Various Course Proposals for: Mathematics with a View Towards (the Theoretical Underpinnings of) Machine Learning

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Abstract

In light of the growing use, acceptance of, and demand for, machine learning in many fields, notably data science, but also other fields such as finance—and this in both industry and academics, some university departments might wish, or find themselves forced to, accord to the winds of change and address this pressing issue. The goal of this document is to assist in designing relevant courses using material at the appropriate mathematical level. It protocols, sorts, evaluates, and contrasts, numerous viable books for a variety of possible courses. The subjects span several levels of, and different avenues in, linear algebra and real analysis, with briefer discussions of material in probability theory and mathematical finance.

*I wish to thank my Ph.D. student, Ni Jiang, for the herculean effort of assisting me in gathering the bibliography information on the books mentioned herein.

Contents

1 Motivation and Impetus

> Linear algebra has moved to the center of machine learning, and we need to be there. Gilbert Strang, *Linear Algebra and Learning from Data*, 2019, page viii.

> Interest in the subject of numerical linear algebra has been growing steadily. It plays an important role, for example, in data science and machine learning, particularly when very large matrices are involved. Bill Satzer (2020).[1]

> Neural networks are not black boxes. They are a big pile of linear algebra.[2]

The explosion in research and application of machine learning for, and availability of, big data, requires no introduction. While some amusing (jaded and embittered) quotes abound from the older generation of academic statisticians, e.g., Brian Ripley's "machine learning is statistics minus any checking of models and assumptions",[3] there are few naysayers remaining. Still, it is important to understand that machine learning is not the same as the field of statistics. Sources abound to address their differences, e.g., Larry Wasserman's blog *Statistics Versus Machine Learning*, https://normaldeviate.wordpress.com/2012/06/12/statistics-versus-machine-learning-5-2/ or the informative *The Actual Difference Between Statistics and Machine Learning* by Matthew Stewart, https://towardsdatascience.com/the-actual-difference-between-statistics-and-machine-learning-64b49f07ea3, among many others. Mandatory reading on this issue is Leo Breiman's (2001) *Statistical Modeling: The Two Cultures*, Statistical Science, 16(3). His position is very clear just from the abstract, which I quote in full:

> There are two cultures in the use of statistical modeling to reach conclusions from data. One assumes that the data are generated by a given stochastic data model. The other uses algorithmic models and treats the data mechanism as unknown. The statistical community has been committed to the almost exclusive use of data models. This commitment has led to irrelevant theory, questionable conclusions, and has kept statisticians from working on a large range of interesting current problems. Algorithmic modeling, both in theory and practice, has developed rapidly in fields outside statistics. It can be used both on large complex data sets and as a more accurate and informative alternative to data modeling on smaller data sets. If our goal as a field is to use data to solve problems, then we need to move away from exclusive dependence on data models and adopt a more diverse set of tools.

I would also include as mandatory reading Galit Shmueli's (2010) *To Explain or to Predict?*, Statistical Science, 25(3). As a quote from that article that I particularly like, given the popularity of machine learning in finance, we read on page 304:

> Finance is an example where practice is concerned with prediction whereas academic research is focused on explaining. In particular, there has been a reliance on a limited number of models that are considered pillars of research, yet have proven to perform very poorly in practice.

It is also worth reminding that machine learning, while not as old as the field of statistics itself, does go back a while. From the preface of MacKay's book (see Section 6.5), we learn that "In the 1960s, a single field, cybernetics, was populated by information theorists, computer scientists, and neuroscientists, all studying common problems."

[1] From the online MAA review of Tom Lyche's book *Numerical Linear Algebra and Matrix Factorizations* (2020), mentioned in Section 4.5.

[2] Found on: https://datascience.stackexchange.com/questions/42621/data-science-related-funny-quotes.

[3] To which Andrew Gelman replied "In that case, maybe we should get rid of checking of models and assumptions more often. Then maybe we'd be able to solve some of the problems that the machine learning people can solve but we can't!" (see https://statmodeling.stat.columbia.edu/2008/12/03/machine_learnin/).

From the viewpoint of university education, it is useful to explicitly outline the underlying mathematical subjects required to penetrate this field. Some of the above quotes serve to emphasize that linear algebra is a *sine qua non* for machine learning, and thus belongs prominently on the list. The rest of the entries are also easy to guess. Much more work is required to sift through the myriad of available books on the various topics, protocol the best ones, comment on their contents, and make some intelligent comparisons and recommendations. This is what I am attempting to do in this document. Before proceeding, allow me to disclaim:

> Given the relatively large scope of topics I have addressed, it is essentially certain that some, if not many, academics, whether professor colleagues, doctoral or master's students, etc., will find their favorite books missing from my lists, or that my assessment of some book is not in accordance with their opinion. While these things are basically unavoidable, what I can hope for is that I have not made any "glaring omissions", or any painful gaffes or verbal blunders. If so, please let me know. If I have omitted your book, or criticized it unfairly, feel free to also let me know: I attempt (in general, in all my academic pursuits) to be as objective and impersonal as humanly possible. This document is meant as a small service to our academic community, and not as a promotional vehicle for anyone's works.

The core undergraduate material required to pursue virtually *any* scientific discipline that uses quantitative methods consists of basic linear algebra, and (univariate and multivariate) calculus. The next stage builds on those two elements to cover the entities I deem necessary for a solid working understanding of basic machine learning methodology. These are (with notable exclusion of computer programming skills):

1. Probability Theory and Modern Statistical Inference,
2. (Intermediate) Linear Algebra,
3. (Convex) Optimization Theory,
4. (Real) Analysis,
5. Information Theory.

This last entry is (briefly, if not far too inadequately) discussed in Section 6.5. This document concentrates on linear algebra and real analysis, with some attention paid to probability theory (and mathematical finance).

Real analysis is an undisputable fundament for anything mathematical at the advanced undergraduate and graduate level, including precisely the first three listed topics. This is obvious for (beyond basic) probability theory, optimization, and (not formally on the above list, but discussed below in Sections 4.6 and 8.1) numerical analysis of computing and algorithms; while for linear algebra, note recent books such as Yang's 2015 *A Concise Text on Advanced Linear Algebra*, in which, quoting from the Preface,

> "Throughout the book, methods and ideas of analysis are greatly emphasized";

or Bisgard's 2021 *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*, in which the title already makes the content clear, and from the Preface, we read

> "[It] is assumed that readers have had a course in basic analysis".

Further interconnections between these subjects of course exist. As one of many examples, consider the class of multivariate heuristic optimization methods *that do not require continuity* (let alone differentiability) of the objective function: Their study is deeply entrenched in probability theory and analysis. Another example is the necessity of linear algebra for studying stochastic processes, e.g., Markov chains and the now ubiquitous class of Hidden Markov models. A recent book in stochastic processes that explicitly makes extensive use of linear algebra (and the rudiments of real analysis, but not measure theory) is *Discrete Stochastic Processes and Applications*, by Jean-François Collet, 2018. In fact, besides having excellent content, the description of the prerequisites in the preface is exemplary. Quoting,

> The prerequisites for this text are as follows.
>
> - From linear algebra: multiplication of matrices, scalar product, and the concept of eigenvalue (no knowledge of diagonalization results is needed); the section on the Perron–Frobenius theorem is totally self-contained.
> - From calculus: sequences and proofs by induction, manipulation of convergent series, the chain rule, computation of one-dimensional integrals by change of variables and integration by parts. From calculus in several variables, the notions of gradient and Hessian.
> - From probability theory: discrete and absolutely continuous random variables; conditional probability, computations of expectations, moments, Gaussian variables.

As another example of linear algebra playing a central role—perhaps where you might not expect it—is in mathematical finance. In *Market-Consistent Prices: An Introduction to Arbitrage Theory* (2020), the authors (who happen to be my colleagues) Pablo Koch-Medina and Cosimo Munari write: "[W]e put more emphasis on the linear algebraic structure of the theory than other authors, who may prefer to highlight the probabilistic aspects." (I discuss their book in Section 6.4.) We also see linear algebra extensively used in Steven Roman's *Introduction to the Mathematics of Finance: Arbitrage and Option Pricing*, 2nd edition, 2012 (also in Section 6.4). Interestingly, Roman is also the author of *Advanced Linear Algebra*, 3rd edition, 2007 (see Section 4.3).

Continuing our discussion of broad topics required to understand (and certainly to do high quality academic research in) machine learning, the next level of sophistication entails material typically associated with beginning graduate-level mathematics, notably measure theoretic probability and functional analysis. For academics sitting in social science faculties spanning business, computer science, economics, and finance (such as myself), it is noteworthy that the study of metric spaces and fixed point theorems (and arguably much of functional analysis) as well as Lebesgue integration becomes very relevant, not just in (the underpinnings of) machine learning, but also in mathematical economics. The latter realization is notably very clear in the highly praised books aimed at doctoral economics students, *Real Analysis with Economic Applications*, by Efe A. Ok, 2007; and *An Introduction to Mathematical Analysis for Economic Theory and Econometrics*, by Dean Corbae, Maxwell B. Stinchcombe, and Juraj Zeman, 2009.[4]

Undergraduates in mathematics, but also other fields, such as engineering, and applied mathematics and statistics, are compelled to take mathematical courses that extend well beyond rudimentary linear algebra and basic college calculus. In faculties such as mine, it is more likely that students at the master's level (and only a modest percentage, namely those with ambitions for doing a Ph.D., and/or intent to work on highly quantitative subjects such as graduate-level mathematical finance, mathematical economics, econometrics, insurance, or quant risk management) would take such courses. I claim (no doubt influenced by top-selling books purchased at the airport book shop, and voraciously read during the long flight, such as Yuval Noah Harari's trinity, notably *Homo Deus: A Brief History of Tomorrow* and *21 Lessons for the 21st Century*) that quantification and sophistication of most all areas of human inquiry will continue to increase: The demand for students versed in higher mathematics will

[4]The latter authors clearly have a sense of humor: The preface begins with "The objective of this volume is to provide a simple introduction to mathematical analysis with applications in economic theory and econometrics". An inspection of the table of contents indicates that the topics are, at least chapters 6 to 11, germane to a beginning graduate level course in real and functional analysis.

correspondingly increase; and the demand for those not comfortable with what appears to them to be abstruse intellectual perversions will (along with salary offerings) decrease.

With the ever-growing popularity of machine learning, and the growing presence of artificial intelligence (both as a subject of study, but also as a motivation to learn more advanced mathematics—otherwise your target job gets replaced by a microchip on the cloud), it can be expected that interest in courses in real analysis, linear algebra, numerical analysis, probability theory, modern statistical inference and high-dimensional statistics, etc., will increase. As the adoption, sophistication, and availability of AI continues to spread, the demand for those with "a college education" whose terminal mathematics are first-year calculus and a "101 course" in basic (and I would argue, outdated, and even wrong)[5] statistics, will dwindle even further. Being able to solve simple problems by rote, or "understand" basic linear regression, or the rudimentary linear algebra (and trivial parameter estimation) associated with Markowitz' baseline asset allocation theory, are no longer enough for a student to look enticing enough to get hired in the modern job marketplace—at least not in the job they were applying for, but there will always be some demand for manual labor, such as with the office cleaning team, or cafeteria cooking service. Not to mention: that basic Markowitz portfolio methodology does not even work—if you want to beat the market, you either need incredible luck, incredible friends, the genius and obsession of Warren Buffett, or graduate-level training in probability and statistical theory, and quantitative finance.

This idea has surely been written about in many venues, but I find it expressed very well in a book review (actually of three books, to assist the transition to higher mathematics, with emphasis on proofs) by Joseph H. Silverman, appearing in *The American Mathematical Monthly*, Vol. 106, No. 3 (1999), pp. 272-274. (I mention the three books he evaluates in Section 5.1.) He writes:

> By rote problem solving I mean, of course, the sort of process used in most calculus classes whereby students are shown standard problem templates and, after absorbing a sufficient number of examples, learn to solve similar problems in a color-by-number fashion.
>
> For mathematicians, problem solving of the sort just described is related to mathematics much as doing a crossword puzzle is related to writing a novel. Both activities require a good vocabulary and some mental agility, but only the latter requires creativity.

There are now numerous books available on machine learning, and my goal is not to inspect and protocol them. Instead, I just mention the stated prerequisites from a few of them; one being among the most popular *and* most basic books, and the others being (highly rated and) relatively mathematical treatises on the topic. The first is *An Introduction to Statistical Learning* (2013, corrected 7th printing, 2017), by Gareth James, Daniela Witten, Trevor Hastie, and Robert Tibshirani. (Some refer to this book as the baby version of *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, 2nd edition, 2009, by Trevor Hastie, Robert Tibshirani, and Jerome Friedman.) From the preface of James et al., we read:

[5] I have in mind the blind, misguided, and ubiquitous application of Neyman-Pearson hypothesis testing, and the similar use and reporting of (Fisherian) p-values. Adding insult to injury, the two methods are not only both flawed, but—often if not nearly always, at the undergraduate level—presented together as though it were a unified methodology, while their respective advocates literally hated each other and felt that the opponent's methodology was wrong. I agree with them both, in aggregate: Both methods are flawed. However, both can be correctly used in certain situations, as the original authors meant them to be, as opposed to how they are taught and deployed since, say, the 1970s. To my (and many academics') great relief, this is finally starting to change. I discuss this at length in (and shamelessly refer to) my textbook, *Fundamental Statistical Inference: A Computational Approach* (2018, Sec. 2.8). For further critique (and quite some vitriol) against the incorrect use of p-values, see the discussion of MacKay's (2003) book in Section 6.5.

