UNIVERSITY OF TROMSØ UiT

Linear codes and Betti numbers of Stanley-Reisner rings associated to matroids.

Based on parts of joint work with Jan N. Roksvold and H. Verdure

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Content

Matroids, equivalent definitions

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— Example

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

over any field. Then $E = \{1, 2, 3, 4, 5\}$, and the maximal elements in $\mathcal{N}$ are

$$\{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}.$$

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The elements I of $\mathcal{N}$ satisfy:

— 1. $\emptyset \in \mathcal{N}$

— 2. If $I \in \mathcal{N}$, and $I' \subset I$, then $I' \in \mathcal{N}$.

— 3. If I_1 and I_2 are in $\mathcal{N}$, and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{N}$.

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DEFINITION OF A MATROID:
A (finite) matroid is a pair $(E = \{1, 2, \cdots, n\}, \mathcal{N})$, where $\mathcal{N} \subset 2^E$ satisfies $(1), (2), (3)$. The basis $\mathcal{B}$ of a matroid are the maximal elements of $\mathcal{N}$. They all have the same cardinality and this cardinality is the RANK of the matroid. The elements of $\mathcal{N}$ are the called the INDEPENDENT subsets of E.

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CAUTION: There are matroids that do not come from matrices (over any field). Example for $n = 9$.

— THE DUAL MATROID $M^* = (E = \{1, 2, \cdots, n\}, \mathcal{N}^*)$ is the one whose basis $\mathcal{B}^*$ consists of the complements of the elements of $\mathcal{B}$. In the example above: $\mathcal{B}^* = \{\{4, 5\}, \{2, 4\}, \{1, 5\}, \{1, 2\}.\}$ This is well defined. The rank of M^* is $n - rk(M)$.

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— MATROID OF A LINEAR CODE:
If C is a linear code over a finite field let $M(C)$ be the matroid assosiated to any parity check matrix of C (well defined). Then C is an $[n, n - rk(M)]$-code, and $M(C)^*$ is the matroid associated to any generator matrix of C, i.e. it is $M(C^*)$, where C^* is the orthogonal complement of C.
Also the minimum distance and all higher weights of C are only dependent on, and are easily expressible in terms of properties of the matroid $M(C)$ (and/or $M(C)^*$.)

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If $X \subset E$, then $rk(X) =$ largest cardinality of an independent subset of X. Moreover $rk(M) = rk(E)$. The rank function $2^E \to N_0$ satisfies:

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COMMENT: Any function $2^E \to N_0$ satisfying $(R1), (R2), (R3)$ determines a matroid: $\mathcal{N} =$ the set of those I with $rk(I) = |I|$.

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Then rk^* is the rank function of the dual matroid M^*.

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$$\text{Independent sets} : \mathcal{I} = \{\sigma \in 2^E |\ C \not\subseteq \sigma,\ \forall C \in \mathcal{C}\}$$

$$\text{Rank function} : r(\sigma) = max\{\#I |\ I \in \mathcal{I}, I \subset \sigma\}.$$

$$\text{Nullity function} : n(\sigma) = \#\sigma - r(\sigma)$$

Generalized Hamming weights

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Definition

The Stanley-Reisner ideal of M is

$$I_M = \left\langle \mathbf{X}^\sigma = \prod_{e \in \sigma} X_e \,|\, \sigma \in \mathcal{C} \right\rangle.$$

and the Stanley-Reisner ring

$$R_M = S/I_M.$$

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3 types of Betti numbers:

$$\mathbb{N}^E\text{- or multigraded Betti numbers: } \beta_{i,\alpha}$$

$$\mathbb{N}\text{-graded Betti numbers: } \beta_{i,d} = \sum_{|\alpha|=d} \beta_{i,\alpha}$$

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$$F_0 = S$$

$$\beta_{1,\sigma} = 1 \Leftrightarrow \sigma \in \mathcal{C}.$$

Betti table

The Betti table of a matroid is a matrix together with an integer n where the number in the i-th column and the j-th row represents $\beta_{i,i+j+c-2}$. The suffix c on the table denotes the minimal absolute values of a twist occuring.

