Linear codes and Betti numbers of Stanley-Reisner rings associated to matroids.
Based on parts of joint work with Jan N. Roksvold and H. Verdure

Trygve Johnsen
Department of Mathematics and Statistics
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Matroids, equivalent definitions

— Matroids initially arose from matrices $M$ over a field $F$. The matroid associated to $M$ is a pair

$$(E = \{1, 2, \cdots, n\}, \mathcal{N}),$$

where $\mathcal{N}$ is the set of subsets of $E$ indexing those sets of columns of $M$ that are linearly independent.
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where $\mathcal{N}$ is the set of subsets of $E$ indexing those sets of columns of $M$ that are linearly independent.

— Example

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

over any field. Then $E = \{1, 2, 3, 4, 5\}$, and the maximal elements in $\mathcal{N}$ are

$$\{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 4\}, \{3, 4, 5\}.$$

The set $B$ with these 4 subsets as elements, certainly determines $\mathcal{N}$, which contain of 14 additional, smaller subsets of $E$. 
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The set $B$ with these 4 subsets as elements, certainly determines $\mathcal{N}$, which contain of 14 additional, smaller subsets of $E$. 

The elements $I$ of $\mathcal{N}$ satisfy:

— 1. $\emptyset \in \mathcal{N}$
— 2. If $I \in \mathcal{N}$, and $I' \subset I$, then $I' \in \mathcal{N}$.
— 3. If $I_1$ and $I_2$ are in $\mathcal{N}$, and $|I_1| < |I_2|$, then there is an element $e$ of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{N}$.
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**DEFINITION OF A MATROID:**

A (finite) matroid is a pair \( (E = \{1, 2, \ldots, n\}, \mathcal{N}) \), where \( \mathcal{N} \subset 2^E \) satisfies (1), (2), (3). The basis \( B \) of a matroid are the maximal elements of \( \mathcal{N} \). They all have the same cardinality and this cardinality is the RANK of the matroid. The elements of \( \mathcal{N} \) are the called the INDEPENDENT subsets of \( E \).
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**CAUTION:** There are matroids that do not come from matrices (over any field). Example for $n = 9$. 
THE DUAL MATROID $M^* = (E = \{1, 2, \ldots, n\}, \mathcal{N}^*)$ is the one whose basis $\mathcal{B}^*$ consists of the complements of the elements of $\mathcal{B}$. In the example above: $\mathcal{B}^* = \{\{4, 5\}, \{2, 4\}, \{1, 5\}, \{1, 2\}\}$. This is well defined. The rank of $M^*$ is $n - rk(M)$. 
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MATROID OF A LINEAR CODE:
If $C$ is a linear code over a finite field let $M(C)$ be the matroid associated to any parity check matrix of $C$ (well defined). Then $C$ is an $[n, n - rk(M)]$-code, and $M(C)^*$ is the matroid associated to any generator matrix of $C$, i.e. it is $M(C^*)$, where $C^*$ is the orthogonal complement of $C$.
Also the minimum distance and all higher weights of $C$ are only dependent on, and are easily expressible in terms of properties of the matroid $M(C)$ (and/or $M(C)^*$).
DEFINITION OF RANK FUNCTION OF A MATROID:

If \( X \subset E \), then \( rk(X) = \) largest cardinality of an independent subset of \( X \). Moreover \( rk(M) = rk(E) \). The rank function \( 2^E \rightarrow \mathbb{N}_0 \) satisfies:

1. \( R_1 \): If \( X \subset E \), then \( 0 \leq rk(X) \leq |X| \).
2. \( R_2 \): If \( X \subset Y \subset E \), then \( rk(X) \leq rk(Y) \).
3. \( R_3 \): If \( X \) and \( Y \) are subsets of \( E \), then \( rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y) \).

