# PRIMARY DECOMPOSITION OF MODULES 

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#### Abstract

Basics of the theory of primary decomposition of modules are outlined here. The notions of graded rings and modules are also introduced, and then the special features of primary decomposition in the case of graded modules are discussed.


## Contents

Introduction ..... 2

1. Associated Primes ..... 2
2. Primary Decomposition ..... 5
3. Graded Rings and Modules ..... 7
4. Primary Decomposition in Graded Modules ..... 11
References ..... 13

This is a slightly revised version of the second chapter of the Lecture Notes of the NBHM sponsored Instructional Conference on Combinatorial Topology and Algebra, held at IIT Bombay in December 1993. As such, these notes may be viewed as a continuation of [Gh], which was a revised version of the first chapter.

## Introduction

In the first two sections, we will discuss an extension of the notions of associated primes and primary decomposition to the case of modules. The classical case of ideals $I$ in (noetherian) rings $A$ corresponds, in this general set-up, to the case of $A$-modules $A / I$; remembering this may be helpful in understanding some of the concepts and results below. Later, in Section 3, we describe the notion of graded rings and graded modules and prove some basic results concerning them. Finally, in Section 4 we discuss some special properties of primary decomposition in the case of graded modules over graded rings.

## 1. Associated Primes

Taking into consideration the modern viewpoint that the notion of associated primes is more fundamental than primary decomposition, we shall derive in this section basic results about associated primes without mentioning primary submodules or primary decomposition.

Throughout this section, we let $A$ denote a ring and $M$ an $A$-module. By $\operatorname{Spec} A$ we shall denote the set of all prime ideals of $A ; \operatorname{Spec} A$ is sometimes called the spectrum ${ }^{1}$ of $A$.
Definition: A prime ideal $\mathfrak{p}$ of $A$ is called an associated prime of $M$ if $\mathfrak{p}=(0: x)$ for some $x \in M$. The set of all associated primes of $M$ is denoted by $\operatorname{Ass}_{A}(M)$, or simply by $\operatorname{Ass}(M)$. Minimal elements of $\operatorname{Ass}(M)$ are called the minimal primes of $M$, and the remaining elements of $\operatorname{Ass}(M)$ are called the embedded primes of $M$.

Note that if $\mathfrak{p}=(0: x) \in \operatorname{Ass}(M)$, then the map $a \mapsto a x$ of $A \rightarrow M$ defines an embedding (i.e., an injective $A$-module homomorphism) $A / \mathfrak{p} \hookrightarrow M$. Conversely, if for $\mathfrak{p} \in \operatorname{Spec} A$, we have an embedding $A / \mathfrak{p} \hookrightarrow M$, then clearly $\mathfrak{p} \in \operatorname{Ass}(M)$. It may also be noted that if $M$ is isomorphic to some $A$-module $M^{\prime}$, then $\operatorname{Ass}(M)=\operatorname{Ass}\left(M^{\prime}\right)$.
(1.1) Lemma. Any maximal element of $\{(0: y): y \in M, y \neq 0\}$ is a prime ideal. In particular, if $A$ is noetherian, then $\operatorname{Ass}(M) \neq \emptyset$ iff $M \neq 0$.
Proof: Suppose $(0: x)$ is a maximal element of $\{(0: y): y \in M, y \neq 0\}$. Then $(0: x) \neq A$ since $x \neq 0$. Moreover, if $a, b \in A$ are such that $a b \in(0: x)$ and $a \notin(0: x)$, then $a x \neq 0$ and $b \in(0: a x) \subseteq(0: x)$. Since $(0: x)$ is maximal, $(0: a x)=(0: x)$. Thus $b \in(0: x)$. Thus $(0: x)$ is a prime ideal. The last assertion is evident.

Definition: An element $a \in A$ is said to be a zerodivisor of $M$ if $(0: a)_{M} \neq 0$, i.e., if $a x=0$ for some $x \in M$ with $x \neq 0$. The set of all zerodivisors of $M$ is denoted by $\mathcal{Z}(M)$.
(1.2) Exercise: Check that the above definition of $\mathcal{Z}(M)$ is consistent with that in (2.4) of [Gh]. Show that if $A$ is noetherian, then $\mathcal{Z}(M)=\bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$.

[^0](1.3) Lemma. For any submodule $N$ of $M, \operatorname{Ass}(N) \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M / N)$. More generally, if $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq M_{n}=M$ is any chain of submodules of $M$, then $\operatorname{Ass}(M) \subseteq \cup_{i=1}^{n} \operatorname{Ass}\left(M_{i} / M_{i-1}\right)$.
Proof: The inclusion $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$ is obvious. Let $x \in M$ be such that $(0: x) \in \operatorname{Ass}(M)$. If $(0: x) \notin \operatorname{Ass}(N)$, then we claim that $(0: x)=(0: \bar{x}) \in \operatorname{Ass}(M / N)$, where $\bar{x}$ denotes the image of $x$ in $M / N$. To see this, note that $(0: x) \subseteq(0: \bar{x})$ and if $a \in A$ is such that $a \bar{x}=0 \neq a x$, then $a x \in N$ and $a \notin(0: x)$, and since $(0: x)$ is prime, we have $b \in(0: a x) \Leftrightarrow b a \in(0: x) \Leftrightarrow b \in(0: x) ;$ consequently, $(0: x)=(0: a x) \in \operatorname{Ass}(N)$, which is a contradiction. Thus $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M / N)$. The last assertion follows from this by induction on $n$.

The inclusions $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M)$ and $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(N) \cup \operatorname{Ass}(M / N)$ in the above Lemma are, in general, proper. This may be seen, for instance, when $A$ is a domain and $M=A$ by taking $N=0$ and $N=$ a nonzero prime ideal of $A$, respectively.
(1.4) Corollary. Suppose $M_{1}, \ldots, M_{h}$ are $A$-modules such that $M \simeq \oplus_{i=1}^{h} M_{i}$. Then $\operatorname{Ass}(M)=\cup_{i=1}^{h} \operatorname{Ass}\left(M_{i}\right)$.
Proof: Follows using induction on $h$ by noting that the case of $h=2$ is a consequence of the first assertion in (1.3).

Example: Let $G$ be a finite abelian group of order $n$. Suppose $n=p_{1}^{e_{1}} \ldots p_{h}^{e_{h}}$, where $p_{1}, \ldots, p_{h}$ are distinct prime numbers and $e_{1}, \ldots, e_{h}$ are positive integers. Then $G$ is a $\mathbb{Z}$-module, and the $p_{i}$-Sylow subgroups $P_{i}, 1 \leq i \leq h$, are $\mathbb{Z}$-submodules of $G$ such that $G \simeq P_{1} \oplus \cdots \oplus P_{h}$. Clearly, $\operatorname{Ass}\left(P_{i}\right)=p_{i} \mathbb{Z}$ [indeed, if $y$ is any nonzero element of $P_{i}$ of order $p_{i}^{e}$, then elements of $(0: y)$ are multiples of $p_{i}^{e}$, and if $x=p_{i}^{e-1} y$, then $\left.(0: x)=p_{i} \mathbb{Z}\right]$. Thus by $(1.4), \operatorname{Ass}(G)=$ $\left\{p_{1} \mathbb{Z}, \ldots, p_{h} \mathbb{Z}\right\}$. More generally, if $M$ is a finitely generated abelian group, then $M=\mathbb{Z}^{r} \oplus T$ for some $r \geq 0$ and some finite abelian group $T$, and it follows from (1.4) that if $r>0$ then $\operatorname{Ass}(M)=\left\{l_{0} \mathbb{Z}, l_{1} \mathbb{Z}, \ldots, l_{s} \mathbb{Z}\right\}$, where $l_{0}=0$ and $l_{1}, \ldots l_{s}$ are the prime numbers dividing the order of $T$.

