}_q^N$ spanned by the packets and they define a code as a nonempty subset $C \subseteq G(n, N)(\mathbb{F}_q)$ of the Grassmannian of $n$-dimensional $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q^N$ and endowed $G(n, N)(\mathbb{F}_q)$ with the metric

$$\text{dist}(V_1, V_2) := \dim_{\mathbb{F}_q}(V_1 + V_2) - \dim_{\mathbb{F}_q}(V_1 \cap V_2).$$

- Showed that a minimal distance decoder for this metric achieves correct decoding if the dimension of the intersection of the transmitted and received vector-space is sufficiently large.

- Obtained Hamming, Gilbert-Varshamov and Singleton coding bounds.
Terracini’s lemma - tangent spaces tend to be in general position

- Algebraic varieties are osculating. By Terracini’s lemma their tangent spaces tend to be in general position. The tangent space at a generic point $P \in Q_1 Q_2$ on the secant variety of points on some secant is spanned by the tangent spaces at $Q_1$ and $Q_2$.

- In general, the secant variety have the expected maximal dimension and therefore the tangent spaces generically span a space of maximal dimension. (F. Zak, R. Lazarsfeld et. al.)

- We suggest osculating spaces for constructing linear subspaces in general position for linear network coding.

- Any network code comes from an algebraic curve taking osculating spaces (E. Ballico, [Bal])
Notation

- $\mathbb{F}_q$ is the finite field with $q$ elements of characteristic $p$.
- $\mathbb{F} = \overline{\mathbb{F}_q}$ is an algebraic closure of $\mathbb{F}_q$.
- $R_d = \mathbb{F}[X_0, \ldots, X_n]_d$ and $R_d(\mathbb{F}_q) = \mathbb{F}_q[X_0, \ldots, X_n]_d$ the homogenous polynomials of degree $d$ with coefficients in $\mathbb{F}$ and $\mathbb{F}_q$.
- $R = \mathbb{F}[X_0, \ldots, X_n] = \bigoplus_d R_d$ and $R(\mathbb{F}_q) = \mathbb{F}_q[X_0, \ldots, X_n] = \bigoplus_d R_d(\mathbb{F}_q)$
- $\text{AffCone}(Y) \subseteq \mathbb{F}^{M+1}$ denotes the affine cone of $Y \subseteq \mathbb{P}^M$ and $\text{AffCone}(Y)(\mathbb{F}_q)$ its $\mathbb{F}_q$-rational points.
- $O_{k,X,P} \subseteq \mathbb{P}^M$ is the embedded $k$-osculating space of a variety $X \subseteq \mathbb{P}^M$ at the point $P \in X$ and $O_{k,X,P}(\mathbb{F}_q)$ its $\mathbb{F}_q$-rational points.
- $\mathcal{V} = \sigma_d(\mathbb{P}^n) \subseteq \mathbb{P}^M$ with $M = \binom{d+n}{n} - 1$ is the Veronese variety.
Codes from osculating spaces

**Definition**

Let $X \subseteq \mathbb{P}^M$ be a smooth projective variety of dimension $n$ defined over $\mathbb{F}_q$.

The elements of the $k$-osculating linear network code $C_{k,X}$ are the linear subspaces in $\mathbb{F}_q^{M+1}$ which are the affine cones of the $k$-osculating subspaces $O_{k,X,P}(\mathbb{F}_q)$ at $\mathbb{F}_q$-rational points $P$ on $X$.

$$C_{k,X} = \{ \text{AffCone}(O_{k,X,P})(\mathbb{F}_q) \mid P \in X(\mathbb{F}_q) \}.$$  

The number of elements in $C_{k,X}$ is the number of $\mathbb{F}_q$-rational points on $X$. The vector spaces in $C_{k,X}$ have dimension at most $\binom{k+n}{n}$. 

Osculating Spaces of Varieties, Forms and Linear Network Codes

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Principal parts

Let $X$ be a smooth variety of dimension $n$ defined over the field $K$ and let $\mathcal{F}$ be a locally free $\mathcal{O}_X$-module. The sheaves of $k$-principal parts $\mathcal{P}^k_X(\mathcal{F})$ are locally free and if $\mathcal{L}$ is of rank 1, then $\mathcal{P}^k_X(\mathcal{L})$ is a locally free sheaf of rank $\binom{k+n}{n}$. There are the fundamental exact sequences

$$0 \to S^k\Omega_X \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{P}^k_X(\mathcal{F}) \to \mathcal{P}^{k-1}_X(\mathcal{F}) \to 0,$$

where $\Omega_X$ is the sheaf of differentials on $X$ and $S^k\Omega_X$ its $k$th symmetric power.
If $\mathcal{L}$ is of rank 1, then $P^k_X(\mathcal{L})$ is a locally free sheaf of rank $\binom{k+n}{n}$.

If $X$ is affine with coordinate ring $A = K[x_1, \ldots, x_n]$, then

- $X$ and $\mathcal{L}$ can be identified with $A$.
- $S^k \Omega_X$ can be identified with the forms of degree $k$ in $A[dx_1, \ldots, dx_n]$ in the indeterminates $dx_1, \ldots, dx_n$.
- $P^k_X(\mathcal{L})$ can be identified with the polynomials of total degree $\leq k$ in the indeterminates $dx_1, \ldots, dx_n$.