COMMENT: Any function \( 2^E \rightarrow \mathbb{N}_0 \) satisfying \( R_1 \), \( R_2 \), \( R_3 \) determines a matroid: \( \mathbb{N} = \) the set of those \( I \) with \( rk(I) = |I| \).
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— R1. If \( X \subset E \), then \( 0 \leq r(X) \leq |X| \).
DEFINITION OF RANK FUNCTION OF A MATROID:

If $X \subset E$, then $rk(X) =$ largest cardinality of an independent subset of $X$. Moreover $rk(M) = rk(E)$. The rank function $2^E \to N_0$ satisfies:

— R1. If $X \subset E$, then $0 \leq r(X) \leq |X|$.
— R2. If $X \subset Y \subset E$, then $rk(X) \leq rk(Y)$.
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- **R1.** If \( X \subset E \), then \( 0 \leq r(X) \leq |X| \).
- **R2.** If \( X \subset Y \subset E \), then \( rk(X) \leq rk(Y) \).
- **R3.** If \( X \) and \( Y \) are subsets of \( E \), then 
  \[ rk(X \cup Y) + rk(X \cap Y) \leq rk(X) + rk(Y). \]
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COMMENT: Any function \( 2^E \to N_0 \) satisfying \((R1), (R2), (R3)\) determines a matroid: \( N = \) the set of those \( l \) with \( rk(l) = |l| \).
Given a matroid $M$ with rank function $rk$. Put

$$rk^*(X) = |X| - rk(E) + rk(E - X).$$
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Then $rk^*$ is the rank function of the dual matroid $M^*$. 
Matroids defined through circuits

Definition

$E$ is a finite set and $C \subset 2^E$ such that:

1. $\emptyset \notin C$
2. If $C_1, C_2 \in C$ and $C_1 \subseteq C_2$, then $C_1 = C_2$
3. If $C_1, C_2 \in C$, $\forall e \in C_1 \cap C_2$, there exists $C_3 \in C$ such that $C_3 \subseteq C_1 \cup C_2 - \{e\}$.

Independent sets:

$I = \{\sigma \in 2^E | C \not\subseteq \sigma, \forall C \in C\}$

Rank function:

$r(\sigma) = \max\{\#I | I \in I, I \subset \sigma\}$

Nullity function:

$n(\sigma) = \#\sigma - r(\sigma)$
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Generalized Hamming weights

$C$ a $[n, k]$- linear code over a field $\mathbb{K}$. The generalized Hamming weights are

$$d_i = \text{Min}\{\# \text{Supp}(D), \ D \subseteq C \text{ subcode of dimension } i\}.$$
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**Definition**

Let $M$ be a matroid on the ground set $E$, The generalized Hamming weights of $M$ are

$$d_i = \text{Min}\{\#\sigma \mid n(\sigma) = i\}.$$
Stanley-Reisner rings associated to matroids

Let $M$ be a matroid on the ground set $E$. 
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Let $M$ be a matroid on the ground set $E$. Let $\mathbb{K}$ be any field, and $S = \mathbb{K}[X] = \mathbb{K}[X_e, e \in E]$. 

Definition

The Stanley-Reisner ideal of $M$ is $I_M = \langle \prod_{e \in \sigma} X_e \mid \sigma \in C \rangle$.

and the Stanley-Reisner ring $R_M = S/I_M$. 


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and the Stanley-Reisner ring

$$R_M = S/I_M.$$
Betti numbers

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$$0 \leftarrow S/I_M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_s \leftarrow 0$$

where

$$F_i = \bigoplus_{\alpha \in \mathbb{N}^E} S(-\alpha)^{\beta_{i,\alpha}}.$$
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3 types of Betti numbers:

- \( \mathbb{N}^E \)- or multigraded Betti numbers: \( \beta_{i,\alpha} \)

- \( \mathbb{N} \)-graded Betti numbers: \( \beta_{i,d} = \sum_{|\alpha|=d} \beta_{i,\alpha} \)