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$$\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}_2 .$$

Hochster's formula

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$$0 \leftarrow \mathbb{K} \xleftarrow{\partial_0} \bigoplus_{\substack{F \in M \\ \#F = 1}} \mathbb{K} \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{r-1}} \bigoplus_{\substack{F \in M \\ \#F = r}} \mathbb{K} \xleftarrow{\partial_r} \cdots \leftarrow 0.$$

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The boundary maps are: if $F = \{x_0 < \cdots < x_i\}$,

$$\partial_i(e_F) = \sum_{j=0}^{i} (-1)^j e_{\{x_0, \cdots, \check{x}_j, \cdots, x_i\}}.$$

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Definition

The i-th reduced homology of M over $\mathbb{K}$ is the $\mathbb{K}$ vector space

$$\tilde{H}_i(M, \mathbb{K}) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$$

and its dimension is denoted by $\tilde{h}_i(M, \mathbb{K})$.

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Let M be a matroid on E of rank r. Then

$$\tilde{h}_i(M, \mathbb{K}) = \begin{cases} (-1)^r \chi(M) & \text{if } i = r-1 \\ 0 & \text{otherwise} \end{cases}$$

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Corollary

The Betti numbers of a matroid are independent of the field $\mathbb{K}$.

Goal

Look at relations between matroids, their Betti numbers and their generalized Hamming weights.

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$$\beta_{1,\sigma} = \begin{cases} 1 & \text{if } \sigma \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

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8. Algebraic geometric codes

Content

Non-redundancy

An ingredient in understanding the role of cirquits for the nullity of subsets of E.

Definition

Let M be a matroid, and $\Sigma \subseteq \mathcal{C}$. We say that Σ is not redundant if

$$\forall \sigma \in \Sigma, \quad \bigcup_{\tau \in \Sigma - \{\sigma\}} \tau \subsetneq \bigcup_{\tau \in \Sigma} \tau.$$

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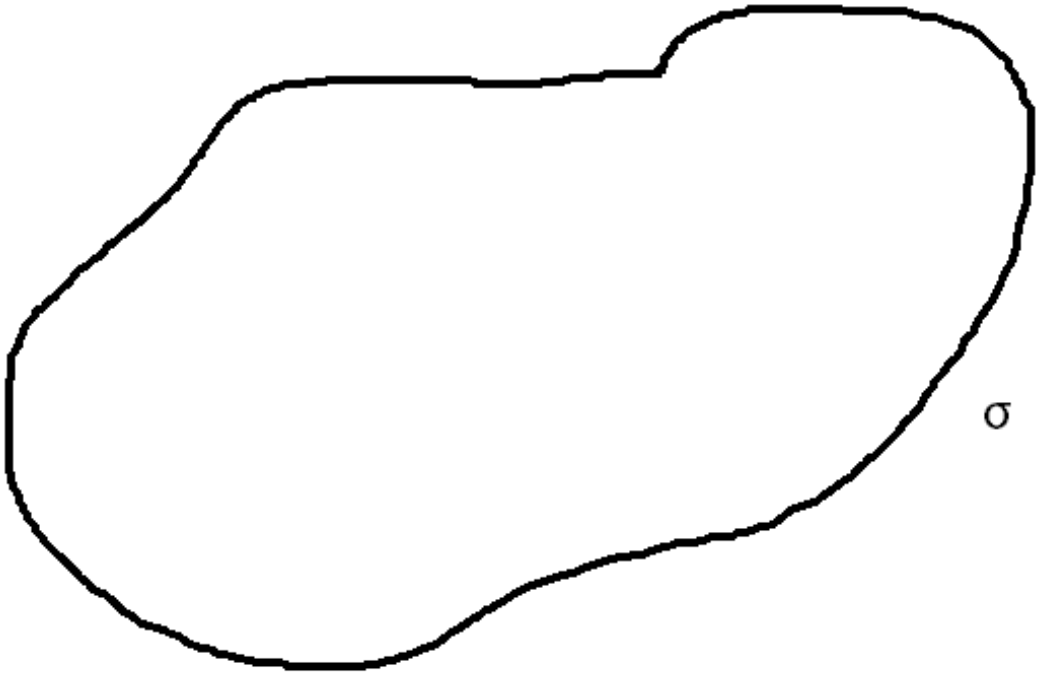
Definition