For an enjoyable account of the animosity between Fisher and (well nearly everyone who dared criticize his work, but, in particular) Jerzy Neyman and Egon Pearson (that being Karl Pearson's son), see David Salsburg's *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century*, (2002). Regarding the (in)compatibility of the two approaches, see E. L. Lehmann, *The Fisher, Neyman-Pearson Theories of Testing Hypotheses: One Theory or Two?*, JASA (1993), 88(424).

> This book is appropriate for advanced undergraduates or master's students in statistics or related quantitative fields.

This assessment, impressively, and unlike many books, is reasonably correct, and in fact is a bit on the conservative side: Much of the (very well presented for beginners) material is accessible for undergraduates having had a first course in something modest like "a first course in statistics and empirical methods", though for sure, the richer a student's background is in probability and statistics, the more she will get out of the book. Despite the authors having stated the prerequisites, some still try to cut corners, and then complain: From a disappointed purchaser's review at Amazon, "It might well be an introduction to the topic but if you have no maths/statistical background beforehand do not buy this book."

Turning now to more mathematically refined presentations—and, indeed, where "the action is" for academics and graduate students, from the preface of Shai Shalev-Shwartz and Shai Ben-David, *Understanding Machine Learning*, 2014, 410 pages, we read that "the reader is assumed to be comfortable with basic notions of probability, linear algebra, analysis, and algorithms." Notice the change, from the previous book, requesting the reader to have some familiarity with basic statistics, to *probability*—that previously mentioned Amazon reviewer would be aghast (though perhaps, and hopefully, speechless) at the difference. Moreover, requested is also *analysis*, which I presume to be at least familiarity and adeptness at the material in a first course in real analysis, and most surely more, notably multivariate real analysis and metric space theory.

From Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar, *Foundations of Machine Learning*, Second Edition, 2018, 486 pages, we learn from the preface that:

> The reader is assumed to be familiar with basic concepts in linear algebra, probability, and analysis of algorithms. However, to further help, we have included an extensive appendix presenting a concise review of linear algebra, an introduction to convex optimization, a brief probability review, a collection of concentration inequalities useful to the analyses and discussions in this book, and a short introduction to information theory.

An inspection of their five page linear algebra review in appendix A indicates that the material associated with a second course in undergraduate linear algebra (see Section 4) is expected. From their appendix B, on convex optimization, the student best have had a course on multivariate real analysis (advanced calculus; see Section 7.1), and some exposure to convex optimization. (There are many books on that latter topic, including a well known one from Stephen Boyd and Lieven Vandenberghe, whose book on linear algebra, *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares*, is discussed below, in Section 4.8.) Appendix C, on probability, is surprisingly modest, and does not exceed anything in a first undergraduate course, though appendix D, on concentration inequalities, is far more serious looking, as is appendix E, on information theory (see Section 6.5).

Amid numerous 5-star reviews on Amazon, we read from one disgruntled reviewer's 2-star review, entitled "A book from Phds to Phds", that "The content of the book is too heavy. Lots of mathematical proofs where it appears the target audience is definitely people already in the field." I think this nicely supports my above claim regarding demand for university graduates versed in higher mathematics (and the understandable frustration that will increasingly be encountered by those who are not).

The last example I give is from another (advanced, extensive, and very attractive) machine learning book, namely Xian-Da Zhang, *A Matrix Algebra Approach to Artificial Intelligence*, 2020, 820 pages. From the preface, we read:

> The development of AI is built on mathematics. For example, multivariant calculus deals with the aspect of numerical optimization, which is the driving force behind most machine learning algorithms. The main math applications in AI are matrix algebra, optimization, and mathematical statistics, but the latter two are usually

described and applied in the form of matrix. Therefore, matrix algebra is a vast mathematical tool of fundamental importance in most AI subjects.

The aim of this book is to provide the solid matrix algebra theory and methods for four basic and important AI fields, including machine learning, neural networks, support vector machines, and evolutionary computation.

I now list several recent books not explicitly on machine learning, but rather on high-dimensional probability and statistics, and show excerpts from their prefaces or table of contents, to help indicate what they expect the student to know in order to venture further with their material. (Bold facing was added by me):

1. *Analysis of Multivariate and High-Dimensional Data*, by Inge Koch, 2013.

    > To understand the underlying theoretical ideas, the reader should have a solid background in the theory and application of statistical inference and multivariate regression methods and should be able to apply confidently ideas from **linear algebra and real analysis**.

2. *Large Sample Covariance Matrices and High-Dimensional Data Analysis*, by Jianfeng Yao, Shurong Zheng, and Zhidong Bai, 2015.

    > It is assumed that the reader is familiar with the usual theory of **mathematical statistics**, especially methods dealing with multivariate normal samples. Other prerequisites include knowledge of elementary **matrix algebra** and **limit theory** (the law of large numbers and the central limit theorem) for independent and identically distributed samples. A special prerequisite is some familiarity with contour integration; however, a detailed appendix on this topic has been included.

    Of relevance for academics in finance is the last chapter, 12, entitled "Efficient Optimisation of a Large Financial Portfolio".

3. *Introduction to High-Dimensional Statistics*, by Christophe Giraud, 2015.

    The author does not directly say what the prerequisites are in the preface. However, already in the introductory first chapter, we are immediately in a discussion about the volume of a p-dimensional ball, a topic directly out of books on multivariate analysis. We then soon subsequently read:

    > Concentration inequalities are central tools for the non asymptotic analysis of estimators, and we will meet them in every major proof of this book.

    Thus we are relatively deep into probability theory and also aspects of real (multivariate) analysis. The appendices of the book also give indications of what is required. In particular, in addition to the concentration inequalities, we see the emphasis also on topics from intermediate linear algebra, not to mention the necessity of studying Hilbert spaces, motivating having prerequisites of metric spaces and some functional analysis. From the table of contents, we list the entries for the appendices:

C.3 Matrix Norms
C.4 Matrix Analysis
Appendix D Subdifferentials of Convex Functions
Appendix E Reproducing Kernel Hilbert Spaces Notations

4. *Mathematical Foundations of Infinite-Dimensional Statistical Models*, by Evarist Giné and Richard Nickl, 2016.

    Throughout this book, we assume familiarity with material from **real and functional analysis, measure and probability theory** on the level of a US graduate course on the subject.

5. *High-Dimensional Probability: An Introduction with Applications in Data Science*, by Roman Vershynin, 2018.

    The essential prerequisites for reading this book are a **rigorous course in probability theory** (of the Masters or Ph.D. level), **an excellent command of undergraduate linear algebra**, and general familiarity with basic notions about **metrics, normed and Hilbert spaces, and linear operators**. A knowledge of measure theory is not essential but would be helpful.

6. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, by Martin J. Wainwright, 2019.

    ... it assumes a strong undergraduate background in basic aspects of mathematics, including the following:

    A course in **linear algebra**, including material on matrices, eigenvalues and eigendecompositions, singular values, and so on.

    A course in **basic real analysis**, at the level of Rudin's elementary book (Rudin, 1964), covering convergence of sequences and series, metric spaces and abstract integration.

    A course in **probability theory**, including both discrete and continuous variables, laws of large numbers, as well as central limit theory. A measure-theoretic version is not required, but the ability to deal with the abstraction of this type is useful. Some useful books include Breiman (1992), Chung (1991), Durrett (2010) and Williams (1991).

    A course in classical **mathematical statistics**, including some background on decision theory, basics of estimation and testing, maximum likelihood estimation and some asymptotic theory. Some standard books at the appropriate level include Keener (2010), Bickel and Doksum (2015) and Shao (2007).

7. *A First Course in Random Matrix Theory: for Physicists, Engineers, and Data Scientists*, by Marc Potters and Jean-Philippe Bouchaud, 2021.

    The reader is assumed to have a background in undergraduate mathematics taught in science and engineering: **linear algebra, complex variables and probability theory**. Important results from probability theory are recalled [...] while stochastic calculus and Bayesian estimation are not assumed to be known.

    The first page of the book starts off with eigenvalues and singular values. Chapter 20 is on applications in finance.

2 On Books and Teaching Mathematics

> Typically textbooks are written for professors, not students, because professors choose the book and students then have to buy it and what professors want in a book is often very different then what a student wants in a book, especially when the majority of students are not in the course in order to become mathematics major but rather are studying engineering, computer science, physics, economics or some other subject.
>
> Bruce Cooperstein, *Elementary Linear Algebra: An eTextbook* (2016, p. 2)

> To be sure, there is something special about holding a physical book that cannot be replaced with a digital version. However, we should not over-romanticize the qualities of the print book and let this stand in the way of significant advantages, not the least of which is much better pedagogy.
>
> Bruce Cooperstein, *A Textbook Education*, Solutions Journal, Feb 2016.

2.1 Books and PDF Files

With respect to the choice of textbook, if a book is in fact used: I advise the instructor to embrace the fact that the 21st century is fully here, and students are tech-savvy, and prefer to carry a digital tablet containing possibly hundreds of (mostly free) books and collections of lecture notes, instead of a backpack laden with traditional, heavy, printed (and quite possibly rather expensive) books.

Besides their obvious anachronistic aspects and "nerd factor", books wear out, pages get torn, take insulting coffee stains, etc.. PDF files do not suffer any of these drawbacks, and are portable and (outside of scans of old books) searchable. They are also easier to read by not requiring optimal (or any) light (albeit battery power), and being able to trivially expand the font size (notably for aging eyes such as mine). Text can be copied from the source, and pdf viewers allow annotation (often from multiple users).

It might be deemed an insulting and horrible annoyance to authors (such as myself, of five books, four in probability and statistics) that nearly all books are now available, for free, as perfect pdf files on the web. It is however a blessing for students, but also researchers wishing to have a quick peak before engaging their own or taxpayer money. The availability of a pdf file for the book is a crucial issue of decisive relevance to students (and their evaluation of the professor). The idea of students buying physical textbooks during their university studies is now completely dated, and so the availability of a pdf file of the book (whether free or not) should play a serious role in the choice. The fact that most all books (from the last 40 or so years, and more so for recent books) are available (perhaps not legally, but that does not deter students) as *free* pdf files, and students having come to expect this, should also play a role, perhaps secondary, in the decision.

For a bit of trivia, there are some books well over 40 years old that have quite readable pdf files available. Here are two excerpts from George Shoobridge Carr's famous *A Synopsis of Elementary Results in Pure Mathematics: Containing Propositions, Formulae, and Methods of Analysis, with Abridged Demonstrations*, from 1886 (not 1986!), 935 pages (plus several pages of quite reasonable graphics), available here: https://archive.org/details/synopsisofelemen00carrrich/page/n5/mode/2up. The typesetting is shockingly readable, and there are graphics, and nearly 1000 pages!—how did they do that back then? Some things look completely familiar to modern presentations, while other things, such as the definition of expectation, are a bit dated. For example:

> The operations of *differentiation* and *integration* are the converse of each other. Let $f(x)$ be the *derivative* of $\phi(x)$; then $\phi(x)$ is called the *integral* of $f(x)$ with respect to x. These converse relations are expressed in the notations of the Differential and Integral Calculus, by
>
> $$\frac{d\phi(x)}{dx} = f(x) \quad \text{and by} \quad \int f(x)dx = \phi(x).$$

> Definition.—When a sum of money is to be received if a certain event happens, that sum multiplied into the probability of the event is termed the expectation.
>
> G. S. Carr (1886), pages 313, and 102.

Before leaving this topic, note the increasing popularity of open source books (e.g., Creative Commons License), whereby you can have the electronic book for free, and even the underlying LaTeX, and modify it! I cite several such books below. In one such book, a very attractive one no less, *Linear Algebra with Applications*, 2021, by W. Keith Nicholson, he motivates his choice to go open source by writing in the preface:

> Mathematics education at the beginning university level is closely tied to the traditional publishers. In my opinion, it gives them too much control of both cost and content. The main goal of most publishers is profit, and the result has been a sales-driven business model as opposed to a pedagogical one. This results in frequent new "editions" of textbooks motivated largely to reduce the sale of used books rather than to update content quality. It also introduces copyright restrictions which stifle the creation and use of new pedagogical methods and materials. The overall result is high cost textbooks which may not meet the evolving educational needs of instructors and students.
>
> To be fair, publishers do try to produce material that reflects new trends. But their goal is to sell books and not necessarily to create tools for student success in mathematics education. Sadly, this has led to a model where the primary choice for adapting to (or initiating) curriculum change is to find a different commercial textbook. My editor once said that the text that is adopted is often everyone's third choice.