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$F_0 = S$

$$\beta_{1,\sigma} = 1 \iff \sigma \in \mathcal{C}.$$
The Betti table of a matroid is a matrix together with an integer $n$ where the number in the $i$-th column and the $j$-th row represents $\beta_{i,i+j+c-2}$. The suffix $c$ on the table denotes the minimal absolute values of a twist occurring.
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Example

Let $M$ be the matroid with circuits

\{\{1, 2, 4\}, \{1, 2, 3\}, \{3, 4\}\}.
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Betti table:
\[
\begin{bmatrix}
1 & 0 \\
2 & 2
\end{bmatrix}_2.
\]
Hochster’s formula

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The chain complex of $M$ over $K$ is

$$
0 \leftarrow K \xleftarrow{\partial_0} \bigoplus_{F \in M} K \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{r-1}} \bigoplus_{F \in M} K \xleftarrow{\partial_r} \cdots \leftarrow 0.
$$

Definition

The $i$-th reduced homology of $M$ over $K$ is the $K$ vector space $\tilde{H}_i(M, K) = \ker(\partial_i) / \text{im}(\partial_{i+1})$ and its dimension is denoted by $\tilde{h}_i(M, K)$. 
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The boundary maps are: if $F = \{x_0 < \cdots < x_i\}$,

$$\partial_i(e_F) = \sum_{j=0}^{i} (-1)^j e_{\{x_0, \ldots, \hat{x_j}, \ldots, x_i\}}.$$
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Hochster’s formula

Theorem

\[ \beta_{i,\sigma}(K) = \tilde{h}_{|\sigma|-i-1}(M_{|\sigma} , K) \]
Hochster’s formula

Theorem

\[ \beta_{i,\sigma}(K) = \tilde{h}_{|\sigma|-i-1}(M|_{\sigma}, K) \]

Theorem

Let \( M \) be a matroid on \( E \) of rank \( r \). Then

\[ \tilde{h}_i(M, K) = \begin{cases} (-1)^r \chi(M) & \text{if } i = r - 1 \\ 0 & \text{otherwise} \end{cases} \]
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Corollary

The Betti numbers of a matroid are independent of the field \( K \).
Goal

Look at relations between matroids, their Betti numbers and their generalized Hamming weights.
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Remark 1: The $\mathbb{Z}_E$-graded case:
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Remark 1: The $\mathbb{N}^E$-graded case:

$$\beta_{1,\sigma} = \begin{cases} 1 & \text{if } \sigma \in C \\ 0 & \text{otherwise} \end{cases}$$
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Non-redundancy

An ingredient in understanding the role of circuits for the nullity of subsets of $E$.

**Definition**

Let $M$ be a matroid, and $\Sigma \subseteq C$. We say that $\Sigma$ is not redundant if

$$\forall \sigma \in \Sigma, \quad \bigcup_{\tau \in \Sigma - \{\sigma\}} \tau \subsetneq \bigcup_{\tau \in \Sigma} \tau.$$
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**Definition**

Let $M$ be a matroid on the ground set $E$, and $\sigma \subseteq E$. The degree of non-redundancy of $\sigma$ is

$$\text{deg}(\sigma) = \text{Max}\{\#\Sigma | \Sigma \text{ non-redundant and } \bigcup_{\tau \in \Sigma} \tau \subseteq \sigma\}.$$ 

We have $n(\sigma) = \text{deg}(\sigma)$. 
\[ n(\sigma) \leq \text{deg}(\sigma) \]
$n(\sigma) \leq \deg(\sigma)$
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\[ r(\sigma) = \#I \Rightarrow n(\sigma) = \#(\sigma - I) \]
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Important elements in proof:

**Proposition**

Let $M$ be a matroid and $X, Y \subseteq E$. Then

$$n(X \cup Y) + n(X \cap Y) \geq n(X) + n(Y)$$
\( n(\sigma) \geq \text{deg}(\sigma) \)

Important elements in proof:

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**Corollary**

If \( \Sigma \subset C \) is non redundant, then

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 n\left( \bigcup_{\tau \in \Sigma} \tau \right) \geq \#\Sigma.
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\[ n(\sigma) \geq \deg(\sigma) \]