Having discussed some properties of associated primes of quotient modules, we now describe what happens to associated prime upon localisation. Before that, let us recall that if $S$ is m. c. subset of $A$, then the map $\mathfrak{p} \mapsto S^{-1} \mathfrak{p}$ gives a one-to-one correspondence of $\{\mathfrak{p} \in \operatorname{Spec} A: \mathfrak{p} \cap S=$ $\emptyset\}$ onto Spec $S^{-1} A$.
(1.5) Lemma. Suppose $A$ is noetherian and $S$ is a m.c. subset of $A$. Then

$$
\operatorname{Ass}_{S^{-1} A}\left(S^{-1} M\right)=\left\{S^{-1} \mathfrak{p}: \mathfrak{p} \in \operatorname{Ass}(M) \text { and } \mathfrak{p} \cap S=\emptyset\right\}
$$

Proof: If $\mathfrak{p} \in \operatorname{Ass}(M)$ and $S \cap \mathfrak{p}=\emptyset$, then we have an embedding $A / \mathfrak{p} \hookrightarrow M$, which, in view of (1.2) of [Gh], induces an embedding $S^{-1} A / S^{-1} \mathfrak{p} \hookrightarrow S^{-1} M$. And $S^{-1} \mathfrak{p} \in \operatorname{Spec} S^{-1} A$ since $S \cap \mathfrak{p}=\emptyset$. Thus $S^{-1} \mathfrak{p} \in \operatorname{Ass}_{S^{-1} A}\left(S^{-1} M\right)$. On the other hand, if $\mathfrak{p}^{\prime} \in \operatorname{Ass}_{S^{-1} A}\left(S^{-1} M\right)$, then $\mathfrak{p}^{\prime}=S^{-1} \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \cap S=\emptyset$, and $\mathfrak{p}^{\prime}=\left(0: \frac{x}{1}\right)$ for some $x \in M$. Write $\mathfrak{p}=\left(a_{1}, \ldots, a_{n}\right)$. Now $\frac{a_{i}}{1} \cdot \frac{x}{1}=0$ in $S^{-1} M$ for $1 \leq i \leq n$, and thus there exists $t \in S$ such that $t a_{i} x=0$ for $1 \leq i \leq n$. This implies that $\mathfrak{p} \subseteq(0: t x)$. Further, if $a \in(0: t x)$, then $\frac{a}{1} \in\left(0: \frac{x}{1}\right)=S^{-1} \mathfrak{p}$ so that $s a \in \mathfrak{p}$ for some $s \in S$, and hence $a \in \mathfrak{p}$. Thus $\mathfrak{p} \in \operatorname{Ass}(M)$.

Note that the above result implies that if $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p} A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}}\left(M_{\mathfrak{p}}\right)$. So, in particular, $M_{\mathfrak{p}} \neq 0$.

Definition: The set $\left\{\mathfrak{p} \in \operatorname{Spec} A: M_{\mathfrak{p}} \neq 0\right\}$ is called the support of $M$ and is denoted by Supp ( $M$ ).
(1.6) Lemma. $\operatorname{Supp}(M) \subseteq\{\mathfrak{p} \in \operatorname{Spec} A: \mathfrak{p} \supseteq \operatorname{Ann}(M)\}$. Moreover, if $M$ is f.g., then these two sets are equal.

Proof: If $\mathfrak{p} \in \operatorname{Spec} A$ and $M_{\mathfrak{p}} \neq 0$, then there is $x \in M$ such that $\frac{x}{1} \neq 0$ in $M_{\mathfrak{p}}$. Now $a \in \operatorname{Ann}(M) \Rightarrow a x=0 \Rightarrow a \notin A \backslash \mathfrak{p} \Rightarrow a \in \mathfrak{p}$. Thus $\mathfrak{p} \supseteq \operatorname{Ann}(M)$. Next, suppose $M$ is f.g. and $\mathfrak{p} \in \operatorname{Spec} A$ contains $\operatorname{Ann}(M)$. Write $M=A x_{1}+\cdots+A x_{n}$. If $M_{\mathfrak{p}}=0$, we can find $a \in A \backslash \mathfrak{p}$ such that $a x_{i}=0$ for $1 \leq i \leq n$. But then $a M=0$, i.e., $a \in \operatorname{Ann}(M)$, which is a contradiction.
(1.7) Theorem. Suppose $A$ is noetherian and $M$ is finitely generated. Then there exists a chain $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq M_{n}=M$ of submodules of $M$ such that $M_{i} / M_{i-1} \simeq A / \mathfrak{p}_{i}$, for some $\mathfrak{p}_{i} \in \operatorname{Spec} A(1 \leq i \leq n)$. Moreover, for any such chain of submodules, we have $\operatorname{Ass}(M) \subseteq\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}\right\} \subseteq \operatorname{Supp}(M)$; furthermore, the minimal elements of these three sets coincide.

Proof: The case of $M=0$ is trivial. If $M \neq 0$, then there exists $\mathfrak{p}_{1} \in \operatorname{Spec} A$ such that $A / \mathfrak{p}_{1}$ is isomorphic to a submodule $M_{1}$ of $M$. If $M_{1} \neq M$, we apply the same argument to $M / M_{1}$ to find $\mathfrak{p}_{2} \in \operatorname{Spec} A$, and a submodule $M_{2}$ of $M$ such that $M_{2} \supseteq M_{1}$ and $A / \mathfrak{p}_{2} \simeq M_{2} / M_{1}$. By (3.1), $M$ has no strictly ascending chain of submodules, and therefore the above process must terminate. This yields the first assertion. Moreover, $\operatorname{Ass}\left(M_{i} / M_{i-1}\right)=\operatorname{Ass}\left(A / \mathfrak{p}_{i}\right)=\left\{\mathfrak{p}_{i}\right\}$, and so by (1.3), we see that $\operatorname{Ass}(M) \subseteq\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}\right\}$. Also, in view of (1.2) of [Gh], we have $\left(M_{i} / M_{i-1}\right)_{\mathfrak{p}_{i}} \simeq A_{\mathfrak{p}_{i}} / \mathfrak{p}_{i} A_{\mathfrak{p}_{i}} \neq 0$. Hence $\left(M_{i}\right)_{\mathfrak{p}_{i}} \neq 0$. Thus $\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}\right\} \subseteq \operatorname{Supp}(M)$. Lastly, if $\mathfrak{p} \in \operatorname{Supp}(M)$, then $M_{\mathfrak{p}} \neq 0$ and so $\operatorname{Ass}_{A_{\mathfrak{p}}}\left(M_{\mathfrak{p}}\right) \neq \emptyset$. Now (1.5) shows that there exists $\mathfrak{q} \in \operatorname{Ass}(M)$ with $\mathfrak{q} \cap(A \backslash \mathfrak{p})=\emptyset$, i.e., $\mathfrak{q} \subseteq \mathfrak{p}$. This implies the last assertion.
(1.8) Corollary. If $A$ is noetherian and $M$ is f.g., then $\operatorname{Ass}(M)$ is finite. Furthermore, the minimal primes of $M$ are precisely the minimal elements among the prime ideals of $A$ containing $\operatorname{Ann}(M)$.