Let M be a matroid on the ground set E, and $\sigma \subseteq E$. The degree of non-redundancy of σ is

$$\deg(\sigma) = Max\{\#\Sigma |\ \Sigma \text{ non-redundant and } \bigcup_{\tau \in \Sigma} \tau \subseteq \sigma\}.$$

We have $n(\sigma) = deg(\sigma)$.

$$n(\sigma) \leqslant deg(\sigma)$$



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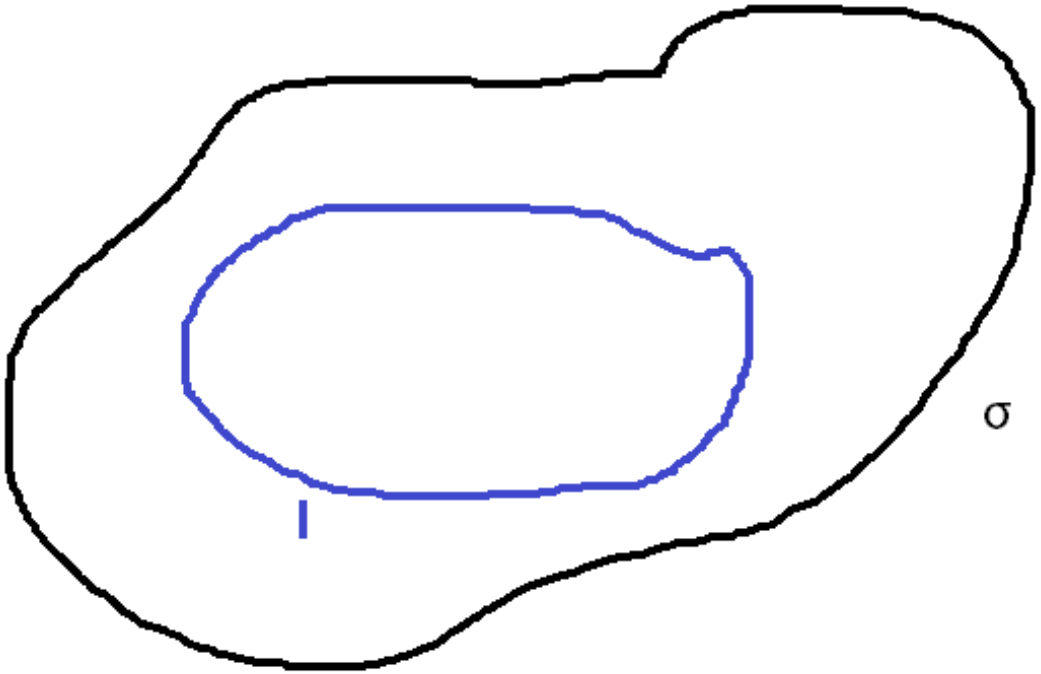

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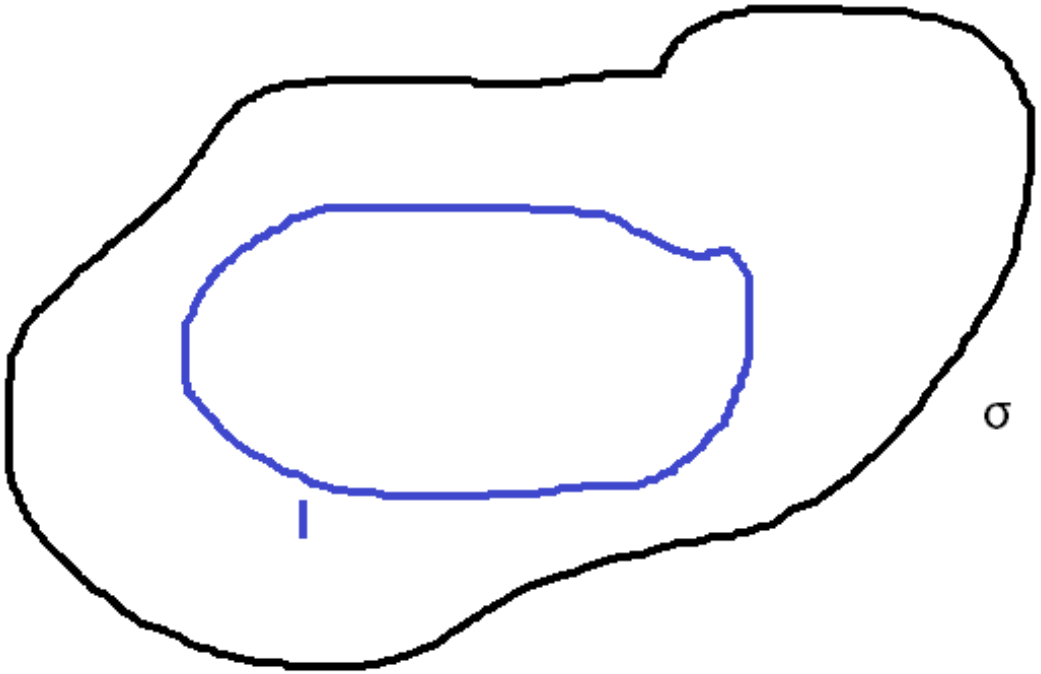