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\end{align*}
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\begin{align*}
n \left( \bigcup_{\tau \in \Sigma} \tau \right) & \geq \# \Sigma. \\
n \left( \bigcup_{\tau \in \Sigma} \tau \right) & \geq n \left( \bigcup_{\tau \in \Sigma - \{\sigma\}} \tau \right) + n(\sigma) - n \left( \bigcup_{\tau \in \Sigma - \{\sigma\}} \tau \cap \sigma \right).
\end{align*}
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Moreover, \( \beta_{n(\sigma),\sigma} = (-1)^{r(\sigma) - 1} \chi(M|_\sigma) \).
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Let $M$ be a matroid on the ground set $E$, and let $\sigma \subseteq E$. Then

$$\beta_{i,\sigma} \neq 0 \iff \sigma \text{ is minimal with } n(\sigma) = i.$$
The Betti numbers decide the weight hierarchy

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**Theorem**

Let $M$ be a matroid on the ground set $E$ of rank $r$. Then the generalized Hamming weights are given by

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**Example**

Let $C = \{\{1, 2, 3, 4\}, \{1, 4, 5\}, \{1, 6\}, \{2, 3, 4, 6\}, \{2, 3, 5\}, \{4, 5, 6\}\}$. The Betti table is

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \\ 2 & 7 & 4 \end{bmatrix}_2$$
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The Betti table is

$$
\begin{bmatrix}
  1 & 0 & 0 \\
  3 & 2 & 0 \\
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\end{bmatrix}
$$

The weight hierarchy is therefore 2, 4, 6.
MDS-codes

A linear \([n, k]\)-code satisfies \(d \leq r + 1 = n - k + 1\). If equality, the code is called MDS \((n - k)\) is pr. def. the redundancy \(r\) of the code). Such codes correspond to uniform matroids \(U(r, n)\).
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\[
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\begin{align*}
0 & \leftarrow R_{U(r,n)} \leftarrow S \leftarrow S(-(r + 1))^{(r)}(\binom{n}{r+1}) \\
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and the Betti diagram is:

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
r & 1 & \ldots & s & \ldots & n - r \\
\hline
\binom{r}{r} & \binom{n}{r+1} & \ldots & \binom{r+s-1}{r} & \binom{n}{r+s} & \ldots & \binom{n-1}{r} & \binom{n}{n}
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\[
\begin{array}{cccc}
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\end{array}
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Hence the weight hierarchy is \(\{n - k + 1, \ldots, n - 1, n\}\).
Some negative results

\[ \beta_i \not\Rightarrow d_i \]
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— $\beta_i \not\Rightarrow d_i$
— $d_i \not\Rightarrow \beta_i, \beta_{i,d}, \beta_{i,\sigma}$
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Definition

Let $M$ be a matroid. Its dual $\overline{M}$ has the same ground set, and its set of bases is the set of complements of bases of $M$.
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**Corollary**
The $\mathbb{N}$-graded Betti numbers of $\overline{M}$ give the weight hierarchy of $M$. 
Alexander dual of a matroid

Definition
Let $\Delta$ be a simplicial complex, with set of faces $\mathcal{F}$. Then its Alexander dual $\Delta^*$ has set of faces $\mathcal{F}^* = \{\overline{\tau} \mid \tau \not\in \mathcal{F}\}$.

Eagon-Reiner: the Alexander dual of a matroid has a linear resolution.

Example
$C = \{\{1, 4\}, \{2, 3\}\}$

Betti table of the Alexander dual $M^*$:

$\begin{bmatrix} 4 & 4 & 1 \\ 2 \end{bmatrix}$

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A linear code $C$ of length $n$ and dimension $k$ is $h$-MDS if $d_h = n - k + h$
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$C$ is $h$-MDS if and only if the right part

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### Corollary
If $C$ is non-degenerate, then it is MDS if and only if the Alexander dual of $M(C)$ is also a matroid.
An example from algebraic codes

Let $X$ be an algebraic curve over $\mathbb{F}_q$ of genus $g$ in $\mathbb{P}^{g-1}$ embedded by the canonical divisor $K$. 