For arbitrary $X$, the local picture is similar, taking local coordinates $x_1, \ldots, x_n$ at the point in question replacing $A$ by the completion of the local ring at that point.
In general, for each $k$ there is a canonical morphism

$$d_k : \mathcal{F} \rightarrow \mathcal{P}_X^k(\mathcal{F}).$$

For $\mathcal{L}$ of rank 1, using local coordinates as above, $d_k$ maps an element in $A$ to its truncated Taylor series

$$f = f(x_1, \ldots, x_n) \mapsto \sum_{|\alpha| \leq k} \frac{1}{|\alpha|!} \frac{\partial^{|\alpha|} f}{\partial x^\alpha},$$

where $\alpha = i_1 i_2 \ldots i_n$ and $|\alpha| = i_1 + i_2 + \cdots + i_n$. 
The osculating subspaces

Let \( X \) be a smooth of dimension \( n \) and \( f : X \to \mathbb{P}^M \) an immersion. For \( \mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n}(1) \), let \( \mathcal{P}^k_X(\mathcal{L}) \) denote the sheaf of principal parts of order \( k \). There are homomorphisms

\[
a^k : \mathcal{O}_X^{M+1} \to \mathcal{P}^k_X(\mathcal{L}).
\]

**Definition**

For \( P \in X \) the morphism \( a^k(P) \) defines the \( k \)-osculating space \( O_{k,X,P} \) to \( X \) at \( P \) as

\[
O_{k,X,P} := \mathbb{P}(\text{Im}(a^k(P))) \subset \mathbb{P}^M
\]

of projective dimension at most \( \binom{k+n}{n} - 1 \). For \( k = 1 \) the osculating space is the tangent space.
The Veronese variety - an example

Let

- \( R_1 = \mathbb{F}[X_0, \ldots, X_n]_1 \) be the \( n + 1 \) dimensional vector space of linear forms in \( X_0, \ldots, X_n \).
- \( \mathbb{P}^n = \mathbb{P}(R_1) \), the associated projective \( n \)-space over \( \mathbb{F} \).
- \( R_d \) the vector space of forms of degree \( d \). A basis consists of the \( \binom{n+d}{d} \) monomials \( X_0^{d_0} X_1^{d_1} \cdots X_n^{d_n} \) with \( d_0 + d_1 + \cdots + d_n = d \).
- \( \mathbb{P}^M = \mathbb{P}(R_d) \) the associated projective space of dimension \( M = \binom{n+d}{d} - 1 \).
**Definition**

The $d$-uple morphism of $\mathbb{P}^n = \mathbb{P}(R_1)$ to $\mathbb{P}^M = \mathbb{P}(R_d)$ is the morphism

$$\sigma_d : \mathbb{P}^n = \mathbb{P}(R_1) \to \mathbb{P}^M = \mathbb{P}(R_d)$$

$L \mapsto L^d$

with image the Veronese variety

$$\mathcal{X}_{n,d} = \sigma_d(\mathbb{P}^n) = \{L^d | L \in \mathbb{P}(R_1)\} \subseteq \mathbb{P}^M.$$
Osculating subspaces of the Veronese variety

For the Veronese variety $\mathcal{V}_{n,d}$, the $k$-osculating subspaces $(1 \leq k < d)$ at the point $P \in \mathcal{V}_{n,d}$ corresponding to the 1-form $L \in R_1$, can be described explicitly as

$$O_{k,\mathcal{V}_{n,d},P} = \mathbb{P}(\{L^{d-k}F | F \in R_k\}) = \mathbb{P}(R_k) \subseteq \mathbb{P}^M$$

of projective dimension exactly $\binom{k+n}{n} - 1$.

The osculating spaces constitute a flag of linear subspaces

$$O_{1,\mathcal{V}_{n,d},P} \subseteq O_{2,\mathcal{V}_{n,d},P} \subseteq \cdots \subseteq O_{d-1,\mathcal{V}_{n,d},P}.$$
The construction applied to the Veronese variety

**Theorem**

Let \( n, d \) be positive integers and consider the Veronese variety \( \mathcal{X}_{n,d} \subseteq \mathbb{P}^M \), with \( M = \binom{d+n}{n} - 1 \), defined over the finite field \( \mathbb{F}_q \). Let \( C_{k,\mathcal{X}_{n,d}} \) be the associated \( k \)-osculating linear network code. The packet length of the linear network code is \( \binom{d+n}{n} \), the dimension of the ambient vector space. The number of vector spaces in the linear network code \( C_{k,\mathcal{X}_{n,d}} \) is

\[
|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \cdots + q^n, \text{ the number of } \mathbb{F}_q\text{-rational points on } \mathbb{P}^n.
\]

The vector spaces \( V \in C_{k,\mathcal{X}_{n,d}} \) are of dimension \( \binom{k+n}{n} \).
Distances in $\mathcal{C}_{k,x_{n,d}}$