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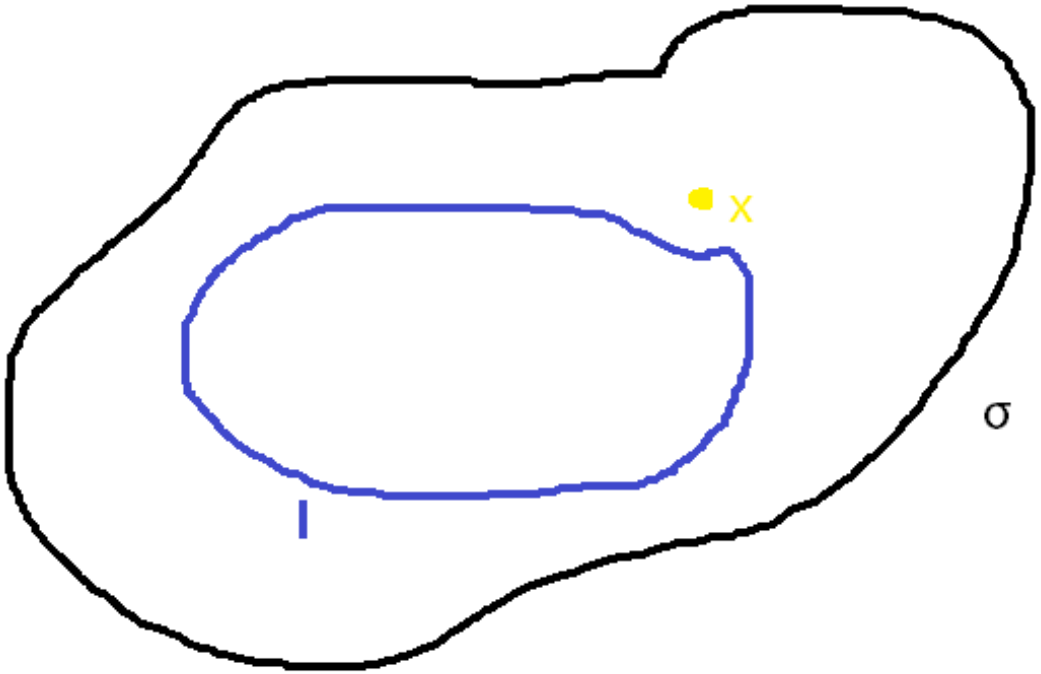