Riemann-Roch:

$$r(A) = l(K) - l(K - A) = n(A)$$

Here $r$ and $n$ are matroids, and $l$ is R.R. notation.

A "quasi-t-gonality" disregarding divisors with repeated points:

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Cl_D(A) := \deg(A) - 2(l(A) - 1)) = \#A - 2n(A).
\]

A "quasi-Clifford index" is:

\[
Cl_D(X) := \min\{CL_D(A) | h^0(A) \geq 2, h^1(A) \geq 2\} = \min\{j - 2i | i \geq 1, j \leq g - 2 + i, \beta_{i,j} \neq 0\}
\]
Summary

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Result: If a linear code $C$ has constant weight, then the cardinalities of the supports of all sub-linear spaces of $C$ of dimension $i$ are the same number, which we of course call $d_i$, for $i = 1, \cdots, k$. So it is "constant weight in all dimensions".
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$$d_i = d \frac{q^k - q^{k-i}}{q^k - q^{k-1}},$$

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Constant weight linear codes are repetitions of simplex codes.
We show:
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Since all subcodes of dimension $i$ have the same support weight, all minimal elements of $N_i$ (the supports of these subcodes) will have the same cardinalities.
We show:
Constant weight codes have pure resolutions (of the Stanley-Reisner ring of the parity check matroid simplicial complex).
Main point: Let $M$ be a matroid on the ground set $E$, and let $\sigma \subseteq E$. Then

$$\beta_{i,\sigma} \neq 0 \iff \sigma \text{ is minimal with } n(\sigma) = i.$$ 

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[J-V]: The Betti-numbers satisfy: $\beta_{i,d_i} = g(i, k)q^{i(i-1)/2}$.

Proof: Special argument using properties of constant
weight codes in a profound way, or:
The Herzog-Kuhl equations for Betti numbers associated
to pure Cohen-Macaulay resolutions.
Last approach: $\beta_{1,\sigma} = 1$, for cirquits $\sigma$. 
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More details: Count circuits (supports of codewords) to obtain $\beta_{1,d_1}$. Set $d_0 = 0$. Then we use a formula

$$\beta_{i,d_i} = \prod_{j \neq 0,i} \left| \frac{d_j - d_0}{d_j - d_i} \right|,$$

for $i \geq 1$ obtained from The Herzog-Kuhl equations, which are the ones imposed on the $\beta_{i,j}(S/I)$ by the vanishing of the first $c$ coefficients of the Hilbert polynomial of $M$, corresponding to the fact that the support of $S/I$ has codimension $c$. 
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We want to study the polynomial

\[ P_{M,j}(Z) = (-1)^j \sum_{|\sigma| = j} \sum_{\gamma \subseteq \sigma} (-1)^{|\gamma|} Z^{n_M(\gamma)} \text{ for } 1 \leq j \leq n, \]

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We want to study the polynomial

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After an argument similar to one of Jurrius/Pellikaan one shows that:

\[ P_{M(H),i}(q^m) \]

is the number of codewords of weight \( i \) in \( C \otimes_{F_q} F_{q^m} \) for a linear code \( C \) with parity check matrix \( H \).
Result: $P_{M,i}(Z)$ is determined by Betti numbers of $M$, and so-called elongations of $M$. 
Result: $P_{M,i}(Z)$ is determined by Betti numbers of $M$, and so-called elongations of $M$. As a simple illustration we show that the constant term of $P_{M,n}(Z)$ is equal to

$$(-1)^{n-r} \beta_{n-r(M),n}.$$
Recall that

\[ P_{M,n}(Z) = \sum_{\gamma \subseteq E} (-1)^{|\gamma|} Z^{n_M(\gamma)}, \]

and note that the constant term of this polynomial is

\[ \sum_{n_M(\gamma)=0} (-1)^{|\gamma|} = (-1)^{n+1} \chi(M), \]

where \( \chi(M) \) is the reduced Euler characteristic of \( M \). Let \( f_i \) be the number of independent sets in \( M \) of cardinality \( i \).