Proof: Follows from (1.7) in view of (1.6).
Remark: A chain $0=M_{0} \subseteq M_{1} \subseteq \cdots \subseteq M_{n}=M$ of submodules of $M$ is sometimes called a filtration of $M$. Using a filtration as in (1.7), it is often possible to reduce questions about modules to questions about integral domains.
(1.9) Exercise: Show that if $A$ is noetherian, $M$ is f.g., and $I$ is an ideal of $A$ consisting only of zerodivisors of $M$, then there exists some $x \in M$ such that $x \neq 0$ and $I x=0$.
(1.10) Exercise: Show that if $A$ is noetherian and $M$ is f.g., then

$$
\sqrt{\operatorname{Ann}(M)}=\bigcap_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}=\bigcap_{\mathfrak{p} \in \operatorname{Supp}(M)} \mathfrak{p}=\bigcap_{\substack{\mathfrak{p} \text { a minimal } \\ \text { prime of } M}} \mathfrak{p} .
$$

## 2. Primary Decomposition

We continue to let $A$ denote a ring and $M$ an $A$-module. As in the case of ideals, the primary decomposition of modules into primary submodules will be achieved using the auxiliary notion of irreducible submodules. To compare the notions and results discussed in this section to those in the classical case, you may substitute $A$ for $M$.
Definition: Let $Q$ be a submodule of $M$. We say that $Q$ is primary if $Q \neq M$ and for any $a \in A$ and $x \in M$, we have

$$
a x \in Q \text { and } x \notin Q \Longrightarrow a^{n} M \subseteq Q \text { for some } n \geq 1 .
$$

We say that $Q$ is irreducible if $Q \neq M$ and for any submodules $N_{1}$ and $N_{2}$ of $M$ we have

$$
Q=N_{1} \cap N_{2} \Longrightarrow Q=N_{1} \text { or } Q=N_{2} .
$$

Clearly, a submodule $Q$ of $M$ is primary iff every zerodivisor of $M / Q$ is nilpotent for $M / Q$. [An element $a \in A$ is said to be nilpotent for $M$ if $a^{n} M=0$ for some $n \geq 1$. In other words, $a$ is nilpotent for $M$ iff $a \in \sqrt{\operatorname{Ann}(M)}$.] If $Q$ is a primary submodule of $M$ and $\mathfrak{p}=\sqrt{\operatorname{Ann}(M / Q)}$, we say that $Q$ is $\mathfrak{p}$-primary.
(2.1) Exercise: Given any submodule $Q$ of $M$, show that $\mathcal{Z}(M / Q)=\sqrt{\operatorname{Ann}(M / Q)} \Longleftrightarrow Q$ is primary $\Longrightarrow \operatorname{Ann}(M / Q)$ is a primary ideal of $A$ Use (1.2), (1.8) and (1.10) to deduce the following characterization.

If $A$ is noetherian and $M$ is f.g., then: $Q$ is primary $\Longleftrightarrow \operatorname{Ass}(M / Q)$ is singleton. And also the following characterization.

If $A$ is noetherian and $M$ is f.g., then: $Q$ is $\mathfrak{p}$-primary $\Longleftrightarrow \operatorname{Ass}(M / Q)=\{\mathfrak{p}\}$.
As we shall see in the sequel, the above characterization of primary submodules [of f.g. modules over noetherian rings] is extremely useful. For this reason perhaps, it is sometimes taken as a definition of primary submodules [of arbitrary modules]. At any rate, we may tacitly use the above characterizations of primary and $\mathfrak{p}$-primary submodules in several of the proofs below.
(2.2) Lemma. Suppose $A$ is noetherian, $M$ is f.g., and $Q_{1}, \ldots, Q_{r}$ are $\mathfrak{p}$-primary submodules of $M$, where $r$ is a positive integer. Then $Q_{1} \cap \cdots \cap Q_{r}$ is also $\mathfrak{p}$-primary.

Proof: Clearly, $Q_{1} \cap \cdots \cap Q_{r} \neq M$. Moreover, there is a natural injective homomorphism of $M / Q_{1} \cap \cdots \cap Q_{r}$ into $M / Q_{1} \oplus \cdots \oplus M / Q_{r}$. Therefore, in view of (1.1), (1.3) and (1.4), we see that

$$
\emptyset \neq \operatorname{Ass}\left(M / Q_{1} \cap \cdots \cap Q_{r}\right) \subseteq \operatorname{Ass}\left(\oplus_{i=1}^{r} M / Q_{i}\right)=\cup_{i=1}^{r} \operatorname{Ass}\left(M / Q_{i}\right)=\{\mathfrak{p}\}
$$

Thus it follows from (2.1) that $Q_{1} \cap \cdots \cap Q_{r}$ is $\mathfrak{p}$-primary.
(2.3) Lemma. If $M$ is noetherian, then every submodule of $M$ is a finite intersection of irreducible submodules of $M$.

Proof: Assume the contrary. Then we can find a maximal element, say $Q$, among the submodules of $M$ which aren't finite intersections of irreducible submodules of $M$. Now $Q$ can't be irreducible. Also $Q \neq M$ (because $M$ is the intersection of the empty family of irreducible submodules of $M$ ). Hence $Q=N_{1} \cap N_{2}$ for some submodules $N_{1}$ and $N_{2}$ of $M$ with $N_{1} \neq Q$ and $N_{2} \neq Q$. By maximality of $Q$, both $N_{1}$ and $N_{2}$ are finite intersections of irreducible submodules of $M$. But then so is $Q$, which is a contradiction.
(2.4) Lemma. Suppose $A$ is noetherian, $M$ is f.g., and $Q$ is an irreducible submodule of $M$. Then $Q$ is primary.

Proof: Since $Q \neq M, \operatorname{Ass}(M / Q) \neq \emptyset$. Suppose $\operatorname{Ass}(M / Q)$ contains two distinct prime ideals $\mathfrak{p}_{1}=\left(0: \bar{x}_{1}\right)$ and $\mathfrak{p}_{2}=\left(0: \bar{x}_{2}\right)$, where $\bar{x}_{1}, \bar{x}_{2}$ denote the images in $M / Q$ of some elements $x_{1}, x_{2}$ of $M$. Clearly $\bar{x}_{1}$ and $\bar{x}_{2}$ are nonzero elements of $M / Q$. We claim that $A \bar{x}_{1} \cap A \bar{x}_{2}=\{0\}$ Indeed, if $a \bar{x}_{1}=b \bar{x}_{2}$, with $a, b \in A$, is nonzero, then $a \notin\left(0: \bar{x}_{1}\right)$ and $b \notin\left(0: \bar{x}_{2}\right)$. Since $\left(0: \bar{x}_{1}\right)$ is prime, we find that $\left(0: \bar{x}_{1}\right)=\left(0: a \bar{x}_{1}\right)$ (check!). Similarly, $\left(0: \bar{x}_{2}\right)=\left(0: b \bar{x}_{2}\right)$. This gives $\mathfrak{p}_{1}=\mathfrak{p}_{2}$, which is a contradiction. Now if $y \in\left(Q+A x_{1}\right) \cap\left(Q+A x_{2}\right)$, then $y=y_{1}+a x_{1}=y_{2}+b x_{2}$ for some $y_{1}, y_{2} \in Q$ and $a, b \in A$. But then $a \bar{x}_{1}=b \bar{x}_{2}$ in $M / Q$ and thus $y \in Q$. It follows that $Q=\left(Q+A x_{1}\right) \cap\left(Q+A x_{2}\right)$. Also since $\bar{x}_{1} \neq 0 \neq \bar{x}_{2}$, we have $\left(Q+A x_{1}\right) \neq Q \neq\left(Q+A x_{2}\right)$. This contradicts the irreducibility of $Q$. Thus $\operatorname{Ass}(M / Q)$ is singleton so that $Q$ is primary.
(2.5) Lemma. Suppose $A$ is noetherian, $M$ is f.g., $Q$ is a $\mathfrak{p}$-primary submodule of $M$. Then the inverse image of $Q_{\mathfrak{p}}$ under the natural map $M \rightarrow M_{\mathfrak{p}}$ (given by $x \mapsto \frac{x}{1}$ ) is $Q$.
Proof: Suppose $x \in M$ is such that $\frac{x}{1} \in Q_{\mathfrak{p}}$. Then $t x \in Q$ for some $t \in A \backslash \mathfrak{p}$. If $x \notin Q$, then $\bar{x}$, the image of $x$ in $M / Q$, is nonzero, and thus $t \in \mathcal{Z}(M / Q)$. Hence from (2.1), we see that $t \in \mathfrak{p}$, which is a contradiction.
(2.6) Remark: Given any $\mathfrak{p} \in \operatorname{Spec} A$ and a submodule $Q^{\prime}$ of $M_{\mathfrak{p}}$, the inverse image of $Q^{\prime}$ under the natural map $M \rightarrow M_{\mathfrak{p}}$ is often denoted by $Q^{\prime} \cap M$. Thus (2.5) can be expressed by saying that if $Q$ is a $\mathfrak{p}$-primary submodule of $M$, then $Q_{\mathfrak{p}} \cap M=Q$. Note that we have been tacitly using the fact that if $Q$ is any submodule of $M$ and $S$ is any m.c. subset of $A$, then $S^{-1} Q$ can be regarded as a submodule of $S^{-1} M$.
(2.7) Theorem. Suppose $A$ is noetherian, $M$ is f.g., and $N$ is any submodule of $M$. Then we have
(i) There exist primary submodules $Q_{1}, \ldots, Q_{h}$ of $M$ such that $N=Q_{1} \cap \cdots \cap Q_{h}$.
(ii) In (i) above, $Q_{1}, \ldots, Q_{h}$ can be chosen such that $Q_{i} \nsupseteq \cap_{j \neq i} Q_{j}$ for $1 \leq i \leq h$, and $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$ are distinct, where $\mathfrak{p}_{i}=\sqrt{\operatorname{Ann}\left(M / Q_{i}\right)}$.
(iii) If $Q_{i}$ and $\mathfrak{p}_{i}$ are as in (ii) above, then $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$ are unique; in fact, $\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}\right\}=$ $\operatorname{Ass}(M / N)$. Moreover, if $\mathfrak{p}_{i}$ is minimal among $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$, i.e. $\mathfrak{p}_{i} \nsupseteq \mathfrak{p}_{j}$ for $j \neq i$, then the corresponding primary submodule $Q_{i}$ is also unique; in fact, $Q_{i}=N_{\mathfrak{p}_{i}} \cap M$.