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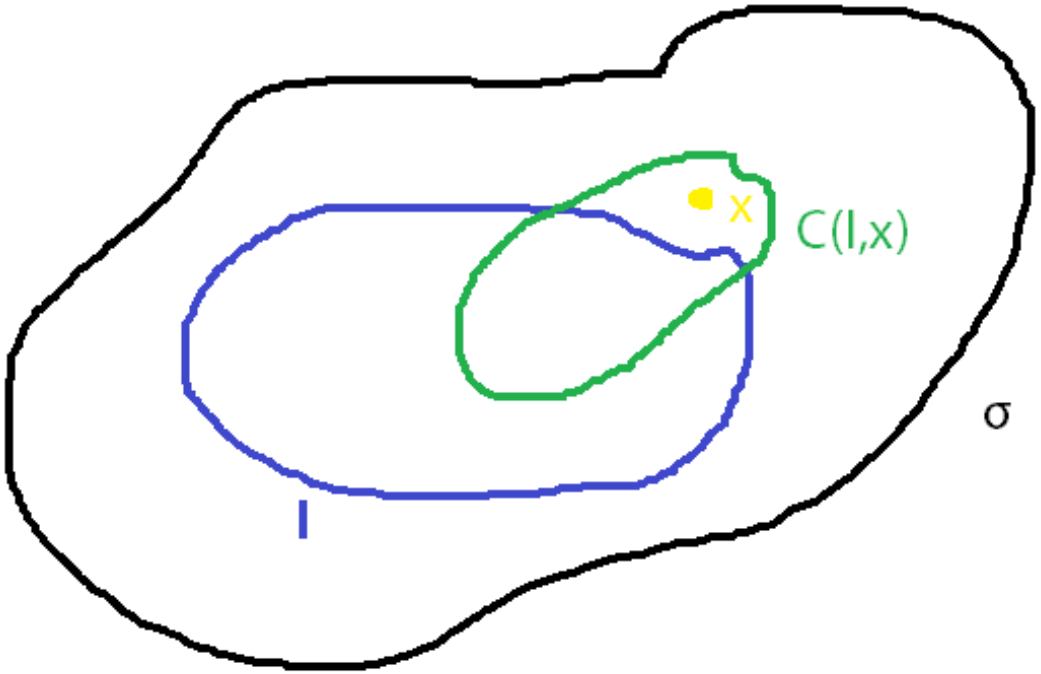

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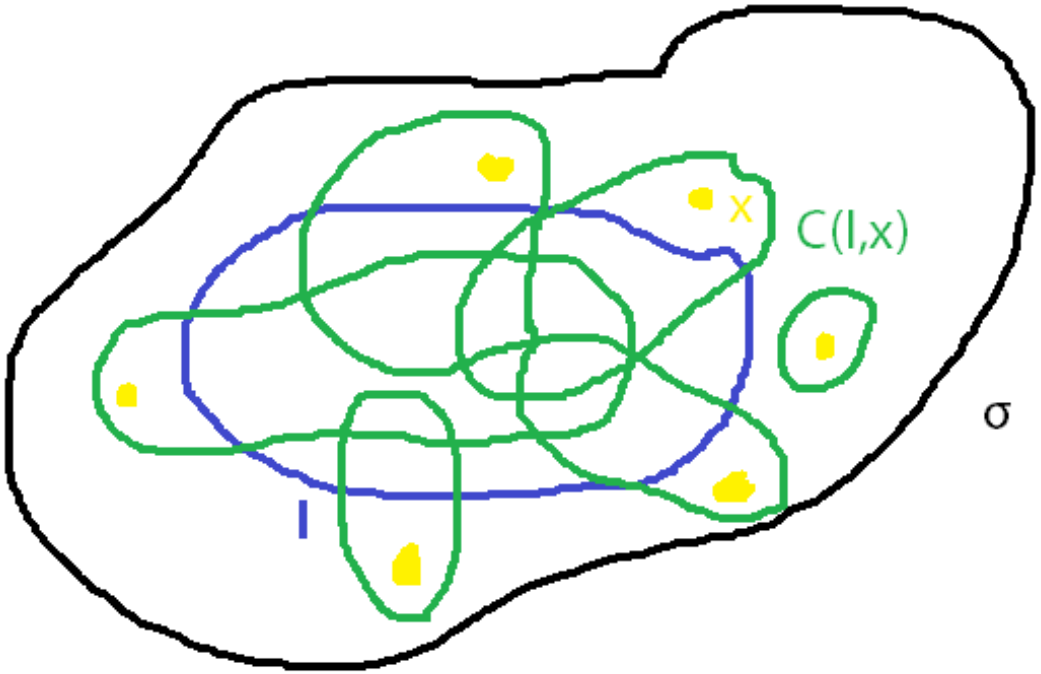

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$$n\left(\bigcup_{\tau \in \Sigma} \tau\right) \geqslant n\left(\bigcup_{\tau \in \Sigma - \{\sigma\}} \tau\right) + n(\sigma) - n\left(\left(\bigcup_{\tau \in \Sigma - \{\sigma\}} \tau\right) \cap \sigma\right).$$

Content

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Moreover, $\beta_{n(\sigma),\sigma} = (-1)^{r(\sigma)-1}\chi(M|_\sigma)$.