There is also now the teaching option based fully on an interactive html presentation, and no pdf file (let alone physical book). An example is the "book" *immersive linear algebra*, by J. Ström, K. Åström, and T. Akenine-Möller, located at `http://immersivemath.com/ila/index.html`. Another example is the "book" *Linear Algebra*, by David Cherney, Tom Denton, and Andrew Waldron, dated March 5, 2021, found here: `https://math.libretexts.org/Bookshelves/Linear_Algebra/Map%3A_Linear_Algebra_(Waldron_Cherney_and_Denton)`. In this case, also available is the related book (the electronic version of which is legally free, via a Creative Commons license) *Linear Algebra*, by David Cherney, Tom Denton, Rohit Thomas and Andrew Waldron, 2016. This book, along with the aforementioned one from Nicholson, appears in my list of recommended books in Section 3.2.

There are also numerous "lecture notes" on the web, freely available. For example, see the large list given here, `https://realnotcomplex.com/algebra/linear-algebra`, this being dedicated to linear algebra; while for a broader (and huge) list of online math resources in a variety of fields, including free open source books, see `https://project-awesome.org/rossant/awesome-math` or `https://github.com/rossant/awesome-math#real-analysis`.

2.2 The Agony of Choice and Die Qual der Wahl

One does not need to have studied microeconomics and psychology to know that, on the one hand, having a large set of (necessarily, due to market competition) "reasonably equivalent" choices that are competing for market share is a bonanza for the consumer, but on the other hand, when confronted with making the choice, this, in its extreme, can be debilitating, leading to mental exhaustion, anxiety, dissatisfaction, and regret (see, e.g., *The Paradox Of Choice: Why More Is Less*, Barry Schwartz, 2016). At the risk of belaboring the obvious, note that randomly choosing one of the many powdered instant coffees available in the supermarket (or any such product, e.g., toilet paper—covid-related shortages notwithstanding) will result in nearly the same utility as the choice made after considerable research; and the cost in this case of choosing an inferior product is vanishingly small. The point I wish to make and emphasize is: The cost of choosing an inferior good is not at all small for the case of selecting a textbook for students to use for a semester or year-long class in—what is for them, the first confrontation with rather challenging—mathematics.

There is a flip side to this: What if you were allowed to take as many of those instant coffee brands home with you, use them as you wish, and not have to pay? That is the case now with textbooks—they are virtually all available (no doubt not, or only quasi, legally) as pdf files on the internet: Every student knows exactly the relevant web sites. And even if this were not the situation, one could also go to the library—assuming it is well stocked, locate the relevant book shelf, and then thumb through various books, hoping each book's index was well and generously constructed. In this latter, old-fashioned way, there are the costs of physically going to the library (and leaving the comfort zone of one's home, wearing a t-shirt and sweat pants, as the recent pandemic and resulting zoom sessions for meetings and teaching have allowed us); the "cost" of the randomness of what books the library happens to have, and not have; and the cost of manually searching for the topic of interest in the book. All those costs are, since years, gone, with the availability of electronic (and searchable) copies of (essentially all relevant) books. However, *both ways* entail the same cost of time and energy required to read and study the relevant topic of interest within the book. So, while there is no free lunch, or zero cost to obtaining knowledge, at least all of the—for a young person, primitive—20th century cost factors have been eliminated, and what remains is the "pure intellectual cost" of digesting the written mathematics.

Nevertheless, I believe it still makes good sense to have one, or, better, two, primary books for a course. With their easy access to other books, students can quickly obtain information on some issue or topic that they deem not detailed extensively enough for their tastes in the primary book(s). Of course, this means, surely for some students, engaging the temptation to search for solutions to exercises; but it also means the ability read more extensively on a particular topic of interest, or explore topics not covered in the primary book, as perhaps initiated by the instructor.

To give a simple example of the potential for this (and arguably also emphasize the value of protocolling lists of good books), assume the student is taking (or you are conducting) a course in basic real analysis, and the specific subject of interest is Euler's (or the Euler–Mascheroni) constant, γ: In particular, interest centers on the convergence of $a_n = 1 + 2^{-1} + 3^{-1} + \cdots + +n^{-1} - \log n$. A student not pleased with the presentation in the primary book can (ideally sit down and work on the problem—of course, but perhaps to double check her calculations) search in various books for this topic. To illustrate, I went and gathered the following information across some of the books I mention in Section 5—an exercise that took me about an hour (without leaving my home, and while wearing ugly, old sweatpants).

Some of the analysis books mentioned in the lists in Sections 5.2 and 5.4 do not discuss or mention Euler's constant. Just that finding alone should already be enough to motivate the construction of a large list of, or even establishing a large collection of, real analysis books, recognizing that no single book can contain all topics, certainly not discussed in detail. Examples of such books include Conway (2018, *A First Course in Analysis*); and Strichartz (2000, *The Way of Analysis*). Both of these books (are, in my opinion, excellent, and) "start from the beginning" of real analysis, covering the usual topics for a first course, namely univariate, but also cover multivariate aspects. Both appear in Section 7.1 in my list of recommended books for multivariate real analysis (and Strichartz also appears in my list in Section 5.2). As these and similar books attempt to cover material enough for two (or possibly three—Strichartz even covers the Lebesgue integral) courses, it does not come as a surprise that there will be topics that do not get any mention but that otherwise get some, or much, attention in analysis books with more modest scope, i.e., just basic univariate analysis.

Some books indeed present Euler's constant, albeit just in an exercise (asking to show convergence of the series, provide bounds, etc.), and without available solution. Examples include (the highly praised and, undebatably, one of the "industry standards") Bartle and Sherbert (2011, page 277); (the fantastic, very lengthy and detailed) Zorich, (2015, page 146, but also see middle of page 580, using Euler-Maclaurin); (the admirable and far shorter than Zorich) Lebl (2020, page 190); (the extremely highly praised—and for good reason) Abbott

(2016, page 237); and (impressive newcomer that I have added to my personal reading list) Field (2017, p. 99, but see also Sec. 6.3.5 using Euler-Maclaurin). As the reader can see, all these books are (by me, but apparently also many others) considered excellent, and each takes a place in my list in Section 5.2. I purposely chose these highly rated books to remind the reader (student or instructor) that many books, notably good ones, lack solutions, and this is very often a source of student frustration, notably for those outside of pure mathematics. It is also an issue that can be easily mitigated by bringing to students' attention some of the many other books in the field.

Finally, some books contain a presentation of the convergence result, such as, alphabetically:

1. Beals (2004), page 57. A very good presentation.
2. Brannan (2006), pages 299-300. Also a very good presentation.
3. Fischer (1983), *Intermediate Real Analysis*, page 331. This is rather detailed and clear presentation (and serves as a case in point to also consider having a look in the older books).
4. Garling, 2013, *A Course in Mathematical Analysis: Volume I*, page 236. The result is mentioned in one sentence, and not highly detailed. (This is not surprising: This first volume, of three, is a bit more advanced than most beginning real analysis books; and this is valuable information for those students who wish to read, during or after a first course, a more sophisticated presentation.)
5. Ghorpade and Limaye (2018), page 279.
6. Hijab (2016), exercise 4.4.18, page 179, with answer on page 372 using the integral test.
7. Jacob and Evans (2016), *A Course in Analysis Volume l*, page 252. Here is a good case in point: The presentation is detailed and clear, but there appears to be a mistake. Thus, as a check (also to avoid the peril of falsely assuming the book is wrong), one can easily check the presentation in other books.
8. Lang (1997), *Undergraduate Analysis*, page 89, exercises IV.2.21 and IV.2.22, with solutions detailed in Shakarchi (1998), *Problems and Solutions for Undergraduate Analysis*, pages 62-63.
9. Little, Teo, and van Brunt (2015), pages 325-6. This is another excellent, detailed presentation.
10. Magnus (2020), page 407, as an application of the Maclaurin–Cauchy theorem, and thus as part of a more general derivation.
11. Sasane (2015), page 249, exercise 5.47, with solution given on pages 443-4.
12. Stoll (2001), section 7.1, page 293, exercise 19, with solution outline (or, if you prefer, generous hint) on page 533.
13. Trench (2003), section 4.3, exercise 14, page 230, with the solution given in the solutions manual, page 92.

Obviously, one could make similar lists for a large multitude of topics (or, better, design a computer algorithm to do it). I now give a very abbreviated further example, not just to drive the point home, but also to show how a practical, important topic, very well developed, can appear in a book one might not have expected to see it in, and not appear in a verbose, detailed, longer book, where it would have been expected.

In the aforementioned Jacob and Evans book (which I know well, having read and enjoyed it during the assembly of this document, and praise in Section 5.2 below), they give a very nice and detailed example for practicing with convergence of sequences, and also to demonstrate (the application of the axiom of completeness to show) the existence of square roots of non-negative reals, on page 239. Perhaps surprisingly, they do not say or motivate where the recursion comes from—they literally just state it, out of nowhere. Of course, someone versed in the field immediately recognizes it to be Newton-Raphson. That topic does not occur at all in the book (but might occur in a subsequent volume, perhaps when fixed point theorems are discussed, I did not check). Perhaps the authors just assume the instructor using the book will mention this during class, and possibly make a short digression about Newton-Raphson? I am of the opinion that the whole point of using a book (or two) is so that *I do not have to reinvent the wheel* and construct class notes for lecturing on a highly established topic.[6]

Exercise 8 on their page 241 gives the general recursion for the kth root (and asks the reader to confirm that it converges). The answers to all exercises are provided (this one being on page 642, and fully detailed). Changing books, Garling's aforementioned book proves the general kth root case in the main text (page 88), and mentions it is Newton Raphson, saying "Thus it not only proves the existence of $y^{1/k}$, but also enables good approximations to it to be calculated." A search (or the index) reveals that Newton-Raphson is developed on pages 203-5 (complete with the ubiquitous graphic illustrating the method). Interestingly, Jacob and Evans include Garling's book in their reference list. Thus, we have a case in point that a book (Jacob and Evans, volume I, 744 pages) positioned to be modest in its scope of material for a first course in real analysis, and very detailed and student friendly, can be outdone (I was going to say "trumped", but prefer not to), at least with respect to a particular topic, by a much shorter and technically more ambitious book (Garling, volume I, 300 pages).

2.3 Thoughts on Teaching

A worthy debate for another time regards how the material is best taught. Keeping the discussion short, I start with a quote from the preface of *A Course in Abstract Analysis*, by John B. Conway, 2012, regarding lectures:

> I think my job as an instructor in a graduate course is to guide the students as they learn the material, not necessarily to slog through every proof. In the book, however, I have given the details of the most tedious and technical proofs; but when I lecture I frequently tell my class, "Adults should not engage in this kind of activity in public."

In my own lectures, I ask, beg, beseech, implore and admonish the students to read the next set of notes *before* class, so they get much more out of lecture, can ask intelligent questions, and have a more sophisticated dialog with me about the material. Some do, some of the time. It seems Robert Strichartz agrees (and even thinks about how to enforce this): We read from the preface of his well received *The Way of Analysis*, 2000:

> My recommendation is that students be required to read the material before it is discussed in class. (This may be difficult to enforce in practice, but here is one suggestion: have students submit brief written answers to a question based on the reading and also a question they would like to have answered in class.) The ability to read and learn from a mathematical text is a valuable skill for students to develop. This book was written to be read—not deciphered. If I have perhaps coddled the students too much, I'm sure they won't complain about that!

Next consider the question about if students should have access to the solutions of the exercises in the book. We will meet several books below that provide detailed solutions in the book, or even have an entire large separate book with solutions (e.g., Harville, 2008;

[6] If there happens to be no book I deem satisfactory, then I go and write my own: I did this four times already, with probability, statistical inference, and time-series analysis. I am therefore so relieved that, for the subjects I am inspecting in this document, there is no need whatsoever to attempt to improve upon what is already available.

see Section 4.7; Lang via Rami Shakarchi, 1998; see Section 5.4; Grimmett and Stirzaker, 2020; see Section 6.1; and Yeh, 2014; see Section 8.3), while others provide some solutions, or hints, and others make available an instructor's manual with solutions (not intended to be easily accessible by students). Finally, other books give nothing. Nobody debates that students need to try the exercises, notably the ones involving proofs, in order to enhance the learning effect and build confidence and competence. One might argue though, that students not aiming to become professional mathematicians, but rather just want to understand the material at a working level, while they are learning other material closer to their interest, are better served with being able to see at least some solutions. Knowing they are available, the student might actually take *more* interest in the material, really try the exercises, and compare their thoughts with the given solutions. The alternative is to be discouraged, annoyed, and feel tortured by the material, ultimately losing interest and motivation (and respect for the instructor imposing such ludicrous demands on their time and patience).