Proof: Clearly, (i) is a direct consequence of (2.3) and (2.4). Given a decomposition as in (i), we can use (2.2) to reduce it by grouping together the primary submodules having the same associated prime so as to ensure that the associated primes become distinct. Then
we can successively remove the primary submodules contained in the intersections of the remaining submodules. This yields (ii). Now let $Q_{1}, \ldots, Q_{h}$ be as in (ii). Fix some $i$ with $1 \leq i \leq h$. Let $P_{i}=\cap_{j \neq i} Q_{i}$. Clearly, $N \subset P_{i}$ and $N \neq P_{i}$. Thus we have $0 \neq P_{i} / N=$ $P_{i} / P_{i} \cap Q_{i} \simeq P_{i}+Q_{i} / Q_{i} \hookrightarrow M / Q_{i}$, and hence, in view of (1.1) and (1.3), we find that $\emptyset \neq$ $\operatorname{Ass}\left(P_{i} / N\right) \subseteq \operatorname{Ass}\left(M / Q_{i}\right)=\left\{\mathfrak{p}_{i}\right\}$. Thus $\left\{\mathfrak{p}_{i}\right\}=\operatorname{Ass}\left(P_{i} / N\right) \subseteq \operatorname{Ass}(M / N)$. On the other hand, since $N=Q_{1} \cap \cdots \cap Q_{h}, M / N$ is isomorphic to a submodule of $\oplus_{j=1}^{h} M / Q_{j}$, and so by (1.4), $\operatorname{Ass}(M / N) \subseteq \cup_{j=1}^{h} \operatorname{Ass}\left(M / Q_{j}\right)=\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}\right\}$. This proves that $\operatorname{Ass}(M / N)=\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}\right\}$. In particular, $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$ are unique. Now suppose, without loss of generality, that $\mathfrak{p}_{1}$ is minimal among $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$. Then for $j>1, \mathfrak{p}_{1} \nsupseteq \mathfrak{p}_{j}$, i.e., $\left(A \backslash \mathfrak{p}_{1}\right) \cap \mathfrak{p}_{j} \neq \emptyset$, and hence by (1.5), we find that $\operatorname{Ass}_{A_{\mathfrak{p}_{1}}}\left(\left(M / Q_{j}\right)_{\mathfrak{p}_{1}}\right)=\emptyset$; thus by (1.3), $\left(M / Q_{j}\right)_{\mathfrak{p}_{1}}=0$, i.e., $M_{\mathfrak{p}_{1}}=\left(Q_{j}\right)_{\mathfrak{p}_{1}}$. It follows that $N_{\mathfrak{p}_{1}}=\left(Q_{1}\right)_{\mathfrak{p}_{1}} \cap \cdots \cap\left(Q_{h}\right)_{\mathfrak{p}_{1}}=\left(Q_{1}\right)_{\mathfrak{p}_{1}}$, and, in view of (2.5) and (2.6), we obtain that $Q_{1}=\left(Q_{1}\right)_{\mathfrak{p}_{1}} \cap M=N_{\mathfrak{p}_{1}} \cap M$. This proves (iii).

Definition: A decomposition $N=Q_{1} \cap \cdots \cap Q_{h}$, as in (i) above is called a primary decomposition of $N$. If $Q_{1}, \ldots, Q_{h}$ satisfy the conditions in (ii), then it is called an irredundant (primary) decomposition of $N$.

Example: Let $G$ be a finite abelian group of order $n$. Let the notation be as in the Example preceding (1.5). For $1 \leq i \leq h$, let $Q_{i}=P_{1}+\cdots+P_{i-1}+P_{i+1}+\cdots+P_{h}$. Then $G / Q_{i} \simeq P_{i}$, and thus $Q_{i}$ is $p_{i} \mathbb{Z}$-primary. Observe that $(0)=Q_{1} \cap \cdots \cap Q_{h}$ is an irredundant primary decomposition of (0); in fact, this decomposition is unique because each of the associated primes $p_{1} \mathbb{Z}, \ldots, p_{h} \mathbb{Z}$ is clearly minimal. In general, if $N$ is a subgroup, i.e., a $\mathbb{Z}$-submodule, of $G$, and $\Lambda=\left\{i: 1 \leq i \leq h\right.$ and $\left.N+Q_{i} \neq G\right\}$, then $N=\cap_{i \in \Lambda}\left(N+Q_{i}\right)$ is an irredundant primary decomposition of $N$, and this too is unique. Verify!

It may be remarked that the examples of primary ideals, primary decomposition of ideals, etc., discussed in the last chapter, constitute examples in this general set-up as well. Thus the pathologies which arise in the case of ideals [see, for instance, the Remark and the Example preceding (2.3) of Ch. 1] continue to exist for f. g. modules over noetherian rings.

## 3. Graded Rings and Modules

In this section, we shall study some basic facts about graded rings and modules. This will allow us to discuss, in the next section, some special properties of associated primes and primary decomposition in the graded situation.

The prototype of a graded ring is the polynomial ring $A\left[X_{1}, \ldots, X_{n}\right]$ over a ring $A$. Before giving the general definition, we point out that the set of all nonnegative integers is denoted by $\mathbb{N}$.

Definition: A ring $R$ is said to be $\mathbb{N}$-graded, or simply graded, if it has additive subgroups $R_{d}$, for $d \in \mathbb{N}$, such that $R=\oplus_{d \in \mathbb{N}} R_{d}$ and $R_{d} R_{e} \subseteq R_{d+e}$, for all $d, e \in \mathbb{N}$. The family $\left\{R_{d}\right\}_{d \in \mathbb{N}}$ is called an $\mathbb{N}$-grading of $R$, and the subgroup $R_{d}$ is called the $d^{\text {th }}$ graded component of $R$.

Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring. Elements of $R_{d}$ are said to be homogeneous of degree $d$. Every $a \in R$ can be uniquely written as $a=\sum_{d \in \mathbb{N}} a_{d}$ with $a_{d} \in R_{d}$ such that all except finitely many $a_{d}$ 's are 0 ; we call $a_{d}$ 's to be the homogeneous components of $a$. Given a homogeneous element $b \in R$, we sometimes write $\operatorname{deg}(b)$ to denote its degree. An ideal of $R$ generated by
homogeneous elements is called a homogeneous ideal. Note that an ideal is homogeneous iff it contains the homogeneous components of each of its elements. If $I$ is a homogeneous ideal of $R$, then $R / I$ has the induced $\mathbb{N}$-grading given by $(R / I)_{d}=R_{d} /\left(I \cap R_{d}\right)$. A subring $S$ of $R$ is said to be graded if $S=\oplus_{d \in \mathbb{N}}\left(S \cap R_{d}\right)$. Note that $R_{0}$ is a graded subring of $R$ and each $R_{d}$ is an $R_{0}$-module. If $R^{\prime}=\oplus_{d \in \mathbb{N}} R_{d}^{\prime}$ is any graded ring, then a homomorphism $\phi: R \rightarrow R^{\prime}$ is called a graded ring homomorphism, or a homomorphism of graded rings, if $\phi\left(R_{d}\right) \subseteq R_{d}^{\prime}$ for all $d \in \mathbb{N}$; note that in this case the kernel of $\phi$ is a homogeneous ideal of $R$ and the image of $\phi$ is a graded subring of $R^{\prime}$.

Examples: 1. Let $A$ be a ring and $I$ be a homogeneous ideal of $A\left[X_{1}, \ldots, X_{n}\right]$. Then $R=A\left[X_{1}, \ldots, X_{n}\right] / I$ is a graded ring, with its $d^{\text {th }}$ graded component being given by $R_{d}=$ $A\left[X_{1}, \ldots, X_{n}\right]_{d} / I_{d}$, where $I_{d}=I \cap A\left[X_{1}, \ldots, X_{n}\right]_{d}$. Note that in this case $R$ as well as $R_{d}$ are f.g. $A$-algebras, and in particular, $A$-modules. Graded rings of this type are usually called graded $A$-algebras.
2. Let $A$ be a ring and $I$ be any ideal of $A$ such that $I \neq A$. Then $\operatorname{gr}_{I}(A) \stackrel{\text { def }}{=} \oplus_{d \in \mathbb{N}} I^{d} / I^{d+1}$ is a graded ring. It is called the associated graded ring of $A$ w.r.t. $I$.
(3.1) Exercise: Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring, and $I, J$ be homogeneous ideals of $R$. Then show that $I+J, I J, I \cap J,(I: J)$ and $\sqrt{I}$ are homogeneous ideals.
(3.2) Exercise: Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring, and $I$ be a homogeneous ideal of $R$. Show that $I$ is prime iff $I$ is prime in the graded sense, that is, $I \neq R$ and

$$
a, b \text { homogeneous elements of } R \text { and } a b \in I \Rightarrow a \in I \text { or } b \in I \text {. }
$$

Given an ideal $I$ of a graded ring $R=\oplus_{d \in \mathbb{N}} R_{d}$, we shall denote by $I^{*}$ the largest homogeneous ideal contained in $I$. Note that $I^{*}$ is precisely the ideal generated by the homogeneous elements in $I$.
(3.3) Lemma. If $R=\oplus_{d \in \mathbb{N}} R_{d}$ is a graded ring and $\mathfrak{p}$ is a prime ideal of $R$, then $\mathfrak{p}^{*}$ is a prime ideal of $R$.

Proof: Follows from (3.2).
(3.4) Lemma. Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring. Then $1 \in R_{0}$.

Proof: Write $1=\sum_{d \in \mathbb{N}} a_{d}$ with $a_{d} \in R_{d}$. Then for any $b \in R_{e}, b=\sum_{d \in \mathbb{N}} b a_{d}$. Equating the degree $e$ components, we find $b=b a_{0}$. This implies that $c a_{0}=1$ for all $c \in R$. Hence $1=a_{0} \in R_{0}$.

Given a graded ring $R=\oplus_{d \in \mathbb{N}} R_{d}$, we define $R_{+}=\oplus_{d>0} R_{d}$. Note that $R_{+}$is a homogeneous ideal of $R$ and $R / R_{+} \simeq R_{0}$. Thus if $R_{0}$ is a field, then $R_{+}$is a homogeneous maximal ideal of $R$; moreover, $R_{+}$is also maximal among all homogeneous ideals of $R$ other than $R$, and so it is the maximal homogeneous ideal of $R$.
(3.5) Lemma. Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring and $x_{1}, \ldots, x_{n}$ be any homogeneous elements of positive degree in $R$. Then $R_{+}=\left(x_{1}, \ldots, x_{n}\right)$ iff $R=R_{0}\left[x_{1}, \ldots, x_{n}\right]$. In particular, $R$ is noetherian iff $R_{0}$ is noetherian and $R$ is a f.g. $R_{0}$-algebra.

Proof: Suppose $R_{+}=\left(x_{1}, \ldots, x_{n}\right)$. We show by induction on $d$ that each $x \in R_{d}$ is in $R_{0}\left[x_{1}, \ldots, x_{n}\right]$. The case of $d=0$ is clear. Suppose $d>0$. Write $x=a_{1} x_{1}+\cdots+a_{n} x_{n}$, where $a_{1}, \ldots, a_{n} \in R$. Since $x$ is homogeneous, we may assume, without loss of generality, that each $a_{i}$ is homogeneous. Then for $1 \leq i \leq n$, we must have $\operatorname{deg}\left(a_{i}\right)=d-\operatorname{deg}\left(x_{i}\right)<d$, and so, by induction hypothesis, $a_{i} \in R_{0}\left[x_{1}, \ldots, x_{n}\right]$. It follows that $R=R_{0}\left[x_{1}, \ldots, x_{n}\right]$. Conversely, if $R=R_{0}\left[x_{1}, \ldots, x_{n}\right]$, then it is evident that $R_{+}=\left(x_{1}, \ldots, x_{n}\right)$. The second assertion in the Lemma follows from the first one by noting that f.g. algebras over noetherian rings are noetherian.
(3.6) Exercise: Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring. Show that $R$ is noetherian iff it satisfies a.c.c. on homogeneous ideals.
(3.7) Exercise: Let $A$ be a ring and $I=\left(a_{1}, \ldots, a_{r}\right)$. If $\bar{a}_{1}, \ldots, \bar{a}_{r}$ denote the images of $a_{1}, \ldots, a_{r} \bmod I^{2}$, then show that $\operatorname{gr}_{I}(A)=(A / I)\left[\bar{a}_{1}, \ldots, \bar{a}_{r}\right] \simeq(A / I)\left[X_{1}, \ldots, X_{r}\right] / J$, for some homogeneous ideal $J$ of $(A / I)\left[X_{1}, \ldots, X_{r}\right]$. Deduce that if $(A / I)$ is noetherian and $I$ is f. g., then $\operatorname{gr}_{I}(A)$ is noetherian.
(3.8) Graded Noether Normalisation Lemma. Let $k$ be an infinite field and $R=$ $k\left[x_{1}, \ldots, x_{n}\right]$ be a graded $k$-algebra such that $R_{0}=k$ and $\operatorname{deg}\left(x_{i}\right)=m$ for $1 \leq i \leq n$. Then there exist homogeneous elements $\theta_{1}, \ldots, \theta_{d}$ of degree $m$ in $R$ such that $\theta_{1}, \ldots, \theta_{d}$ are algebraically independent over $k$ and $R$ is integral over the graded subring $k\left[\theta_{1}, \ldots, \theta_{d}\right]$. In particular, $R$ is a finite $k\left[\theta_{1}, \ldots, \theta_{d}\right]$-module.
Proof: If $k$ is an infinite field, then in (4.13) of [Gh], the elements $\theta_{1}, \ldots, \theta_{d}$ can be chosen to be $k$-linear combinations of $x_{1}, \ldots, x_{n}$. The result follows.