Theorem

Let M be a matroid on the ground set E, and let $\sigma \subseteq E$. Then

$$\beta_{i,\sigma} \neq 0 \Leftrightarrow \sigma \text{ is minimal with } n(\sigma) = i.$$

The Betti numbers decide the weight hierarchy

[J-V]:

Theorem

Let M be a matroid on the ground set E of rank r. Then the generalized Hamming weights are given by

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Let $\mathcal{C} = \{\{1,2,3,4\},\{1,4,5\},\{1,6\},\{2,3,4,6\},\{2,3,5\},\{4,5,6\}\}$.
The Betti table is

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The weight hierarchy is therefore $2, 4, 6$

MDS-codes

A linear $[n, k]$-code satisfies $d \leq r + 1 = n - k + 1$. If equality, the code is called MDS ($n - k$ is pr. def. the redundancy r of the code). Such codes correspond to uniform matroids $U(r, n)$.

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	1	$\cdots$	s	$\cdots$	$n-r$
r	$\binom{r}{r}\binom{n}{r+1}$	$\cdots$	$\binom{r+s-1}{r}\binom{n}{r+s}$	$\cdots$	$\binom{n-1}{r}\binom{n}{n}$

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Hence the weight hierarchy is $\{n-k+1, \ldots, n-1, n\}$.

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Content

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Definition

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$$\{d_1, \cdots, d_s, n - d'_1 + 1, \cdots, n - d'_{n-s} + 1\} = E.$$

Corollary

The $\mathbb{N}$-graded Betti numbers of $\overline{M}$ give the weight hierarchy of M.

Alexander dual of a matroid

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Let Δ be a simplicial complex, with set of faces $\mathcal{F}$. Then its Alexander dual $\Delta^\star$ has set of faces $\mathcal{F}^\star = \{\overline{\tau} | \, \tau \notin \mathcal{F}\}$.

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The $\mathbb{N}$-graded Betti numbers of the Alexander dual don't in general give the weight hierarchy of M.

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Corollary

If C is non-degenerate, then it is MDS if and only if the Alexander dual of $M(C)$ is also a matroid.

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A "quasi-t-gonality" disregarding divisors with repeated points:

$$t_D := \min\{\deg A | l(A) = t + 1\} = d_t = \min\{j | \ \beta_{j,t} \neq 0\}.$$

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Let X be the same algebraic curve of genus g.
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A "quasi-Clifford index" is:

$$\begin{aligned} Cl_D(X) &:= \min\{CL_D(A) |\ h^0(A) \geqslant 2, h^1(A) \geqslant 2\} \\ &= \min\{j - 2i |\ i \geqslant 1, j \leqslant g - 2 + i, \beta_{i,j} \neq 0\} \end{aligned}$$

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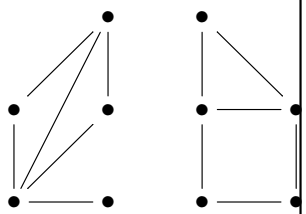
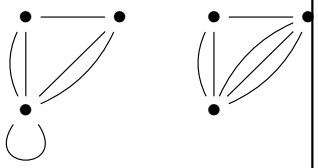
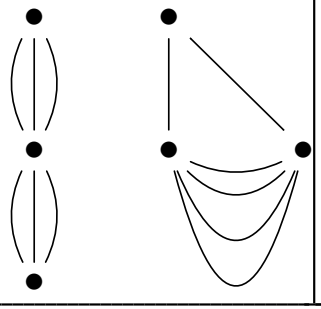
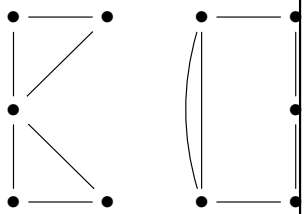
β_i				

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Constant weight linear codes are repetitions of simplex codes.