Not everyone agrees. A good example can be taken from the book by Bernd S. W. Schröder, *Mathematical Analysis: A Concise Introduction*, 2008, 550 pages, which is, from the preface, "a self-contained introduction to the fundamentals of analysis." Regarding what the student needs beforehand, "[The] only prerequisite is some experience with mathematical language and proofs". This is a *very* concise, and advanced, book, e.g., the author reaches outer Lebesgue measure by page 127, and after page 225, it is fully graduate level material in analysis. (An MAA book review is here: https://www.maa.org/publications/maa-reviews/mathematical-analysis-a-concise-introduction.)

Anyway, on page xiv of the preface, we read (my boldfacing):

> To get the most out of this text, the reader is encouraged to *not* look for hints and solutions in other background material. In fact, even for proofs that are adaptations of proofs in this text, it is advantageous to try to create the proof *without* looking up the proof that is to be adapted. There is evidence that the struggle to solve a problem, **which can take days for a single proof**, is exactly what ultimately contributes to the development of strong skills.

While the idea is inspiring and thought-provoking, and might well be appropriate for gifted prodigies aiming to become professional mathematicians, I do not imagine such an approach would do any favors for typical students in finance (or engineering, or computer science, or economics, or biology, etc.) along their admirable path to understand some of the higher mathematics employed in their field—or any favors for an instructor who has the audacity to implement such an austere, tone-deaf approach.

As should be clear, I favor books that "spell things out", such as detailed, clear proofs of results, as well as books that add appropriate commentary to help guide the reader (as well as possibly also adding some historical information); as opposed to trying to win an award for "the most minimalist, terse book that still contains just enough information to allow the reader to reproduce—with monumental effort—all the known results". I have more to say about this, and package my comments (diatribe?) in my statements about (and praising) Strichartz' book; see the end of my list of recommended books in Section 5.2.

Nevertheless, just reading a (possibly highly detailed and verbose) book, with full proofs, and/or having immediate access to, and reading, the solutions to exercises without having first tried to answer them, is unequivocally inadequate to truly develop a deep and lasting understanding of the material. This is nicely expressed by Field (2017, page x) from his very attractive recent book *Essential Real Analysis* that we will discuss in Section 5.2:

> On occasions I advise students in my analysis classes not to spend too much time reading mathematics texts. That view is based on my own experience—an effective way to learn mathematics is to do it, play with it but generally avoid spending too much time reading books about it. Reading a mathematics book can give a veneer of superficial understanding that dissolves the moment one tries to use the theory described in the book. An analogy might be learning carpentry, plumbing or a foreign language—knowing the theory is important but not that

> helpful; knowing how to use the tools is crucial. That takes time, practice and serious effort. As an example, think about hiring a personal trainer at the gym. You pay him or her rather a lot of money and sit back two or three times a week and watch them exercise, lift weights and generally work out and suffer. As a result you lose weight and gain a svelte figure.... It is the same with mathematics and learning mathematics. Much more is required than finding the ultimate book (or teacher).

Some further thoughts on this topic are given in item 1 of Section 3.3.

I end this section with a quote from (a book I like, and discussed below in Section 5.4) Donald Estep's, *Practical Analysis in One Variable*, 2002, page 2, as motivation for students to persevere:

> By the way, I should have included frustration in my description of my feelings about analysis at present as well as in the past. I still get mighty bouts of frustration when I try to learn new analysis or solve problems in my research. The great professional cyclist Greg LeMond said about riding a bike, "It doesn't get any easier, you just get faster." I suppose the same is true of mathematics. Now some analysis seems easy in its familiarity and I struggle with more complicated ideas. But the struggle to understand analysis is really never-ending for me. So if it is any consolation to the reader, analysis may never become easy, but at least you will struggle with harder and harder ideas.

3 Course Proposal 1: "A First Course in Linear Algebra"

> The first lower-division text with vector spaces was Kemeny, Snell, Thompson and Mirkel's 1959 *Finite Mathematical Structures*, which combined vector-space theory with matrix-based applications, such as Markov chains and linear programming.
>
> Alan Tucker (1993, p. 7)[7]

> In view of the availability of so many textbooks having the same title and essentially the same chapter headings, we feel that the most useful role this preface can play is that of indicating to one considering it for adoption those features which, in our opinion, set it apart.
>
> R. R. Stoll and E. T. Wong, *Linear Algebra*, (1968, p. 5)

Given the goal of this presentation, I thought it appropriate to investigate when, in history, books began to appear that are reasonably similar in content (though certainly not in appearance) to modern ones on basic linear algebra. The first quote above gives us an answer, while the second quote indicates that, by 1968, the market was filled with such texts. Section 3.4 mentions some of the other older books.

After a very large book inspection, I am able at this stage to propose several books, (amazingly, but perhaps regretfully, *many* books), each of which (i) is at the right level for such a first course; (ii) is optically at least "good", and often great; and, crucially, (iii) is pedagogically excellent, i.e., in terms of content, presentation, organization, details, clear proofs, and examples. Before stating the list, Section 3.1 gives some insight into how an instructor (or self-learning student) might go about the nontrivial task of choosing one or a small set of books to use. Section 3.2 provides the set of books I recommend from which to make this choice. After some further explanatory comments in Section 3.3, I give two further lists of viable books, older and newer, in Section 3.4 that, while not making the primary list, are (i) books that instructors and students should know about, and/or (ii) are well-suited as complementary / supplementary reading. Finally, Section 3.5 is a digression investigating various notations used in linear algebra books.

3.1 On Judging and Selecting Books

A single one (or, for some, such as Meyer, 2010, a part of one) of these books could be used as the core content source, and one or more could serve as secondary reading to augment the main book. Alternatively, two books from this list and that complement each other in some respect, as discussed subsequently, could be declared as primary reading. (The student who complains that use of two books is asking far too much should be reminded that the two books span nearly the same space, and thus is far from being double the work; and also that his future competitors for top jobs, sitting next to him in class, but also globally, are more than happy to get extra reading material, instead of being confined to a single presentation of the subject, with its inherent and unavoidable limitations; see Section 2.2.)

Fortunately, the times are long gone in which the instructor has to decide on a single textbook, possibly with a list of suggested outside reading, and knowing full well that most students have no intention of buying more than one book for a course, quite possibly also because of the prohibitive cost required to do so. With easy (and, let's call it for what it is from the viewpoint of the student, namely, free, i.e., ignoring the negative externalities incurred by the publisher and author) access to the pdf files of dozens of books, it is both feasible and recommended to use two primary books (and possibly occasionally refer to passages from yet others).

[7] The quote is from Tucker's 1993 article "The Growing Importance of Linear Algebra in Undergraduate Mathematics", in: The College Mathematics Journal, 24(1), pp. 3-9. It is well worth reading, for the historical information, and on getting the two Jordan's, Camille and Wilhelm, correct. (The distinction, and some historical information on the two is also given in footnote 4 of Meyer (2010, p. 15).) Tucker was a prominent figure in the AMS (Applied Mathematics and Statistics) faculty at SUNY Stony Brook University while I was an undergraduate there, majoring in AMS (and economics).

For the idea of using two books, instead of one, and such that the two books complement each other in some regard, a first possible "partition" is to have one book that is predominantly algebraic in approach, and the other being predominantly geometric. All books of course have some aspects of both presentation styles, though some clearly emphasize one approach over the other, e.g., Meyer (2010) and Blyth and Robertson (2002) for the algebraic; and Shifrin and Adams (2011), and Banchoff and Wermer (1992), for geometric.[8]

Another possible "partition" of books is into either the computational, or the structural category, where the former means the emphasis is on matrices and matrix algebra; and the latter is on linear mappings and algebraic transformations. Again, most books embody aspects of both, though some are more extreme, such as Axler's *Linear Algebra Done Right* and Valenza's *Linear Algebra: An Introduction to Abstract Mathematics*, both of which take the structural approach (and are discussed below in Sections 3.3 and 3.4.1, respectively).

Other such partitions are possible, e.g., the terseness and sophistication of the presentation. In particular, some books are constructed explicitly to avoid being an onslaught of theorem-proof pairs, and include more chatty, conversational explanations, in addition to proofs (e.g., Anthony and Harvey, 2012; Nicholson, 2021; and Bretscher, 2013). A book from this category could be paired with one for which the book is dominated by, and defines itself via, the proofs of theorems.

Further possible considerations include restricting the selection of books to those for which the electronic versions are legally free. While this choice would rule out some of my personal favorites, there are still a handful of excellent free books, and it would not be a terrible hindrance to the instructor or student to follow this rule. I purposely include several such books in Sections 3.2 and 3.4.

Over the course of inspecting numerous books, I have developed a radar for certain aspects that play a role in my assessment. I include a small and far from exhaustive sample here, and use a small subset of books (chosen essentially randomly) to illustrate.

1. When presenting the three elementary row operations and illustrating row reduction, does the author explicitly explain why each of them, especially adding a multiple of one row to another row, does not change the solution space? This takes one or two simple sentences. For example, Williams (2018, p. 9) admirably writes:

   > Elementary transformations preserve solutions since the order of the equations does not affect the solution, multiplying an equation throughout by a nonzero constant does not change the truth of the equality, and adding equal quantities to both sides of an equality results in an equality.

   This can be contrasted to Robinson (2006, p. 34), "The critical property of such operations is that, when they are applied to a linear system, the resulting system is equivalent to the original one." Undoubtedly correct, but perhaps you should tell the student *why*? Equally uninformative, and arguably worse, Said-Houari (2017, p. 39) writes "We know from elementary algebra that if we add an equation in the system to another one and then replace the original equation by the sum of the two, then the solution does not change", followed on page 40 by "The following elementary row operations, will not change the solution." Meyer (2010, p. 5) at least addresses the issue, and explicitly asks the student to provide "explanations for why each of these operations cannot change the solution set".

2. When introducing matrix multiplication, is it motivated? For example, Robinson (2006, p. 7) does an excellent job in this case, showing (obviously) the composition of two 2×2 systems. Meyer (2010, p. 93) does the same, and includes a discussion about Arthur Cayley.

[8]Students with preference for the geometric presentation might also enjoy Jordan Ellenberg's new pop-math book *Shape: The Hidden Geometry of Information, Biology, Strategy, Democracy, and Everything Else*, 2021, noting that he is also a mathematics professor, and presumably would also advocate exposing students to *both* approaches.

Said-Houari (2017, p. 13) states it as a left-to-right sequence of matrix-times-column operations, themselves being a top-to-bottom sequence of dot products. This is obviously correct, and also a common way of showing several expressions for matrix multiplication, but formally, he does not motivate the definition. While someone who knows the topic of course fully understands what the author is saying, I am skeptical that a student new to the topic would be pleased with that presentation. There are other books that also do not motivate the definition, though some will say something to the effect of "later we will see why this non-obvious definition is meaningful."

3. Does the book present a visual indicating matrix multiplication? While this is more common in larger, colorful, flashy, chatty books filled with basic numeric examples, and suitable for high-school pupils or college freshman outside of STEM-based departments, I still think even a simple graphical illustration is worth a thousand words. Said-Houari (2017, p. 15) (which is optically excellent, but *not* flashy, or suitable for high-school kids) has a large, optically impressive color graphic for matrix multiplication. Unsurprisingly, Klaus Jänich (1994, p. 69) has a full page of (simple, black and white) graphics dedicated to this. His (I think excellent, and still highly recommended as an older, supplementary) book is discussed in Section 3.4.1.

4. Many books show that the determinant in the 2×2 case, $ad - bc$, and the relevance of it being zero or non-zero, results from row reduction or substitution. I was pleased to see that Said-Houari (2017, p. 1) does an outstanding job in this case, by emphasizing what is relevant is the *slopes of the two lines*, $-a/b$, and $-c/d$, and then shows graphics of the three possible cases. Obviously, if the slopes are equal, then $ad - bc = 0$, and, graphically, there is either no solution or an infinite number. (Said-Houari's book is among the "highest variance" books I have inspected, in the sense that, the author really nails it with some things, offering outstanding presentation and explanation, while for other things, he misses the boat.)[9]

5. Motivate Cramer's rule by showing, in the 2×2 case, by the usual simple methods, that each of the two terms of the solution vector can be written as a ratio of determinants, *à la* Cramer's rule. Robinson (2006, p. 58) does this (even more impressive, because it is, relatively speaking, one of the more advanced books), as does Shen, Wang, and Wojdylo (2019, p. 20) (albeit with an annoying typo; their book is discussed in Section 4.2).

6. I found that the level of sophistication deployed for the development of determinants serves as a reasonably good gauge for the overall mathematics level of the presentation, say "low" (nothing is proven, the basic calculations for 2×2 and 3×3 determinants are shown, the general case using a Laplace row or column expansion is given, and not much more), "medium" (some things proven), and "high" (everything is proven). This observation is useful when comparing dozens of books. There is no value judgement here, except to say that I would incorporate at least one book into the curriculum that ranks "high" in this category.