Theorem

The elements in the code above are equidistant in the metric $\text{dist}(V_1, V_2)$ of Ralf Koetter and Frank R. Kschischang. For vector spaces $V_1, V_2 \in \mathcal{C}_{k,x_{n,d}}$ with $V_1 \neq V_2$

i) if $2k \geq d$, then $\dim_{\mathbb{F}_q}(V_1 \cap V_2) = \left(\frac{2k-d+n}{n}\right)$ and

$$\text{dist}(V_1, V_2) = 2 \left(\binom{k+n}{n} - \binom{2k-d+n}{n}\right).$$

ii) if $2k \leq d$, then $\dim_{\mathbb{F}_q}(V_1 \cap V_2) = 0$ and

$$\text{dist}(V_1, V_2) = 2 \binom{k+n}{n}.$$
Proof.

The associated affine cone of the $k$-osculating space is

$$\text{AffCone}(O_k,\mathcal{X}_{n,d},P)(\mathbb{F}_q) = \{ L^{d-k}F \mid F \in R_k \}$$

of dimension $\binom{k+n}{n}$, proving the claim on the dimension of the vector spaces in the linear network code $C_{k,\mathcal{X}_{n,d}}$.

As there is one element in $C_{k,\mathcal{X}_{n,d}}$ for each $\mathbb{F}_q$-rational point on $\mathbb{P}^n$, it follows that the number of elements in $C_{k,\mathcal{X}_{n,d}}$ is

$$|C_{k,\mathcal{X}_{n,d}}| = |\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \cdots + q^n.$$
Proof.

Finally, let $V_1, V_2 \in C_k, x_{n,d}$ with $V_1 \neq V_2$ and

$$V_i = \{ L_i^{d-k} F_i \mid F_i \in R_k \}$$

If $2k \geq d$, we have

$$V_1 \cap V_2 = \{ L_1^{d-k} F_1 \mid F_1 \in R_k \} \cap \{ L_2^{d-k} F_2 \mid F_2 \in R_k \}$$

$$= \{ L_1^{d-k} L_2^{d-k} G \mid G \in R_{2k-d} \}.$$ 

Otherwise the intersection is trivial, proving the claims on the dimension of the intersections and the derived distances.
Let

\[ \mathcal{N}(e) = \mathbb{F}_q[X_0, \ldots, X_n]_e/ \sim \]

be the normalized homogenous polynomials over \( \mathbb{F}_q \) of degree \( e \), where \( F_1 \sim F_2 \) iff \( F_1 = cF_2 \) for some constant \( c \in \mathbb{F}_q^* \).

Let \( \mathcal{I}(e) \subseteq \mathcal{N}(e) \) be the irreducible normalized homogenous polynomials.

For any subset \( \mathcal{B} \subseteq \mathbb{F}_q[X_0, \ldots, X_n]_e/ \sim \) of normalized homogenous polynomials of degree \( e \), we define the linear network code \( \mathcal{C}_\mathcal{B} \) as a collection of \( \mathbb{F}_q \)-linear subspaces \( V_G \), one for each \( G \in \mathcal{B} \), in the vector space of all homogenous forms of degree \( d \).
The linear network codes $C_B$

Let $G \in \mathbb{F}_q[X_0, \ldots, X_n]_e/\sim$. Assume that $G \neq 0$. Let $d \geq e$ and consider the $\mathbb{F}_q$-linear injective morphism

$$F \mapsto G \cdot F$$

with image

$$V_G := G \cdot \mathbb{F}_q[X_0, \ldots, X_n]_{d-e} \subseteq \mathbb{F}_q[X_0, \ldots, X_n]_d = \mathbb{F}_q^N,$$

which is a $\mathbb{F}_q$-linear subspace of dimension $l = \binom{n+d-e}{n}$ in the ambient vector space of dimension $N = \binom{n+d}{n}$.
The linear network codes $C_B$

**Definition**

For any subset $B \subseteq F_q[X_0, \ldots, X_n]_e/\sim$ of normalized homogenous polynomials of degree $e$, the linear network code $C_B \subseteq G(l, N)(F_q)$ consists of all the linear subspaces in the vector space $F_q[X_0, \ldots, X_n]_d$ of homogenous forms of degree $d$ with $d \geq e$, that are realized as images (2) for some $G \in B$.

$$C_B = \{ V_G = G \cdot F_q[X_0, \ldots, X_n]_{d-e} \mid G \in B \} \subseteq G(l, N)(F_q) .$$ (3)
Applications of the construction

- In [Han1], we studied the resulting linear network codes $C_B$, when $B$ is the set of normalized homogenous polynomials which are powers of linear terms. This amounted to the study of the osculating spaces of Veronese varieties as presented above.

- In [Han2] we present the resulting linear network codes $C_B$, when $B$ is any set of normalized homogenous polynomials in $\mathbb{F}_q[X_0, \ldots, X_n]_e/\sim$, where each pair of unequal polynomials has the constants as their only common divisors generalising the above result.

- In particular we treat the case where $B$ is the set of all irreducible normalized polynomials.
E. Ballico.

Any network codes comes from an algebraic curve taking osculating spaces.


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