Now let us turn to modules.
Definition: Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring, and $M$ be an $R$-module. Then $M$ is said to be $\mathbb{N}$-graded or simply, graded, if it contains additive subgroups $M_{d}$, for $d \in \mathbb{N}$, such that $M=\oplus_{d \in \mathbb{N}} M_{d}$, and $R_{d} M_{e} \subseteq M_{d+e}$, for all $d, e \in \mathbb{N}$. Such a family $\left\{M_{d}\right\}_{d \in \mathbb{N}}$ is called an $\mathbb{N}$-grading of $M$, and $M_{d}$ the $d^{\text {th }}$ graded component of $M$.

Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring and $M=\oplus_{d \in \mathbb{N}} M_{d}$ be a graded $R$-module. By a graded submodule of $M$ we mean a submodule $N$ of $M$ such that $N=\oplus_{d \in \mathbb{N}}\left(N \cap M_{d}\right)$. Note that if $N$ is a graded submodule of $M$, then $M / N$ is a graded $R$-module with the induced grading given by $(M / N)_{d}=M_{d} /\left(N \cap M_{d}\right)$. If $M^{\prime}=\oplus_{d \in \mathbb{N}} M_{d}^{\prime}$ is any graded $R$-module, then by graded module homomorphism, or a homomorphism of graded modules, we mean a $R$-module homomorphism $\phi: M \rightarrow M^{\prime}$ such that $\phi\left(M_{d}\right) \subseteq M_{d}^{\prime}$, for all $d \in \mathbb{N}$. For any such $\phi$, the kernel of $\phi$ and the image of $\phi$ are graded submodules of $M$ and $M^{\prime}$ respectively.

Examples: 1. Let $R=A\left[X_{1}, \ldots, X_{n}\right]$ and $I$ be a homogeneous ideal of $R$. Then $M=R / I$ is a graded $R$-module. In this example, the graded components $M_{d}$ are finite $A$-modules.
2. Let $A$ be a ring and $I$ be an ideal of $A$. If $M$ is any $A$-module, then $\operatorname{gr}_{I}(M)=$ $\oplus_{d \in \mathbb{N}} I^{d} M / I^{d+1} M$ is a graded $\operatorname{gr}_{I}(A)$-module. It is called the associated graded module of $M$ w.r.t. $I$.
(3.9) Exercise: Let the notation be as in (3.7). Let $M$ be an $A$-module. Suppose $M=$ $A x_{1}+\cdots+A x_{m}$, and $\bar{x}_{1}, \ldots, \bar{x}_{m}$ denote the images of $x_{1}, \ldots, x_{m}$ in $M / I M$, then show that $\operatorname{gr}_{I}(M)=\operatorname{gr}_{I}(A) \bar{x}_{1}+\cdots+\operatorname{gr}_{I}(A) \bar{x}_{m}$. Deduce that if $A$ is noetherian and $M$ is f.g., then $\operatorname{gr}_{I}(M)$ is a noetherian $\operatorname{gr}_{I}(A)$-module.
(3.10) Graded Nakayama's Lemma. Let $R=\oplus_{d \in \mathbb{N}} R_{d}$ be a graded ring and $M=\oplus_{d \in \mathbb{N}} M_{d}$ be a graded $R$-module. If $R_{+} M=M$, then $M=0$.

Proof: If $M \neq 0$, we can find a nonzero element $x$ of least degree in $M$. Now the assumption $x \in R_{+} M$ leads to a contradiction.
(3.11) Artin-Rees Lemma. Let $A$ be a noetherian ring, $I$ an ideal of $A$ and $M$ a f.g. A-module. Suppose $M=M_{0} \supseteq M_{1} \supseteq M_{2} \supseteq \cdots$ is a chain of submodules of $M$ such that $I M_{n} \subseteq M_{n+1}$ for all $n \geq 0$ with equality for $n \geq r$ (for some $r$ ). Then for any submodule $M^{\prime}$ of $M$, there exists $s \geq 0$ such that $I M_{n}^{\prime}=M_{n+1}^{\prime}$ for all $n \geq s$, where $M_{i}^{\prime}$ denotes $M^{\prime} \cap M_{i}$.

Proof: Let $t$ be an indeterminate over $A$ and let $R$ be the subring of $A[t]$ defined by $R=$ $\oplus_{n \geq 0} I^{n} t^{n}=A \oplus I t \oplus I^{2} t^{2} \oplus \cdots$. If $I=\left(a_{1}, \ldots, a_{m}\right)$, then $R=A\left[a_{1} t, \ldots, a_{m} t\right]$ and so $R$ is a f.g. graded $A$-algebra [called the Rees algebra of $I$ ]. Thus by (3.5), $R$ is noetherian. Moreover, $L=\oplus_{n \geq 0} M_{n} t^{n}$ is naturally a graded $R$-module. Now clearly $I M_{n}^{\prime} \subseteq M^{\prime} \cap I M_{n} \subseteq M_{n+1}^{\prime}$, and hence $L^{\prime}=\oplus_{n \geq 0} M_{n}^{\prime} t^{n}$ is an $R$-submodule of $L$. By (3.1) of [Gh], each $M_{n}$ is a f.g. $A$-module, and hence so is $\widetilde{M}_{n}=M_{0} \oplus M_{1} t \oplus \cdots \oplus M_{n} t^{n}$. Consequently, the $R$-module $\tilde{L}_{n}=M_{0} \oplus M_{1} t \oplus \cdots \oplus M_{n} t^{n} \oplus I M_{n} t^{n+1} \oplus I^{2} M_{n} t^{n+2} \oplus \cdots$, generated by $\widetilde{M}_{n}$, is f.g. Since $I M_{n}=M_{n+1}$ for $n \geq r$, it follows that $L=\tilde{L}_{n}$ for $n \geq r$, and so $L$ is a noetherian $R$-module. Hence the chain $\tilde{L}_{0}^{\prime} \subseteq \tilde{L}_{1}^{\prime} \subseteq \tilde{L}_{2}^{\prime} \subseteq \cdots$ of $R$-submodules of $L$ terminates, i.e., there exists $s \geq 0$ such that $\tilde{L}_{n}^{\prime}=\tilde{L}_{s}^{\prime}$ for $n \geq s$. It follows that $I M_{n}^{\prime}=M_{n+1}^{\prime}$ for $n \geq s$.
Remark: By taking $M_{n}^{\prime}=I^{n} M$ in (3.11), we get $M^{\prime} \cap I^{n} M=I^{n}\left(M^{\prime} \cap I^{n-s} M\right)$ for $n \geq s$. This, or the particular case when $M=A$ and $M^{\prime}=J$, an ideal of $A$, is often the version of Artin-Rees Lemma used in practise. For example, putting $M^{\prime}=\cap_{n \geq 0} I^{n} M$ and using Nakayama's Lemma, we get a proof of Krull's Intersection Theorem [viz., (3.3) of Ch. 1]. Artin-Rees Lemma is most useful in the theory of completions. For more on completions, which may be regarded as the fourth fundamental process, one may refer to [AM, Ch. 10].

Sometimes, it is useful to consider gradings which are more general than $\mathbb{N}$-grading.
Definition: Let $R$ be a ring and $s$ be a positive integer. By a $\mathbb{Z}^{s}$-grading on $R$ we mean a family $\left\{R_{\alpha}\right\}_{\alpha \in \mathbb{Z}^{s}}$ of additive subgroups of $R$ such that $R=\oplus_{\alpha \in \mathbb{Z}^{s}} R_{\alpha}$ and $R_{\alpha} R_{\beta} \subseteq R_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{Z}^{s}$. Elements of $R_{\alpha}$ are said to be homogeneous of degree $\alpha$. A ring with a $\mathbb{Z}^{s}$-grading is called a $\mathbb{Z}^{s}$-graded ring.