[9]A further case in point is on page 29, before which we, understandably at page 29, have not heard of, or learned a thing about, linear combinations, linear independence, dimension, (row and column) rank, the normal form of an $m \times n$ matrix, matrix equivalence, null space, nullity, rank + nullity, spanning sets, basis, *or even* row reduction; yet we are presented with Theorem 1.2.9: Let A be a square matrix in $\mathcal{M}_n(\mathbb{K})$. Then the following two properties are equivalent:
1. the matrix A is invertible;
2. the homogeneous system associated to the matrix A has the trivial solution $X = \mathbf{0}$ as the unique solution.
The author proves (1) $\Rightarrow$ (2) by writing $X = A^{-1}b$, using $b = \mathbf{0}$, with inverses indeed having been defined, and appealing to the uniqueness of matrix inverses, which also was previously proven (top of page 24). So far so good. We then read: "We leave it to the reader to show that (2) implies (1), which can be done in several ways." Without having covered the aforementioned topics, the reader is supposed to do this on her own, and is even told, with just the meager tools so far developed, that it *can be done in several ways*. I emailed with the author about this, and he had in mind using basic principles with 2×2 matrices. He also agreed that it would be better to spell things out, or at least reference results that are proved later in the book.

The only book that gets "low" (for determinants, but applies to all topics) and that I still include in the list in Section 3.2 is Cohen (2021), its reason for inclusion discussed there. Some books in the "high" category include Blyth and Robertson (2002), Lang (1987), Meyer (2010), Robinson (2006), and Schneider and Barker (1973).

7. Notation. See Section 3.5.

8. While not a must, I find it a nice added benefit if the book discusses linear programming. This is not only a great application of linear algebra and an important topic in optimization and computer science, but also is used in finance for the arbitrage theorem. I do not review books for this topic, but I came across what appears to be an excellent book, namely *Understanding and Using Linear Programming*, by Jiří Matoušek and Bernd Gärtner (2007), 222 pages, the prerequisite of which is a first course in linear algebra.

I conclude this brief sampling of criteria used in my evaluation by noting what I hope is now obvious: It is best to use (a linear combination of) at least two, perhaps three, books. It is tempting to state an answer to a reader's obvious impulse to ask "spare me the chit chat, and tell me, what three books should I use?" Among other problems that arise in wishing to state an answer, I would also need to condition on the mathematical level of the intended course.

I refrain from giving an exact recipe, first because I have not completed the herculean effort to read all the 20 books I give in my recommendation list in Section 3.2 (let alone the 34 additional books I discuss in Section 3.4), and second, because there is some "exchangeability" between some of the books, and it is not obvious which book in the equivalence class is the representative element. (In plain speak: I do not wish to favor one book over another when they are nearly equivalent.) What I *can* say is: For a course with a mathematical level that would be acceptable to beginning students in a math department, or for master's students in the social sciences, then the books that I would proverbially beg, borrow or steal to ensure having on my shelf if I were to teach such a course (and would be part of a list of recommended books to students) would include (given in the order that I present them below): { Meyer (2010), Ricardo (2010), Blyth and Robertson (2002), Lang (1987) }.

The linear-algebra-book-savvy reader might immediately protest (and is another reason I really do not wish to explicitly state my favorites) that I do not include the titans, Axler and Strang. The books by these outstanding authors are discussed in Section 3.3. Next, that just-stated list is *not* necessarily a short list of books to be recommended for teaching the course! The reason I have 20 books in my top list in the subsequent section is precisely because they are all excellent in many regards, and my just-stated short list consists of (i) those books that are among the higher-level ones (but are not the only) in terms of mathematical presentation *for a first course* (see Section 4 for the set of books I favor for teaching a second course in linear algebra); and (ii) include, but are not the only, ones that I have carefully read. The set of books that I would designate as the "primary list", discuss in class, and explicitly ask students to read, would certainly include one or two others from Section 3.2, and there are so many good ones, exacerbating the choice.

Making matters yet more complicated, I have yet to address an arguable elephant in the room, namely that these books I have so far mentioned are not legally free! This behooves me to favor Selinger (2020) and Nicholson (2021).

> **Remark**: There are lists of recommended books on the internet, e.g., https://bookauthority.org/books/best-advanced-linear-algebra-books, this one being shorter, and having perhaps a 40% overlap with my list. This indicates (i) just how many linear algebra books are out there, and one cannot possibly review a large percentage of them, let alone all of them; and (ii) in my opinion, how faulty and incomplete many of these lists in the internet are.
>
> I have omitted from my discussion some books that I found problematic, e.g., poor reviews giving critique that I confirmed. For example, consider the otherwise attractive book *Linear Algebra: Algorithms, Applications, and Techniques*, by

Richard Bronson, Gabrial Costa, and John Saccoman, 2nd edition, 2007. Amazon readers say it has many typos and mistakes. I found one while looking for something, namely on page 204, it refers to example 10 of Section 2.4, but this does not exist (that section has examples 1 through 4, none of which is the one referred to). There is also no full table of contents in the book (!), though each chapter begins with a chapter outline structure. Theorems use a new counter in each section of each chapter, as opposed to a numbering that begins with the chapter number. Finally, I noticed that many proofs are relegated to the exercises, which I (and surely students) do not appreciate. There is a 3rd edition from 2013, but I could not find a pdf of it on the web. From the Amazon preview, it now has a half-page table of contents, just listing the 6 chapter names and pages, and the 5 appendices. Interestingly, the MAA review of this 3rd edition, https://www.maa.org/press/maa-reviews/linear-algebra-algorithms-applications-and-techniques, is very positive! I still am not convinced, as the book appears to be too dumbed down: The MAA reviewer writes "The authors have made every effort to make this book as easy for students to use as possible. Theorems and computational methods are boxed and easy to find. Some definitions are boxed and some terms are italicized in the text with the definition repeated in blue in the margin. Throughout there are notes in the margin to highlight important ideas."

Another (seemingly popular) book I omit is *Elementary Linear Algebra* (the one with or without "Applications Version" in the title), 2013, by Howard Anton and Chris Rorres. (Interestingly for me, I used Howard Anton's calculus book as a freshman.) This book appears too rudimentary, with numerous poor reviews. In my opinion, it cannot compete with several of the books I list below.

Another is what appears to be the very popular (presumably in the USA) book *Linear Algebra and Its Applications*, by David C. Lay, Steven R. Lay, and Judi J. McDonald, 5th edition, 2015. It is from Pearson, over 200 USD, and involves use of Pearson's MyMathLab. The MAA review https://www.maa.org/press/maa-reviews/linear-algebra-and-its-applications makes it clear that "This is a techniques book and not a proofs book".

As a final example of another (and very recent) book, the MAA review of Przemyslaw Bogacki, 2019, *Linear Algebra: Concepts and Applications* indicates the book is just "Yet Another...", with nothing remarkable, nor does it have any color, and even the latexing (parenthesis size) has issues. The reviewer goes on to state "at the end of the day there simply isn't very much about the book that stands out from the *many* other linear algebra books on the market." (his emphasis). On a positive note about that book, the reviewer does say:

> [The author] moreover gives several genuinely interesting applications of the material to multivariable calculus. My favorites involve applications of the SVD to surface integrals and to Jacobian matrices and the geometry of change of variables for double integrals. In fact, I think that the author's 30 page treatment of the SVD and its applications is probably the high point of the text.

That comment earned the book a mention in my supplementary text list below in Section 3.4.

3.2 List of Suggested Primary Books

> For some, matrices have a life of their own, that is, an existence apart from representing linear transformations. R. R. Stoll and E. T. Wong (1968, p. 145)

> For the last thirty years there has been a vigorous and sometimes acrimonious discussion between the proponents of matrices and those of linear transformation.
> Hans Schneider and George Phillip Barker (1973, p. vii)

> [T]he challenge is to try to find a middle ground blending vector spaces and matrix methods and at a level that does not scare off users and yet smooths the transition for mathematics majors to advanced courses. Alan Tucker (1993, p. 8)

The particular ordering of the books in this next list is not indicative of anything important, and correlates somewhat with the ordering in which I inspected them. Abbreviation "LA-1" stands for: A first course in Linear Algebra.

LA-1 Book Choice 1: *Linear Algebra: Concepts and Methods*, by Martin Anthony and Michele Harvey, 2012, 516 pages.

I have carefully read most of the book (and emailed with the authors about some notational issues).[10] The setup, explanations, examples, and proofs are outstanding in terms of a first exposure to the subject for non-mathematics university students. There is arguably some amusing "hand holding" in terms of extra comments and spelling out obvious things—no doubt a welcome attribute for many beginning students. However, it is not a child's book (or, better said, it is not a high-school book): They have serious coverage of, e.g., diagonalization, including detailed, full proofs (that require complex numbers). I consider this to be an excellent book for its target audience.

Regarding the mathematical level, it could be placed into a similar group with books such as DeFranza and Gagliardi (2009), Williams (2018), Johnston (2021), and Norman and Wolczuk (2020) (all discussed herein), though not in terms of appearance or style of presentation. The first three of these four books use (for lack of a better word, and I do not mean it disparagingly) "modern high school format", meaning, generous use of color, margins, extraneous lines, very clear, colorful graphics, super clear fonts, lots of basic examples, and (for DeFranza and Gagliardi, and Williams) numerous (albeit basic) discussions on applications. Anthony and Harvey's book does not use color, or margins. They also do not have (real life) "applications", and I think it was a good choice to omit them: They concentrate on the math, and explain it very well. The font is crystal clear (as is the available pdf file of the book) and the book is a pleasure to read. From the preface,

> We have attempted to write a user-friendly, fairly interactive and helpful text, and we intend that it could be useful not only as a course text, but for self-study. To this end, we have written in what we hope is an open and accessible — sometimes even conversational — style, and have included 'learning outcomes' and many 'activities' and 'exercises'.

The following MAA published review is very positive on the book, and begins by addressing the "dumbing down" of math courses, and how this book reflects this modern tendency (no doubt exacerbated by the aforementioned hand-holding comments strewn throughout the book) but also praises the book for still having solid content (which goes in fact a bit beyond the minimal core of topics) and, along with examples, motivations, and explanations, has equal emphasis on full proofs of results: https://www.maa.org/press/maa-reviews/linear-algebra-concepts-and-methods.

[10] It is in general good to know that the authors of an adopted book respond to emails. The notational issue (and my only critique so far of this book) is discussed below in Section 3.5. The authors (politely though adamantly) defended their notation, and we continue to go back and forth in emails...

As a possible critique of this book (and you will have to work hard to find one), it does not discuss several results one might have wished for; a notable one being the Cayley-Hamilton result (and thus the minimal polynomial and diagonalization), though it does have a whole (short) chapter on diagonalization (leading to the result they prove, namely "A matrix is diagonalisable if and only if all its eigenvalues are real numbers and, for each eigenvalue, the geometric multiplicity equals the algebraic multiplicity"). Omission of Cayley-Hamilton is also the case for the books by DeFranza and Gagliardi (2009), Williams (2018), Norman and Wolczuk (2020), and Johnston (2021) (with Williams mentioning it in an exercise on his page 195, just stating the result and asking the reader to verify a numerical example with a 2×2 matrix; and Norman and Wolczuk asking the reader to prove it, on page 382, as part of their set of "Further Problems", which, as they state, "are intended to be challenging".)

This is not a decisive critique of any of these books, as results such as Cayley-Hamilton can be added by the instructor if desired. As discussed above in Section 3.1, I anyway recommend using more than one primary book, such as one that is more elementary in terms of mathematical presentation, and one that is more sophisticated, e.g, Ricardo, Meyer, or Blyth and Robertson.

LA-1 Book Choice 2: *An Introduction to Linear Algebra*, Third Edition, by Daniel Norman and Dan Wolczuk, 2020, 574 pages. From the preface and having looked a bit at the content, the authors indeed make a strong attempt to communicate with and motivate the student. The book has a very pleasant design, quite similar to Poole (2011) and (but not as flashy as) DeFranza and Gagliardi (2009), Williams (2018), and Johnston (2021). I would put all of these mentioned books in the same equivalence class, in terms of level and pedagogical design. Similar to Leon (2015) (that being another book that is rather close to this grouping, but is slightly more mathematically advanced), Norman and Wolczuk include a section on the SVD.

The discussion of determinants is "medium" level, as is the case with all of these books, and overall well done. As yet another case in point of the benefit of multiple text books, Norman and Wolczuk (2020, p. 329) give clear details for the proof of what they term the "False Expansion Theorem", which is required to show that the off-diagonal elements of $A \operatorname{adj}(A)$ are zero, $\operatorname{adj}(A)$ being the adjugate of square matrix A. This can be contrasted to Robinson (2006, p. 81), who (arguably appropriately for the level and nature of his book) just writes "if $i \neq j$, the sum is also a row expansion of a determinant, but one in which rows i and j are identical. By 3.2.2 the sum will vanish in this case."

I found Section 8.3 to be very good, on graphing quadratic forms in $\mathbb{R}^2$, something not every book does as well. I have also emailed with author Wolczuk; see one of the comments in Section 3.3 below.