We show:
Constant weight codes have pure resolutions (of the Stanley-Reisner ring of the parity check matroid simplicial complex).

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[J-V]: The Betti-numbers satisfy: $\beta_{i,d_i} = g(i,k)q^{\frac{i(i-1}{2}}$.
Proof: Special argument using properties of constant weight codes in a profound way, or:
The Herzog-Kuhl equations for Betti numbers associated to pure Cohen-Macaulay resolutions.

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$$\beta_{i,d_i} = \prod_{j \neq 0,i} |\frac{d_j - d_0}{d_j - d_i}|,$$

for $i \geq 1$ obtained from The Herzog-Kuhl equations, which are the ones imposed on the $\beta_{i,j}(S/I)$ by the vanishing of the first c coefficients of the Hilbert polynomial of M, corresponding to the fact that the support of S/I has codimension c.

Content

We want to study the polynomial

$$P_{M,j}(Z) = (-1)^j \sum_{|\sigma|=j} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Z^{n_M(\gamma)} \text{ for } 1 \leq j \leq n,$$

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which we refer to as the generalized weight enumerator of M. Why ?
After an argument similar to one of Jurrius/Pellikaan one shows that:

$$P_{M(H),i}(q^m)$$

is the number of codewords of weight i in $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ for a linear code C with parity check matrix H.

Result: $P_{M,i}(Z)$ is determined by Betti numbers of M, and so-called elongations of M.

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As a simple illustration we show that the constant term of $P_{M,n}(Z)$ is equal to

$$(-1)^{n-r}\beta_{n-r(M),n}.$$

Recall that

$$P_{M,n}(Z) = \sum_{\gamma \subseteq E} (-1)^{|\gamma|} Z^{n_M(\gamma)},$$

and note that the constant term of this polynomial is

$$\sum_{n_M(\gamma)=0} (-1)^{|\gamma|} = (-1)^{n+1} \chi(M),$$

where $\chi(M)$ is the reduced Euler characteristic of M. Let f_i be the number of independent sets in M of cardinality i.

$$\chi(M) = -1 + f_1 - f_2 + \cdots + (-1)^{r(M)-1} f_{r(M)}.$$

Let $H_i(M;\mathbb{K})$ denote the ith reduced homology of M over $\mathbb{K}$. According to Björner (1992), we have

$$(-1)^{n+1}\chi(M) = (-1)^{n-r(M)} \dim H_{r(M)-1}(M).$$

From *Hochster's formula*, we see that the dimension of $H_{r(M)-1}(M)$ is equal to $\beta_{n-r(M),n}((S/I)_M)$, thus

$$\begin{aligned}\sum_{n_M(\gamma)=0} (-1)^{|\gamma|} &= (-1)^{n+1}\chi(M)\\ &= (-1)^{n-r(M)} \dim H_{r(M)-1}(M)\\ &= (-1)^{n-r(M)}\beta_{n-r(M),n}((S/I)_M),\end{aligned}$$

which was what we wanted to prove.
Hochster's formula

$$\beta_{i,\sigma}((S/I)_M) = \dim H_{|\sigma|-i-1}(M_{|\sigma}).$$

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The elongation M_i of M to rank $r(M) + i$. For $1 \leq i \leq n - r(M)$, the set $\mathcal{I}_i = \{\sigma \in E : n(\sigma) \leq i\}$ forms the set of independent sets of a matroid M_i on E. Note that $M = M_0$

Let n_{M_i} denote the nullity function of M_i. Then, for $\sigma \subseteq E$, we have

$$n_{M_i}(\sigma) = \max\{n(\sigma) - i, 0\}$$

We shall make use of the following observation:

$$n^{-1}(l) = n_{M_l}^{-1}(0) \setminus n_{M_{l-1}}^{-1}(0).$$

Let $\beta^{(i)}$ distinguish the Betti numbers of M_i from those of $M(= M_0)$. (And let $\beta^{(i)} = 0$ whenever $i \notin [i, n - r(M)]$.)