The definitions of graded subring, homogeneous ideal, graded ring homomorphism, in the context of $\mathbb{Z}^{s}$-graded rings, are exactly similar to those in the case of $\mathbb{N}$-graded rings. If $R=\oplus_{\alpha \in \mathbb{Z}^{s}} R_{\alpha}$ is a $\mathbb{Z}^{s}$-graded rings, and $M$ is an $R$-module, then $M$ is said to be $\mathbb{Z}^{s}$-graded if it contains additive subgroups $M_{\alpha}$, for $\alpha \in \mathbb{Z}^{s}$, such that $M=\oplus_{\alpha \in \mathbb{Z}^{s}} M_{\alpha}$ and $R_{\alpha} M_{\beta} \subseteq M_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{Z}^{s}$. Corresponding notions of homogeneous element, graded submodule, and graded module isomorphism, etc. are defined in a similar fashion.

If $\left\{R_{\alpha}\right\}_{\alpha \in \mathbb{Z}^{s}}$ is a $\mathbb{Z}^{s}$-grading on a ring $R$ such that $R_{\alpha}=0$ for all $\alpha=\left(\alpha_{1}, \ldots, \alpha_{s}\right) \in \mathbb{Z}^{s}$ with $\alpha_{i}<0$ for some $i$, then it is called an $\mathbb{N}^{s}$-grading; note that in this case $R=\oplus_{\alpha \in \mathbb{N}^{s}} R_{\alpha}$. Similarly, one has the notion of $\mathbb{N}^{s}$-gradings for modules.
Examples: 1. Let $A$ be a ring and $R=A\left[X_{1}, \ldots, X_{n}\right]$. Then $R$ has a $\mathbb{Z}^{n}$-grading $\left\{R_{\alpha}\right\}_{\alpha \in \mathbb{Z}^{s}}$ given by $R_{\alpha}=A X^{\alpha}$, where for $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbb{Z}^{n}$, $X^{\alpha}$ denotes the monomial $X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}$ if $\alpha \in \mathbb{N}^{s}$ and $X^{\alpha}=0$ if $\alpha \notin \mathbb{N}^{s}$. Note that if $A=k$, a field, then the homogeneous ideals of $R$, w.r.t. the above grading, are precisely the monomial ideals of $k\left[X_{1}, \ldots, X_{n}\right]$.
2. Let $A$ be a ring and $R=A\left[X_{1}, \ldots, X_{n}\right]$. Let $d_{1}, \ldots, d_{n}$ be any integers. For $d \in \mathbb{Z}$, let $R_{d}$ be the $A$-submodule of $R$ generated by the monomials $X_{1}^{\alpha_{1}} \cdots X_{n}^{\alpha_{n}}$ for which $d_{1} \alpha_{1}+$ $\cdots+d_{n} \alpha_{n}=d$. Then $\left\{R_{d}\right\}_{d \in \mathbb{Z}}$ defines a $\mathbb{Z}$-grading on $R$ which is an $\mathbb{N}$-grading iff $d_{i}=$ $\operatorname{deg}\left(X_{i}\right) \geq 0$ for $1 \leq i \leq n$. The corresponding homogeneous polynomials and homogeneous ideals are sometimes called weighted homogeneous polynomials and weighted homogeneous ideals respectively.

The notion of $\mathbb{Z}$-gradings permits us to define the following simple but useful operation on graded rings. If $R=\oplus_{d \in \mathbb{N}} R_{d}$ is a graded ring and $m$ is any integer, then we define $R(m)$ to be the graded $R$-module obtained from $R$ by shifting, or by twisting, the grading by $m$, i.e., $R(m)_{d}=R_{m+d}$ for $d \in \mathbb{Z}$ and $R(m)=\oplus_{d \in \mathbb{Z}} R(m)_{d}$. Further, if $M=\oplus_{d \in \mathbb{N}} M_{d}$ is a graded $R$-module, then we define $M(m)=\oplus_{d \in \mathbb{Z}} M(m)_{d}$, where $M(m)_{d}=M_{m+d}$. Note that $M(m)$ is a graded $R$-module.

Remark: Most of the results proved earlier in this section extend to $\mathbb{Z}^{s}$-graded rings and $\mathbb{Z}^{s}$-graded modules over them. The notion of grading can also be extended by replacing $\mathbb{Z}^{s}$ by a monoid; most results extend to this case as well provided the monoid is assumed to be torsionfree. See $[\mathrm{B}]$ or $[\mathrm{N}]$ for more on this.

## 4. Primary Decomposition in Graded Modules

Throughout this section, $R=\oplus_{d \in \mathbb{N}} R_{d}$ denotes a graded ring and $M=\oplus_{d \in \mathbb{N}} M_{d}$ a graded $R$-module. Given a submodule $N$ of $M$, by $N^{*}$ we denote the largest graded submodule of $N$, i.e., $N^{*}$ is the submodule generated by the homogeneous elements in $N$.
(4.1) Lemma. Every associated prime of $M$ is a homogeneous prime ideal and the annihilator of some homogeneous element of $M$.
Proof: Suppose $\mathfrak{p} \in \operatorname{Ass}(M)$ and $\mathfrak{p}=(0: x)$ for some $x \in M$. Then $x \neq 0$, and we can write $x=x_{e}+x_{e+1}+\cdots+x_{d}$ with $x_{j} \in M_{j}$ and $x_{e} \neq 0$. Now for any $a \in \mathfrak{p}$ with $a=a_{r}+a_{r+1}+\cdots+a_{s}$ and $a_{i} \in R_{i}$, the equation $a x=0$ yields the equations

$$
a_{r} x_{e}=0, a_{r} x_{e+1}+a_{r+1} x_{e}=0, a_{r} x_{e+2}+a_{r+1} x_{e+1}+a_{r+2} x_{e}=0, \ldots
$$

which imply that $a_{r} x_{e}=a_{r}^{2} x_{e+1}=a_{r}^{3} x_{e+2}=\cdots=a_{r}^{d-e+1} x_{d}=0$. Hence $a_{r}^{d-e+1} x=0$, i.e., $a_{r} \in \sqrt{(0: x)}$. Since $\mathfrak{p}=(0: x)$ is prime, we have $a_{r} \in \mathfrak{p}$. Now $\left(a-a_{r}\right) x=0$ and using arguments similar to those above, we find that $a_{r+1} \in \mathfrak{p}$. Proceeding in this manner, we see that $a_{i} \in \mathfrak{p}$ for $r \leq i \leq s$. This proves that $\mathfrak{p}$ is homogeneous. Moreover, if we let $I_{j}=\left(0: x_{j}\right)$, then we have $\mathfrak{p} \subseteq I_{j}$ for $e \leq j \leq d$ and $\cap_{j=e}^{d} I_{j} \subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we have $I_{j} \subseteq \mathfrak{p}$ for some $j$ and thus $\mathfrak{p}=\left(0: x_{j}\right)$.
(4.2) Corollary. Given any $\mathfrak{p} \in \operatorname{Ass}(M)$, there exists an integer $m$ and a graded ring isomorphism of $(R / \mathfrak{p})(-m)$ onto a graded submodule of $M$.