LA-1 Book Choice 3: *Matrix Algebra and Applied Linear Algebra*, by Carl Meyer, 2010, 718 pages. I have carefully read (and immensely enjoyed) the first half of the book, including the exercises (most of which are algebraic, and not numeric "plug and chug") and their solutions (available to all readers). A second edition, fixing the errata from the first, and featuring new and larger sections on least squares and the SVD (both of which are already available for downloading from the author's website), is expected to be out soon, as confirmed to me by the author.

> **Remark**: Apropos the author, he co-authored what appears to be, based on the rave reviews, *the* book on the workings of search engines, namely *Google's PageRank and Beyond: The Science of Search Engine Rankings*, by Amy N. Langville and Carl D. Meyer (2011). One can then safely speculate that Meyer's linear algebra textbook also covers the required topics in depth. To check the obvious, chapter 8 of Meyer (2010) is indeed dedicated to the Perron-Frobenius theorem. (The words "Google" and "PageRank" do not

occur in the book, but a two and a half page example on search engines is given, in the context of the SVD, on page 419.)

Many linear algebra books published after a certain date mention and briefly discuss the functioning of internet search engines as one of the showcase examples of the practical applications of linear algebra. Leon (2015, pp. 336-7) is a good example of a non-superficial discussion that includes mention of Markov chains and the Perron-Frobenius theorem.

If such a "first course" were based on Meyer's book, I would offhand recommend to use up to and including section 4.5 (and all the exercises), and also section 6.1, on determinants (noting there is a section 6.2, "Additional Properties of Determinants", that could wait for the second course). This already implies well over 200 (dense!) pages and a large array of (computational, but primarily) proof-based exercises. One could go somewhat beyond my stated offhand recommendation, and/or use a second book to cover all other first-course topics such as eigen-things, and diagonalization. Of course Meyer covers these topics, and in impressive detail, but they would then appear in a second course, and not in the first, if using my stated partition.

Meyer is among the most mathematically sophisticated books at this first-course level, but it is also a far more ambitious project, and a very large book (with more words per page than most other books), essentially comprising "A beginning course in LA" and then (at least a solid part of) an intermediate one. For example, not only is Cayley-Hamilton covered, but also the minimum polynomial, Krylov subspaces, Jordan form, and Perron–Frobenius theory.

This book's pdf file used to be freely and legally available (and was "perfect", e.g., not a scan). According to http://matrixanalysis.com/DownloadChaptersOFF.html, "Unfortunately, due to egregious infringements of copyright provisions and the misuse of distribution rights, the text and the solutions manual can no longer be made freely available for download." It can, of course, still be found on the usual web sites that host thousands of books. One can hope that the pdf file (and not just the hard-copy) of the soon-forthcoming second edition will be available for purchase (remember, most people, especially students, do not want physical books; see Section 2.1), in which case, I would recommend serious students to part with some cash and have this as one of their primary resources.

In addition to very strong content and structure of presentation, the book's layout, font, and optical appearance are very appealing, with the use of blue background shading for highlighting boxes of important text (but otherwise no other optical superfluous embellishments, as used in, e.g., DeFranza and Gagliardi, 2009; Williams, 2018; and Johnston, 2021). I refrain from elaborating my praise for this book, except to say "I wish I wrote it". I say more (positive things) about this brilliant book in one of the comments below in Section 3.3.

I have very few complaints, and list just one. On page 92, exercise 3.3.4, the reader is graphically shown three transformations, all from $\mathbb{R}^2 \to \mathbb{R}^2$, these being rotations, reflections, and projections (onto the line $y = x$). Up to this point, perpendicularity, and the dot product, have not been defined. The question to the reader is, which of these transformations are linear (of course, they all are). I think these cases should be shown in the text, given their importance. (These three mappings indeed appear again, in more detail, on page 239). As another case in point of an alternative book giving a clear presentation, that for rotations is nicely done, with a perfect graphic using unit vectors in the plane, in Lang (1987, p. 85), while for projection, Lang (1987, p. 99) has a very clear (one page) discussion, having previously just shown a very clear graphical motivation for perpendicularity after having defined it via the scalar (dot) product. Admittedly, Meyer gives the solutions: For the projection case, he states the answer and says to use Pythagoras, albeit without details (it is admittedly easy, and a great way of doing things in the $n = 2$ case). It might however be a good for a student to see the intended solution with Pythagoras, and also to see, at this point, dot products, perpendicularity, and projection, and then confirm that projecting onto the vector (k, k), $k \neq 0$, results in the same answer, and which is invariant to the choice of k.

LA-1 Book Choice 4: *A Modern Introduction to Linear Algebra*, by Henry Ricardo, 2010, 654 pages. This book has only two reviews on Amazon, only one of which is slightly informative—and it is not positive. The MAA review of his book, https://www.maa.org/publications/maa-reviews/a-modern-introduction-to-linear-algebra, is decidedly negative: I will go through the critiques below. Further, as of 2009,[11] the author is retired, and there appeared to be no hint at a second edition, no web page for the book, no errata list, nor solutions manual (though keep reading). As such, I anticipated a quick dismissal. Instead, my initial inspection turned into me reading it nearly in its entirety—I think it is a very good book, albeit with some (easily surmountable) flaws.

I hold this book in high regard, due to the presentation and structure of the material, and the clarity of the theorems and their proofs. I claim it is very well, enjoyably, and even passionately written. Also, the author refers generously to Meyer's book (whom he mentions, along with Strang, Halmos, and Hoffman/Kunze in the acknowledgements), which I also hold in high regard, and often uses the same notation as Meyer (such as for the row / column equivalent matrices). He also cites some journal articles (e.g., a new proof of Cramer's rule, by Friedberg, the first author of a book I discuss below in Section 4.1), and occasionally refers to the ubiquitous and magisterial book known to all numerical analysts, Golub and van Loan. (He also occasionally refers to Matlab, as well as Maple, mentioning also that Maple can be accessed via a toolbox from Matlab, which is something I have used for many years and am familiar with.)

His chapter 4 on determinants is excellent, taking an approach based on LU decomposition, and thus is a different approach than used classically, such as in Lang, which is also a very good presentation. (See below for further comments on Lang and his presentation of determinants.) On the web, there is a short set of presentation slides from the author in which he jokingly critiques the critique of Axler about disliking determinants because they are highly non-obvious and very abstract; and shows how determinants are rather easy and elegant when using the LU decomposition aspect.

I've read the book very carefully and found a modest number of typos. Here are some (not all) of the critiques mentioned in the MAA review of his book, given in italics, and, for each, my thoughts.

- *In terms of physical appearance this text is not competitive. The illustrations are not up to the quality we now expect in a main line textbook. In addition, the type face used makes the portions involving vectors and matrices somewhat hard to read. The theorems and corollaries are shaded in gray; otherwise, there is no color. There are wide margins on every page resulting in a much thicker book than necessary. Often, these margins are used to insert historical or other interesting side notes — in the case of this book, they merely take up space.*

  I find the book optically fine in terms of font and typesetting, albeit it is not as impressive as, say, Anthony & Harvey, or the books by Nathaniel Johnston. The font is crystal clear, and the pdf file is perfect—fully expandable (and you can mark and search text), so the book is optically very pleasant.

  I completely agree about the wide margins: They serve no purpose, and are in fact an annoyance when reading the pdf on a tablet and changing pages. And remember, nobody except the library gets the hardcopy anyways anymore—and someday

[11] This is apparent from the December 2009 MAA review of the book *Geometric Linear Algebra 2* by I-Hsiung Lin: The reviewer is Henry Ricardo (he has in fact evaluated numerous books for MAA), and it states that he is retired from Medgar Evers College (CUNY).

that will change too. Its pdf appearance is, going forward, all that counts. As a further remark, authors have little control over this—formatting is usually decided by the publisher.

The graphics are adequate, certainly for conveying the concepts, albeit indeed not impressive, such as those in, say, Saveliev's *Linear Algebra Illustrated*, 2020, mentioned below in Section 3.4.2 (or in Miroslav Josipović's *Geometric Multiplication of Vectors*, 2019, a book I do not otherwise discuss; see, e.g., his pages 45, 46, and 47). I think many mathematics academics, certainly Klaus Jänich, two of whose books I mention in this document, would agree that the use of graphics for teaching is highly important—but, in my opinion, they do not have to be fancy or (unless it is really needed to differentiate objects) in color.

- *Vectors and scalar multiplication are nicely motivated, though not enough attention is paid to their use in physics. ... The vector cross product is defined in an exercise — no motivation is given for this except as a footnote referring students to a textbook on Advanced Engineering Mathematics.*

  I am not convinced more attention needs to be paid to a physics audience, given the numerous other fields that use linear algebra. Regarding the cross product, I agree: Either present it correctly, or not at all.

- *The first set of exercises contains some nice problems, but concludes with the definition of a convex set and of the convex hull of a set followed by a set of problems I would guess few sophomores would make much progress on.*

  I picked up on this too during my reading of the book. Obviously, no book can cover every topic, and at least some topics not covered in the main text get a mention in the exercises. I agree with the reviewer that a sizeable percentage of students will find the exercises related to convexity rather challenging, and without available solutions (or an instructor), this can lead to frustration.

  As an aside, the book mentioned below in this list by Blyth and Robertson does the same thing, mentioning convexity only in an exercise, namely their page 111, exercise 6.34. Lang's book *Linear Algebra* has a whole chapter on convexity. (And Steven Krantz has a whole book dedicated to the topic.) This is of course not to say that Lang's book is superior, but rather provides yet another example of the kind I give in Section 2.2, showing the advantages of working with the (you know I am going to say it) convex hull of information across multiple books.

- *There is only one worked out example of finding the eigenvalues of a matrix. A second example, of a matrix whose characteristic polynomial has a repeated root, merely lists two independent eigenvectors corresponding to that root. In my experience, students have trouble dealing with the underdetermined systems that result in these cases. The concept of multiplicity occurs several sections later.*

  This is my primary complaint of the book: Ricardo (amid an otherwise excellent chapter of material) just gives the eigenvectors in the examples, and does not show how they are computed, no less in cases with repeated roots. This is a book for a first course in linear algebra—what was the author thinking? Perhaps unexpectedly, Lang's books do a good job here, showing numerous examples of eigenvector calculation. Meyer and, not surprisingly, Anthony & Harvey, also emphasize the relevant calculations. (I presume many other books also do, notably the large ones with the ultra-fancy layouts, colors, and designs, but I cannot go and check everything.) Fortunately, this issue is small, and can be filled in during lectures, and augmented from supplementary reading.

The reviewer concludes with:

> In summary, this text is not up to the standards required for a market as large as that for a linear algebra text. There are not enough computational, concrete examples to motivate the big ideas which tie this material together.

> There are lots of good homework problems, so this might well be a good source for exams or extra-credit problems or the like. As a text, I think there are better choices out there.

(Not a very nice retirement gift to Ricardo, who himself has done numerous reviews for MAA, and was the "Secretary of the Metropolitan NY Section of the MAA".)

There are indeed many good exercises in the book, but it would be expedient[12] (for the student, as well as the instructor) to have a complete set of detailed solutions—as is the case with Meyer, which is, irrespective of the availability of solutions to exercises, an outstanding book.

Given the excellent other choices I present in this list, it is difficult to argue in favor of using Ricardo's book as the primary and only one for a course. Nevertheless, I am glad I took the time to carefully read it. I think it is way better than the reviewer made it out to be, and students would profit from reading it alongside a second text.

Update! I managed to get into contact with the author, having obtained his private email address. To my pleasant surprise, there is an instructor's manual, with *all* solutions to the (recall, numerous and good) exercises, totaling 377 pages. Henry kindly sent this to me, and also mentioned that there is a student version, with just the solutions to odd-numbered exercises. Typographical and other errors are also noted in that document, in red color. I was particularly delighted to have received this, as I think this book is far stronger than a superficial impression, based on reviews, would make it out to be. I also obtained an answer to my rhetorical question posed above, "what was the author thinking?": He explained that he usually taught the class in a computer room, with students using Maple to crank out routine calculations, resulting in less emphasis on basic pen-and-paper calculations that can be done for very small matrices. Fair enough—indeed, computers are here to stay; and as mentioned above, it is easy to supplement the book with some sample eigenvector calculations. Finally, and quite understandably, a second edition is not planned. However, we do have a 2020 book from Ricardo, on differential equations (see Section 4.10), and which *does in fact* enjoy a very positive MAA review. So, some justice done...

In summary, this book is high on my list of favorites.

LA-1 Book Choice 5: The first of these two connected books: *Introduction to Linear and Matrix Algebra*, 482 pages; and *Advanced Linear and Matrix Algebra*, 494 pages; by Nathaniel Johnston, both published in 2021.

The optical presentation is among the best I have ever seen in a textbook, with the author having used some color and judicious spacing. The layout (font, everything) is beautiful. The (findable on the web) pdf file is perfect; and the books have solutions to about half of the exercises. Due to obvious time constraints, I have only skimmed them, and can say that, pedagogically, they seem outstanding. Level-wise, I would put Johnston's first book in the same grouping as DeFranza and Gagliardi (2009), Anthony and Harvey (2012), Williams (2018), and Norman and Wolczuk (2020). Johnston's two books, when taken together, are, roughly, "in the same equivalence class" as Meyer's single large book, as well as (discussed below in Section 4.1) the book by Friedberg, Insel and Spence, though each of these books covers some material not found in the others.