Proposition

The coefficient of Z^l in $P_{M,n}(Z)$ is equal to

$$(-1)^{n-r-l}\left[\beta^{(l-1)}_{n-r-l+1,n} + \beta^{(l-1)}_{n-r-l,n}\right].$$

Since

$$P_{M,n}(Z) = (-1)^n \sum_{\gamma \subseteq E} (-1)^{|\gamma|} Z^{n_M(\gamma)},$$

it is clear that the coefficient of Z^l is equal to

$$(-1)^n \sum_{n_M(\gamma)=l} (-1)^{|\gamma|}.$$

But

$$\sum_{n_M(\gamma)=l} (-1)^{|\gamma|} = \Big[\sum_{n_{M_l}(\gamma)=0} (-1)^{|\gamma|} - \sum_{n_{M_{l-1}}(\gamma)=0} (-1)^{|\gamma|} \Big],$$

and the result follows as for the constant term.

To find $P_{M,j}$ for $0 \leq j \leq n-1$ we proceed in an analogous fashion; first we find the constant term for each j, then we use the elongations to find the remaining coefficients. The final results being:
[J-R-V]:

Theorem

For $1 \leq j \leq n$ the coefficient of Z^l in $P_{M,j}$ is equal to

$$\sum_{i=0}^{n}(-1)^{i+1}\left(\beta_{i+1,j}^{(l-1)} - \beta_{i+1,j}^{(l)}\right).$$

Corollary

Let C be an $[n,k]$-code over $\mathbb{F}_q$. For $1 \leq j \leq n$, the number of words of weight j in $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is

$$\sum_{l=0}^{k}\left(\sum_{i=0}^{n}(-1)^{i+1}\left(\beta_{i+1,j}^{(l-1)} - \beta_{i+1,j}^{(l)}\right)\right)(q^m)^l.$$

Content

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- We define the corresponding code to be the row-space of the matrix

$$G = \begin{pmatrix} P_{1,1} & P_{2,1} & \ldots & P_{n,1} \\ P_{1,2} & P_{2,2} & \ldots & P_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ P_{1,k} & P_{2,k} & \ldots & P_{n,k} \end{pmatrix}.$$

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- How are the d_i determined by the geometry of X?

Find the d_i from properties of X

For each $h = 1, \ldots, k$, let
$J_h = \max\{\#\mathbb{F}_q$-rational points on X in $S \mid S$ is a codim. h subspace in $\mathbb{P}_q^{k-1}\}$. We recall: $n = \#\mathbb{F}_q$-rational points on X.

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Dually: If instead we use G as a parity check matrix H for C, then:
d_i is the smallest t such that there exist t points of X only spanning a $(t - i - 1)$-dimensional subspace of $\mathbb{P}_q^{k-1}$.
Reflects the existence of t-secant $(t - i - 1)$-planes.

Simplex codes

Let X be the set of <u>all</u> points in a projective space $\mathbb{P}^s$. This gives the socalled simplex code $\mathbb{S}_q(s)$. Then all codimension r spaces contain the same number of points, and this code is therefore of constant weight.

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The Betti-numbers follow from the Herzog-Kuhl equations.
Example with $\mathbb{R}_2(3)$. Use

$$\beta_{i,d_i} = \prod_{j \neq 0,i} |\frac{d_j - d_0}{d_j - d_i}|,$$

for $(d_1 = 4, d_2 = 6, d_3 = 7)$. we obtain

$$0 \leftarrow R_M \leftarrow S \leftarrow S(-4)^7 \leftarrow S(-6)^{14} \leftarrow S(-7)^8 \leftarrow 0.$$

The two (only interesting) elongation matroids are uniform, and their Betti-tables are then known. Putting the information from all Betti-tables together and using the Corollary by [J-R-V] above one finds the $A_i(q^m)$ (number of code words of weight m over $\mathbb{F}_q$) for all m, as already done by other methods, e.g. in Example 14 of "Weight enumeration of codes from finite spaces", Jurrius, DCC, 2012. One can then also find the higher weight enumerator polynomials of the code by Theorem 3 in that article.

(The matroid simplicial complexes of) Reed-Muller codes of the first order are not of constant weight (2 weights), but have pure resolutions, as have their elongations, and they may be treated in a very similar way.