\[ \chi(M) = -1 + f_1 - f_2 + \cdots + (-1)^{r(M)-1} f_{r(M)}. \]
Let $H_i(M; \mathbb{K})$ denote the $i^{th}$ reduced homology of $M$ over $\mathbb{K}$. According to Björner (1992), we have

$$(-1)^{n+1} \chi(M) = (-1)^{n-r(M)} \dim H_{r(M)-1}(M).$$

From Hochster’s formula, we see that the dimension of $H_{r(M)-1}(M)$ is equal to $\beta_{n-r(M), n((S/I)_M)}$, thus

$$\sum_{n_M(\gamma)=0} (-1)^{|\gamma|} = (-1)^{n+1} \chi(M)$$

$$= (-1)^{n-r(M)} \dim H_{r(M)-1}(M)$$

$$= (-1)^{n-r(M)} \beta_{n-r(M), n((S/I)_M)},$$

which was what we wanted to prove.

Hochster’s formula

$$\beta_{i,\sigma}((S/I)_M) = \dim H_{|\sigma|-i-1}(M|_{\sigma}).$$
To find the remaining coefficients of $P_{M,n}$, we shall need the Betti numbers of so called *elongations* of $M$:  

The elongation $M_i$ of $M$ to rank $r$ ($M_i(r)$). For $1 \leq i \leq n - r$, the set $I_i = \{ \sigma \in E : n(\sigma) \leq i \}$ forms the set of independent sets of a matroid $M_i$ on $E$. Note that $M = M_0$. 
To find the remaining coefficients of $P_{M,n}$, we shall need the Betti numbers of so called *elongations* of $M$:
The elongation $M_i$ of $M$ to rank $r(M) + i$. For $1 \leq i \leq n - r(M)$, the set $I_i = \{ \sigma \in E : n(\sigma) \leq i \}$ forms the set of independent sets of a matroid $M_i$ on $E$. Note that $M = M_0$. 
Let $n_{Mi}$ denote the nullity function of $M_i$. Then, for $\sigma \subseteq E$, we have

$$n_{Mi}(\sigma) = \max\{n(\sigma) - i, 0\}$$

We shall make use of the following observation:

$$n^{-1}(I) = n^{-1}_{Mi}(0) \setminus n_{Mi-1}^{-1}(0).$$
Let \( \beta^{(i)} \) distinguish the Betti numbers of \( M_i \) from those of \( M(=M_0) \). (And let \( \beta^{(i)} = 0 \) whenever \( i \not\in [i, n - r(M)] \).)

**Proposition**

The coefficient of \( Z^l \) in \( P_{M,n}(Z) \) is equal to

\[
(-1)^{n-r-l} \left[ \beta^{(l-1)}_{n-r-l+1,n} + \beta^{(l-1)}_{n-r-l,n} \right].
\]

Since

\[
P_{M,n}(Z) = (-1)^n \sum_{\gamma \subseteq E} (-1)^{|\gamma|} Z^{n_M(\gamma)},
\]

it is clear that the coefficient of \( Z^l \) is equal to

\[
(-1)^n \sum_{n_M(\gamma) = l} (-1)^{|\gamma|}.
\]

But

\[
\sum_{n_M(\gamma) = l} (-1)^{|\gamma|} = \left[ \sum_{n_M(\gamma) = l} (-1)^{|\gamma|} - \sum_{n_{M_l}(\gamma) = 0} (-1)^{|\gamma|} \right],
\]

and the result follows as for the constant term.
To find $P_{M,j}$ for $0 \leq j \leq n-1$ we proceed in an analogous fashion; first we find the constant term for each $j$, then we use the elongations to find the remaining coefficients. The final results being:

[J-R-V]:

**Theorem**

For $1 \leq j \leq n$ the coefficient of $Z^l$ in $P_{M,j}$ is equal to

$$\sum_{i=0}^{n} (-1)^{i+1} \left( \beta_{i+1,j}^{(l-1)} - \beta_{i+1,j}^{(l)} \right).$$

**Corollary**

Let $C$ be an $[n, k]$-code over $\mathbb{F}_q$. For $1 \leq j \leq n$, the number of words of weight $j$ in $C \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$ is

$$\sum_{l=0}^{k} \left( \sum_{i=0}^{n} (-1)^{i+1} \left( \beta_{i+1,j}^{(l-1)} - \beta_{i+1,j}^{(l)} \right) \right) (q^m)^l.$$
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Construction of codes from algebraic geometry

— $X$ be any subset of the projective space $\mathbb{P}^{k-1}$ over $\mathbb{F}_q$, for example the set of $\mathbb{F}_q$-rational points of a projective variety defined over $\mathbb{F}_q$. 

— We define the corresponding code to be the row-space of the matrix $G = \begin{pmatrix} P_1,1 & P_2,1 & \cdots & P_n,1 \\ P_1,2 & P_2,2 & \cdots & P_n,2 \\ \vdots & \vdots & \ddots & \vdots \\ P_1,k & P_2,k & \cdots & P_n,k \end{pmatrix}$.

The code is only defined up to equivalence, but the code parameters are the same up to such equivalence.

— How are the $d_i$ determined by the geometry of $X$?
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— How are the $d_i$ determined by the geometry of $X$?
Find the $d_i$ from properties of $X$

For each $h = 1, \ldots, k$, let

\[ J_h = \max \{ \# \mathbb{F}_q \text{-rational points on } X \text{ in } S \mid S \text{ is a codim. } h \text{ subspace in } \mathbb{P}^{k-1}_q \} \].

We recall: $n = \# \mathbb{F}_q \text{-rational points on } X$.

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$$d_h = n - J_h. \quad (1)$$

Dually: If instead we use $G$ as a parity check matrix $H$ for $C$, then:

$d_i$ is the smallest $t$ such that there exist $t$ points of $X$ only spanning a $(t - i - 1)$-dimensional subspace of $\mathbb{P}^{k-1}_q$.

Reflects the existence of $t$-secant $(t - i - 1)$-planes.
Simplex codes

Let $X$ be the set of all points in a projective space $\mathbb{P}^s$. This gives the so-called simplex code $S_q(s)$. Then all codimension $r$ spaces contain the same number of points, and this code is therefore of constant weight.

$$d_r = \frac{q^s - 1}{q - 1} - \frac{q^{s-r} - 1}{q - 1}.$$
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The Betti-numbers follow from the Herzog-Kuhl equations.

Example with $\mathbb{R}_2(3)$. Use

$$\beta_i, d_i = \prod_{j \neq 0, i} \left| \frac{d_j - d_0}{d_j - d_i} \right|,$$

for $(d_1 = 4, d_2 = 6, d_3 = 7)$. we obtain

$$0 \leftarrow R_M \leftarrow S \leftarrow S(-4)^7 \leftarrow S(-6)^{14} \leftarrow S(-7)^8 \leftarrow 0.$$
The two (only interesting) elongation matroids are uniform, and their Betti-tables are then known. Putting the information from all Betti-tables together and using the Corollary by [J-R-V] above one finds the $A_i(q^m)$ (number of code words of weight $m$ over $\mathbb{F}_q$) for all $m$, as already done by other methods, e.g. in Example 14 of "Weight enumeration of codes from finite spaces", Jurrius, DCC, 2012. One can then also find the higher weight enumerator polynomials of the code by Theorem 3 in that article.

(The matroid simplicial complexes of) Reed-Muller codes of the first order are not of constant weight (2 weights), but have pure resolutions, as have their elongations, and they may be treated in a very similar way.