LA-1 Book Choice 6: Belkacem Said-Houari, *Linear Algebra*, 2017, 384 pages. This book is optically very appealing, and has all the right topics for a first course, and goes a bit beyond, e.g., the author covers Cayley-Hamilton, the minimal polynomial, and Jordan canonical form. It has clear, complete proofs. I only skimmed it, but did notice a very nice chapter on inequalities, going beyond Cauchy Schwarz and triangle. Further,

[12] One definition of expedient is: "helpful or useful in a particular situation, but sometimes not morally acceptable". I think this word hits the spot.

it discusses and proves some results not seen in other books at this level, such as the Rayleigh–Ritz result on eigenvalues (page 344), Sylvester's Law of Inertia (page 359), and the spectral radius (page 375).

The author provides detailed solutions to all the exercises in the book, some of which are nontrivial and algebraic in nature. I did notice some overlap with exercises involving 2×2 matrices, and the contents of Pop and Furdui (2017). As one of many examples, we have the same exercise from Pop and Furdui (2017, pp. 22, 37) and Said-Houari (2017, pp. 60-1).

I discuss some positives and negatives of this book in Section 3.1. I think this book can be very useful and attractive as the second of a set of primary books (also because it goes a bit beyond the first-course standard syllabus), though in light of some of its competition (in terms of structure of presentation), I would not choose it to be the primary book.

LA-1 Book Choice 7: *Linear Algebra*, by David Cherney, Tom Denton, Rohit Thomas and Andrew Waldron, 2016, (first edition 2013), 436 pages. The electronic version (pdf) is legally free, via a Creative Commons license; `https://open.umn.edu/opentextbooks/textbooks/188` or `https://www.math.ucdavis.edu/~linear/`, with these web pages having the "preface", or description of the book by the authors (it does not appear in the book itself); and the former link also containing two (positive) book reviews (both from professors who give their names). The book is optically excellent, with some color graphics, and also clickable links to web-based material, including lecture videos. In addition to containing all the usual (and mandatory) material, it also contains a short chapter on the simplex algorithm for optimization. Appendix C is a long list of useful online resources (including Beezer's book, and Strang's videos), while other appendices D, E, and F give sample exams, and appendix G, called "movie scripts" is 66 pages of hints and solutions. There is also an online interactive version, with link given in Section 2.1.

LA-1 Book Choice 8: *Linear Algebra with Applications*, by Steven Leon, 9th edition, 2015, 520 pages. Another book, discussed below, that made it to the 9th edition, is Williams (2018). Having been invited by the publisher to continue making so many editions, one can probably assume that the book is successful. While I have not read any of this book in the same detail as I have with a few other linear algebra books discussed herein, its presentation and content place it near the top of my list of favorite books.

The book goes somewhat beyond the core set of topics (e.g., section 6.5 is on the SVD). It has two associated chapters on the web via Pearson (one rather short one, on iterative methods, and a more substantial one on the Jordan canonical form). A solutions manual is available via the publisher (Pearson) for instructors. The book uses a light-blue color for highlighting particular text, but otherwise does not overdo it with colors and superfluous embellishments. The font is crystal clear and the available pdf file is perfect.

The book refers to and uses Matlab, a feature I appreciate. It also has various applications (including the now common, if not nearly mandatory application of the page ranking web search algorithm, page 336; and digital image processing, page 368); but also PCA, as used in psychology for measuring intelligence.[13]

In the discussion below on DeFranza and Gagliardi (2009), I mention some similarities in coverage of examples between DeFranza and Gagliardi (2009), Leon (2015), and Williams (2018). These books (along with Norman and Wolczuk, 2020; and Johnston, 2021) are relatively similar in presentation and appearance, though I would say that Leon

[13]That idea goes back to Karl Pearson, an ardent (atheist, Darwinist, and) eugenicist. An arguably better or more politically correct choice might have been to mention the Big Five Personality Traits, a famous factor analysis in psychology.

(2015) is more mathematically sophisticated than the others, and less "hand-holding". This might be the reason why the book gets a handful of scathing angry reviews at Amazon. For example, one reviewer exclaims:

> This book was incredibly horrendous at explaining basic matrix operations. In an attempt to be succinct?, the author conveys a wide variety of concepts in just a few pages. This is a great refresher for one who already knows the material, but for a beginning student of linear algebra, it simply creates a nightmare.
>
> Students just beginning linear algebra will be terrified by the amount of set theory and logic required to understand the problems and proofs that are presented.

LA-1 Book Choice 9: *Linear Algebra: A Geometric Approach*, 2nd edition, 2011, Theodore Shifrin and Malcolm R. Adams. As expected from the title, the book has many graphics, and in terms of optical presentation, font, spacing, use of color, etc., the book is very attractive (and the pdf file is perfect, with "clickable" entries, e.g., the table of contents, and also the table of notations). The table of contents has all the usual entries, and the book is 372 pages, which is close to the median. We do however see some more advanced topics, such as section 4.4, entitled "Linear Transformations on Abstract Vector Spaces"; and the last chapter, 7, covers complex eigenvalues, Jordan canonical form, "computer graphics and memory", as well as matrix exponentials and differential equations. The book also contains answers to selected exercises.

Given the authors' clear focus on a geometric presentation (which also presumably takes more work, e.g., it necessitates graphics, and also a consistent narrative of geometric interpretation), as well as the aforementioned solid presentation, this book recommends itself as the primary text for a first course in linear algebra, or deserves to at least be considered as a required or recommended secondary text, in conjunction perhaps with one that assumes an algebraic approach.

I was pleased to see that the MAA review is also very positive, https://www.maa.org/press/maa-reviews/linear-algebra-a-geometric-approach, writing:

> The first reaction from those reading this review will most likely be "yet another linear algebra book! why?" I have to admit that was my reaction, too, when I first got my hands on this book. I was more or less expecting yet another watered-down text, with more uninspiring visuals than useful explanations and more tedious matrix computations than clear theoretical interpretations. The good news is that I was wrong. This is a well-written textbook which focuses on the geometric interpretation of the basic concepts of linear algebra but does not hesitate to go into the abstract notions that make the whole subject stand on its own as a glorious chapter of modern mathematics. I am still not sure if I will drop my own favorite text for this one for my next linear algebra course, but it certainly presents a good alternative to the many books out there.
>
> ...
>
> The text makes a serious effort to embed the basic notions of mathematical proof into the main flow. The authors intend it to be used for a course introducing the basics of linear algebra while also preparing the students for more advanced mathematics courses where they will be reading and writing proofs of their own. This makes the text more appropriate for courses which are transitional in nature, where the audience includes students who are looking to become mathematics majors, rather than for courses where the sole purpose is to introduce the main tools of linear algebra to future physicists, engineers and economists. The informal language of the text is interrupted often with

more precisely stated definitions and theorems, and the students are gradually guided into thinking more rigorously and provided with progressively sophisticated exercises to test their developing skills in writing proofs.

...

[The book] does not attempt to revolutionize the teaching of linear algebra. In fact the table of contents is pretty traditional. The emphasis on geometry is also not incredibly novel; there are many other texts which focus on visuals and concrete geometric analogies...

...

I visited the page for this book in amazon.com and the slew of negative comments from students was, for me, a wake-up call. It is not uncommon that students have completely unexpected experiences with a text no matter how scrupulous the instructor may have been in her search for the best textbook to use.

LA-1 Book Choice 10: *A First Course in Linear Algebra*, Robert A. Beezer, 2015, 624 pages.

The electronic version is offered free; see http://linear.pugetsound.edu/, and, via the "GNU Free Documentation License", can also be modified for one's own use, "thus liberating it from some of the antiquated notions of copyright that apply to books in physical form", as the author states on that web page. The book emphasizes algebraic derivations, as opposed to the use of geometric motivations and visualizations, and was explicitly written to also serve as a vehicle to acclimate students to higher mathematics, i.e., more abstraction and proofs, and less rote learning of simple computational procedures. This is emphasized in the preface (where we also, somewhat uniquely for a math book, learn the very comforting fact that it can be appreciated and used by "those of any age"):

This text is designed to teach the concepts and techniques of basic linear algebra as a rigorous mathematical subject. Besides computational proficiency, there is an emphasis on understanding definitions and theorems, as well as reading, understanding and creating proofs. A strictly logical organization, complete and exceedingly detailed proofs of every theorem, advice on techniques for reading and writing proofs, and a selection of challenging theoretical exercises will slowly provide the novice with the tools and confidence to be able to study other mathematical topics in a rigorous fashion.

Most students taking a course in linear algebra will have completed courses in differential and integral calculus, and maybe also multivariate calculus, and will typically be second-year students in university. This level of mathematical maturity is expected, however there is little or no requirement to know calculus itself to use this book successfully. With complete details for every proof, for nearly every example, and for solutions to a majority of the exercises, the book is ideal for self-study, for those of any age.

...

Linear algebra is an ideal subject for the novice mathematics student to learn how to develop a subject precisely, with all the rigor mathematics requires. Unfortunately, much of this rigor seems to have escaped the standard calculus curriculum, so for many university students this is their first exposure to careful definitions and theorems, and the expectation that they fully understand them, to say nothing of the expectation that they become proficient in formulating their own proofs. We have tried to make this text as helpful as possible with this transition. Every definition is stated carefully, set apart from the text. Likewise, every theorem is carefully stated, and almost every one has a complete proof. Theorems usually have just one conclusion, so they can be referenced precisely later.

The MAA review https://www.maa.org/press/maa-reviews/a-first-course-in-linear-algebra is very positive, with its opening statement essentially sufficing: "Robert Beezer's free textbook A First Course in Linear Algebra (FCLA) is an excellent textbook." The only "critique" (and it is, arguably, a compliment), and then augmented with yet another feature of the book is:

> The flip side of this open source flexibility is that the FCLA looks rather plain, especially to students used to the glossy multicolor textbooks commonly used in the calculus sequence. The prose is not as polished as what one sees in published textbooks overseen by diligent editors. Countering this minor fault is the hyperlinked format offered by the electronic versions, in which students can click on a mathematics term to jump back to the definition or click on a theorem reference to jump back to its statement.

We also read:

> The text offers few applications, so some supplements may be needed for a "linear algebra with applications" course.
>
> Overall, while FCLA is not as polished as the popular texts from the major publishers, FCLA offers a very affordable and good quality alternative. We have both used it in our linear algebra courses and highly recommend it.

LA-1 Book Choice 11: *Basic Linear Algebra*, Second Edition, 2002, second printing 2005, 232 pages; and *Further Linear Algebra* (first edition), 2002, 230 pages; by Thomas Scott Blyth and Edmund Frederick Robertson.

The first of these two books is the one suitable for a first course in linear algebra. It gets a disproportionately large amount of attention, simply because it is one of a small handful of linear algebra books discussed in this section that I read, carefully and cover to cover. That in itself is a compliment to the authors, though inevitably, due to the detailed inspection, there are also more critiques of this book than others.

Crucial to mention before discussing these books is that the pdf files for them, as found on the web, are very good, easily good enough to optically enjoy reading the text, though of course they cannot match the staggering caliber of visual clarity inherent in the more recent, well-typeset books.[14] I only discuss the first of the two books, as it is relevant for this section on a first course in linear algebra.

Similar to Beezer's book, Blyth and Robertson are algebraic as opposed to geometric, and will surely come across as terse and austere for the uninitiated—by which I mean, an average student coming out of first class in calculus, having used some glossy, colorful, 600 page, high-school-worthy textbook. I presume the authors had the same goal as Beezer explicitly states, namely, to accustom students to how higher mathematics is presented and taught. *Basic Linear Algebra* starts with a short Foreword, showing an example of systems of equations from the *Nine Chapters of the Mathematical Art* "written during the Han Dynasty".

It then hits the ground running with chapter 1, on the algebra of matrices, as opposed to the more common way of beginning with solutions of systems of equations. The pace is quick but appropriate, and very detailed (though, as mentioned, some students not ready for this kind of presentation might beg to differ). For example, on page 41, the following two "theorems" (arguably nearly obvious results not in need of endowment with a theorem label) are given, and—despite being so easy—proved in clear detail:

[14] Not crucial to mention is that the authors write "role" as "rôle", this being—or at least having been—the UK way of doing things, noting that the authors are Scottish, or at least emeriti from the University of St Andrews. They also use the—or what I will call the—European bracket notation (perhaps Bourbaki would be better) to denote an open or half-open set, i.e., they express the half-open interval $[0, 1)$ as $[0, 1[$, a fact which, unlike the use of "rôle", might indeed be an eyesore for some readers. Admittedly, interval or general open set notation occurs very infrequently in the book, the first occurrence of which is on page 92, in "Supplementary Exercise" 5.39.