Proof: Write $\mathfrak{p}=(0: x)$ for some homogeneous element $x$ of $M$. Let $m=\operatorname{deg}(x)$. The map $a \mapsto a x$ of $R \rightarrow M$ is a homomorphism with $\mathfrak{p}$ as its kernel and it maps $R_{d}$ into $M_{m+d}$. Hence it induces a desired graded ring isomorphism of $(R / \mathfrak{p})(-m)$.
(4.3) Exercise: Show that if $\mathfrak{p} \in \operatorname{Supp}(M)$, then $\mathfrak{p}^{*} \in \operatorname{Supp}(M)$.
(4.4) Theorem. If $R$ is noetherian and $M$ is f.g., then there exists a chain $0=M_{0} \subseteq M_{1} \subseteq$ $\cdots \subseteq M_{n}=M$ of graded submodules of $M$, homogeneous prime ideals $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{n}$ of $R$ and integers $m_{1}, \ldots, m_{n}$ such that $M_{i} / M_{i-1} \simeq\left(R / \mathfrak{p}_{i}\right)\left(-m_{i}\right)$, the isomorphism being that of graded modules.

Proof: Similar to the proof of (1.7) in view of (4.1) and (4.2) above.
Now let us turn to primary submodules of $M$.
(4.5) Lemma. Let $N$ be a graded submodule of $M$ such that $N \neq M$ and for any homogeneous elements $b \in R$ and $y \in N$ we have

$$
b y \in N \text { and } y \notin N \Longrightarrow b^{n} M \subseteq N \text { for some } n \geq 1
$$

Then $N$ is a primary submodule of $M$.
Proof: Let $a \in R$ and $x \in M$ be any elements such that $a x \in N$ and $x \notin N$. Then we can write $x=x^{\prime}+x_{e}+x_{e+1}+\cdots+x_{d}$ with $x^{\prime} \in N, x_{j} \in M_{j}$ and $x_{e} \notin N$. Let $a=a_{r}+a_{r+1}+\cdots+a_{s}$ where $a_{i} \in R_{i}$. Now $a\left(x_{e}+x_{e+1}+\cdots+x_{d}\right) \in N$, and since $N$ is graded, we find that $a_{r} x_{e} \in N$. By assumption, there exists $n_{1} \geq 1$ with $a_{r}^{n_{1}} M \subseteq N$. Now $\left(a-a_{r}\right)^{n_{1}}\left(x_{e}+x_{e+1}+\cdots+x_{d}\right) \in N$, and hence there exists $n_{2} \geq 1$ with $a_{r+1}^{n_{1} n_{2}} M \subseteq N$. Proceeding in this manner, we can find $n_{0} \geq 1$ such that $a_{i}^{n_{0}} M \subseteq N$ for $r \leq i \leq s$. Therefore, we can find $n \geq 1$ [e.g., $n=n_{0}(s-r+1)$ ] such that $a^{n} M \subseteq N$. Thus $N$ is primary.
(4.6) Lemma. If $Q$ is a $\mathfrak{p}$-primary submodule of $M$, then $Q^{*}$ is $\mathfrak{p}^{*}$-primary.

Proof: Since the homogeneous elements of $Q$ (resp: $\mathfrak{p}$ ) are elements of $Q^{*}$ (resp: $\mathfrak{p}^{*}$ ), it follows from (4.5) that $Q^{*}$ is a primary submodule. Further, if $a$ is any homogeneous element of $\mathfrak{p}=\sqrt{\operatorname{Ann}(M / Q)}$, then $a^{n} M \subseteq Q$ for some $n \geq 1$, and since $M$ is graded, $a^{n} M \subseteq$ $Q^{*}$. It follows that $\mathfrak{p}^{*} \subseteq \sqrt{\operatorname{Ann}\left(M / Q^{*}\right)}$. On the other hand, since $M / Q^{*}$ is graded and $\operatorname{Ann}\left(M / Q^{*}\right) \subseteq \operatorname{Ann}(M / Q)$, we see that $\sqrt{\operatorname{Ann}\left(M / Q^{*}\right)}$ is a homogeneous ideal contained in $\mathfrak{p}$, and hence $\sqrt{\operatorname{Ann}\left(M / Q^{*}\right)} \subseteq \mathfrak{p}^{*}$. Thus $Q^{*}$ is $\mathfrak{p}^{*}-$ primary.
(4.7) Theorem. Suppose $A$ is noetherian, $M$ is f.g., and $N$ is a graded submodule of $M$. Let $N=Q_{1} \cap \cdots \cap Q_{h}$ be a primary decomposition of $N$, and $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$ be the associated primes corresponding to $Q_{1}, \ldots, Q_{h}$, respectively. Then we have
(i) $\mathfrak{p}_{i}=\mathfrak{p}_{i}^{*}, Q_{i}^{*}$ is $\mathfrak{p}_{i}$-primary for $1 \leq i \leq h$ and $N=Q_{1}^{*} \cap \cdots \cap Q_{h}^{*}$.
(ii) If the primary decomposition $N=Q_{1} \cap \cdots \cap Q_{h}$ is irredundant, then so is the primary decomposition $N=Q_{1}^{*} \cap \cdots \cap Q_{h}^{*}$.
(iii) If $\mathfrak{p}_{i}$ is minimal among $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$, then $Q_{i}$ is a graded submodule of $N$.

Proof: Applying (4.1) to $M / N$, we get $\mathfrak{p}_{i}=\mathfrak{p}_{i}^{*}$. So by (4.6), we find that $Q_{i}^{*}$ is $\mathfrak{p}_{i}$-primary. Since $N$ is graded and $N \subseteq Q_{i}$, we have $N \subseteq Q_{i}^{*}$. Thus $N \subseteq Q_{1}^{*} \cap \cdots \cap Q_{h}^{*} \subseteq Q_{1} \cap \cdots \cap Q_{h}=N$, which proves (i). If $N=Q_{1} \cap \cdots \cap Q_{h}$ is an irredundant primary decomposition, then $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$ are distinct, and $\operatorname{Ass}(M / N)=\left\{\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}\right\}$. Thus the associated primes corresponding to $Q_{1}^{*}, \ldots, Q_{h}^{*}$ are distinct. Moreover, if the decomposition $N=Q_{1}^{*} \cap \cdots \cap Q_{h}^{*}$ can be shortened, then $\operatorname{Ass}(M / N)$ would have less than $h$ elements, which is a contradiction. This proves (ii). Finally, if $\mathfrak{p}_{i}$ is minimal among $\mathfrak{p}_{1}, \ldots, \mathfrak{p}_{h}$, then the corresponding primary component $Q_{i}$ is unique. Hence from (i), we obtain $Q_{i}=Q_{i}^{*}$, i.e., $Q_{i}$ is graded.
Remark: As a special case of the results of this section, we obtain some useful results about ideals in graded rings. You may find it instructive to write these down explicitly.
(4.8) Exercise: Show that all the results of this section remain valid if $R$ is replaced by a $\mathbb{Z}^{s}$-graded ring and $M$ by a $\mathbb{Z}^{s}$-graded module over it. Deduce from this that if $I$ is a monomial ideal in $k\left[X_{1}, \ldots, X_{n}\right]$, then the associated primes of $I$ are monomial ideals and that $I$ has a primary decomposition such that each of the primary ideals occurring in it is a monomial ideal.

The following exercise indicates a constructive method to obtain primary decompositions of monomial ideals.
(4.9) Exercise: Let $J$ be a monomial ideal of $k\left[X_{1}, \ldots, X_{n}\right]$ and $u, v$ be relatively prime monomials in $k\left[X_{1}, \ldots, X_{n}\right]$. Show that $(J, u v)=(J, u) \cap(J, v)$. Also show that if $e_{1}, \ldots, e_{n}$ are positive integers, then the ideal $\left(X_{1}^{e_{1}}, \ldots, X_{n}^{e_{n}}\right)$ is $\left(X_{1}, \ldots, X_{n}\right)$-primary. Use these facts to determine the associated primes and a primary decomposition of the ideal $I=\left(X^{2} Y Z, Y^{2} Z, Y Z^{3}\right)$ of $k[X, Y, Z]$.

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[^0]:    ${ }^{1}$ For an explanation of this terminology, see the article [Ta] by Taylor.