> Theorem 3.7: If the rows/columns $x_1, \ldots, x_p$ are linearly independent then none can be zero.

and

> Theorem 3.8: The following statements are equivalent:
> (a) $x_1, \ldots, x_p$ ($p \geq 2$) are linearly dependent;
> (b) one of the x_i can be expressed as a linear combination of the others.

The point is, while the book is written in a theorem / proof /example / exercises format, which might appear to the uninitiated as being dry, terse, abstract, unfriendly (and quite intimidating), in fact it is intentional to acclimate the reader to such format; and everything is clearly spelled out in painstaking detail. As another example, we find on page 60 the theorem (and proof):

> Theorem 4.1 Let A be an $m \times n$ matrix. Then:
> (1) A has a right inverse if and only if rank $A = m$;
> (2) A has a left inverse if and only if rank $A = n$.

I do not see this result all that often in other introductory books, though I admittedly have not checked that carefully either, and it could also be in a book, within its exercises.[15]

I could give many other examples, such as their excellent (meaning, clear and detailed) introduction to vector spaces in chapter 5. They differentiate between the number zero from the field, 0_F, and the zero as an element of the vector space, 0_V, and show proofs (as opposed to asking the reader to do so) for the basic results—admittedly easy, once you are familiarized with the idea—that follow from the usual eight assumed properties of a vector space. The naive student who thinks this book is terse and too big of a step away from what she is used to, has no idea of how generous and patient the authors actually are.

Having just praised the book, I now lodge a complaint (perhaps more against the handling editor): These authors should have gotten a solid (albeit proverbial) smack upside the head every time they use the word "clear", as in "it is clear" and "is clearly". I give three representative examples:

Example 1: After the machinery of elementary row operations, including their implementation by pre-multiplication by an elementary matrix, is very well developed—but without having introduced matrix inversion, and without having yet developed the idea that the matrix product **AB** has the same rank as **B** when **A** is square and full rank—we read (at the top of page 47) "Since it is clear that column operations can have no effect on the independence of rows, it follows that column operations have no effect on row rank." This result is crucial for what comes next, namely the proof that row and column rank are the same—a result of major and fundamental importance. I claim that, based on the material given up to that point, this is not clear at all. And they were doing so well up to this point!

Meyer's book, as one example, does a better job with this issue. An online answer that the reader of this document can immediately inspect, https://math.stackexchange.com/questions/3760618/how-prove-that-the-elementary-operations-dont-change-the-rank-of-a-matrix, indicates that the authors' assumption that "it is clear" is not justified. (See also https://math.stackexchange.com/questions/2513352/why-row-operation-does-not-change-the-column-rank.) □

[15] Boyd and Vandenberghe, *Introduction to Applied Linear Algebra*, (2018, Sec. 11.1) has some related material, as does Eves, *Elementary Matrix Theory*, (1968, pp. 98-9). The closest (and basically fully equivalent) presentation is that in Goodaire, *Linear Algebra: Pure and Applied*, (2014, pp. 388-9), where the following theorem is proven: Let A be an $m \times n$ matrix. Then the following statements are equivalent: (1) A has a right inverse. (2) The columns of A span $\mathbb{R}^m$. (3) rank$(A) = m$. (4) The rows of A are linearly independent.

Example 2: Another usage of "clearly" comes up on page 44, line −7: "Any two Hermite matrices obtained from M in this way are clearly row equivalent." Sure, they are! But I am not convinced it is "clear" to a certain percentage of students, notably those whose inherent *forte* is not mathematics. And if the authors want the reader to resolve this themselves, better to say so, making it an exercise. Many students do not like books that delegate proofs to the student as exercises (without available solutions no less), but somehow it is even worse when the author says "It is clear" (whether it is in fact clear, or not). Perhaps there is a clever, insightful way to see this result; while I have my easy, brute force way.[16] □

Example 3: This one gets the authors my *HOP*-award: for most haughty, overweening, and pompous writing: Just the single page 107 sports *five occurrences* of "clear(ly)"! The context of the fifth one is: "Since the inverse of an isomorphism is clearly also an isomorphism...".[17]

It is of value to know that the same page contains the first occurrence of *isomorphism*, in particular, "A bijective linear mapping is called a linear isomorphism, or simply an isomorphism", while adjectives *surjective*, *injective*, and *bijective*, in the context of linear mappings, are defined on page 102 (with admittedly good examples of mappings with one, but not the other, designation, as well as mappings with neither).

Given that the noun isomorphism was just defined on the same page, it is obvious that there was no discussion of the inverse of an isomorphism that I somehow missed, or general discussion of function or mapping inverses. And then to tell the poor reader "the inverse of an isomorphism is clearly also an isomorphism" borders on criminal. What is needed (again, for a book *at this level*, and in agreement with its otherwise very detailed presentation of even nearly obvious results, at least in the first few chapters) is a short digression proving existence, injectivity, surjectivity, and uniqueness, for the inverse of a bijective mapping, ideally in general, or just restricted to linear mappings. This is obviously standard material for, and well spelled out in, many first-course real analysis books (see Section 5), but a course in real analysis, not to mention the ensuing mathematical maturity, are blatantly *not* prerequisites for using this book.

An excellent presentation of this topic can be found, for example, in Jacob and Evans, *A Course in Analysis Volume l: Introductory Calculus, Analysis of Functions of One Variable* (2016, Ch. 5), in particular, pages 71 to 77, with the precisely needed result spelled out and proven very clearly on the top of page 77, namely:

> Corollary 5.18. If $f : D \to F$ is bijective then $f^{-1} : F \to D$ is also bijective and $(f^{-1})^{-1} = f$. Moreover f^{-1} is uniquely determined.

The issue is, while someone trained in mathematics intuitively understands a bijection to mean a correspondence between two sets such that each element of one set is paired with exactly one element of the other set (and note how the word itself is suggestive of its meaning, with "bi" meaning two, and "jection" being something like a projection), the whole point of a book designated as a first passage to higher mathematics for students with essentially a high-school understanding of plug-and-chug calculus *is to teach this intuition, via precise definitions, and clear explanations.* In this case in point, the authors jumped the gun, and failed in their mission (amid hilarious *HOP*-writing no less).

I did some further searching. Lang's *Linear Algebra* (1987, Example 2, page 69) has the details for (amazingly) *precisely the same mapping* (and without ever mention of the

[16] Denote by **A** and **B** two Hermite (that is, reduced row echelon form, as is more common) matrices obtained from matrix **M**, and having required k_1 and k_2 elementary row reductions, respectively, for their construction. Then, apply the reverse k_1 elementary row operations to **M** to get back to **A**, and then the k_2 operations to get to **B**, and thus the sequence of $k_1 + k_2$ elementary row operations indeed takes **A** to **B**, confirming the statement (by the definition of row equivalence).

[17] And being extra picky, I do not like the usage of *since* when its meaning is *because*, though apparently not everyone agrees with this; see, e.g., https://www.grammarphobia.com/grammar-html.

word “clearly”) used in the proof of theorem 6.6, and its corollary, on (that ill-fated) page 107 of B&R. □

The previous example hopefully bolsters my argument from Section 2.2 (there having used Euler’s constant for demonstration) of the benefits of knowing, and having (ideally digital) access to, numerous books on the topic. As yet another case in point (and, coincidentally again referring to Lang’s *Linear Algebra*): Chapter 8 of Blyth and Robertson is on determinants, and unfortunately is (in my opinion) poorly executed in the part on permutations, and in the discussion of the adjoint matrix on page 144. The comparable presentations of these topics, and in fact the entire chapter on determinants, in Lang (1987, Ch. 6), is, certainly in comparison, but also in general, excellent.[18]

As one example (of several I could give) of what went wrong in the discussion of permutations, their Example 8.6 on page 136 shows two rows of numbers, 1 through 6, and below that, a permutation of this, and with lines indicating the mapping between each element. We are told “In this the number of distinct crossings gives the number of inversions.” No further explanation is given to this interesting graphical approach. Perhaps it is obvious. I (and I think most students) would prefer some further discussion and detail. I found three books that do precisely that, namely:

(a) Howard Eves, *Elementary Matrix Theory*, (1968, pp. 160-3), in the section entitled *A Geometric Study of Permutations*;

(b) Hans Schneider and George Phillip Barker, *Matrices and Linear Algebra*, Second Edition, (1973, p. 173), in their lengthy and nicely detailed section entitled *Permutations and Permutation Matrices*, pages 172-9; and

(c) Derek J. S. Robinson, *A Course in Linear Algebra with Applications*, Second Edition, (2006, pp. 60-2), in his terse but excellent chapter on determinants.

All three of these books are discussed in Section 3.4.

As a counterbalance and argument that chapter 8 of B&R is still worth looking at, besides the fact that parts of it are indeed good, it has an excellent selection of exercises (which are endowed with solutions, or at least hints), such as the one on “pivotal condensation” for evaluating determinants. The authors write that it is both useful and “little publicised”, but give no further information. They are referring to what is usually called Chio’s pivotal condensation method, as I learned from reading Eves’ aforementioned book, in which we read on page 129 (from the 1968 version; it first appeared in 1966):

> Such a method of evaluating a determinant is called a *condensation method*, and most of these are variants of a procedure now known as *Chio’s pivotal condensation method*, since a form of this method was given by F. Chio in a paper of 1853, though an earlier trace of the method can be found in an 1849 paper by C. Hermite. Chio’s pivotal condensation method goes considerably beyond its practical aspect, for the process leads to some remarkable determinant identities.

Eves then goes on over four pages to derive it in full, show examples, and continues its discussion and applications in an appendix.[19]

So, in line with my recommendation to use more than one book for teaching, this serves as another case in point whereby a presentation of particular topic is done with more care, skill, and detail, than its counterpart in another (and, in this case, an otherwise very good) book.

[18]I found one typo in Lang’s presentation: On page 172, line -9, “Lemma 7.2” should be “Theorem 7.2”.

[19]The reader interested in this method should read Ken Habgood and Itamar Arel, *A condensation-based application of Cramer’s rule for solving large-scale linear systems*, Journal of Discrete Algorithms, 10(1):98–109, 2012; and also see, via e.g., semanticscholar.org, the numerous, yet more recent papers that cite this one.

From the preface of Blyth and Robertson's first book, we do not get a precise statement of the desired prerequisites, but we do read:

> The word 'basic' in the title of this text could be substituted by 'elementary' or by 'an introduction to'; such are the contents. We have chosen the word 'basic' in order to emphasise our objective, which is to provide in a reasonably compact and readable form a rigorous first course that covers all of the material on linear algebra to which every student of mathematics should be exposed at an early stage.
>
> By developing the algebra of matrices before proceeding to the abstract notion of a vector space, we present the pedagogical progression as a smooth transition from the computational to the general, from the concrete to the abstract.

It appears safe to say (and undoubtedly true, based on the contents) that the reader is expected to have had a basic course in plug-and-chug calculus (and so knows what a derivative and integral are), and no more; certainly not a first course in real analysis, or a previous exposure to linear algebra. This observation gives rise to another form of complaint, packaged as another example:

Example 4: Based on what coursework is expected (and not expected) of the reader to have had before using this book, it is perhaps not ideal that the student encounters the composite function notation "$g \circ f$" without even a brief explanation (first seen on page 117, last line, and line −6), though a few lines before, we do read "We now turn our attention to the matrix that represents the composite of two linear mappings." This is certainly forgivable, but in the same discussion (page 118, line −8), we are told "It follows that $g \circ f = \mathrm{id}_V$", *without ever having introduced or named that latter identity mapping*. There is also no table of notation (list of symbols) in the book.

In this particular incidence of where a bit more elaboration by the authors would have been useful, it happens to occur in the part of the book where a serious transition is being made to considerable abstraction, making it yet worse. In particular, they demonstrate (very well, but this will scare some students) that

$$\mathrm{Lin}_{m,n}(V, W) \simeq \mathrm{Mat}_{n\times m}F,$$

i.e., the mapping $\vartheta : \mathrm{Lin}_{m,n}(V, W) \to \mathrm{Mat}_{n\times m}F$ given by $\vartheta(f) = \mathrm{Mat}f$ (for field F and vector spaces V of dimension m and W of dimension n), is an isomorphism. This is not a good time to skimp on words, and use terms that have not been previously defined.

I mention above the book by Jacob and Evans, and in that same aforementioned set of pages, namely page 76, they clearly define, and detail, the identity mapping id_V, and state and prove that, for $f : D \to F$, $f^{-1} \circ f = \mathrm{id}_D$ and $f \circ f^{-1} = \mathrm{id}_F$. Jacob and Evans also have a notation table (that includes id_D). □

I have given some critiques of the authors' book, and in doing so, showed how other books do a better job in certain cases. Now it is time to turn the tables, and give a case in point where B&R do in fact an excellent job, considerably better than (the otherwise from me highly praised) Jacob and Evans (J&E).

Example 5: I have in mind their very well orchestrated discussion on the Fibonacci sequence (B&R, pp. 165-6), while the presentation in J&E (pp. 211-2), is, in contrast, rather poor. After the Fibonacci example, B&R give another well done example, this one based on continued fractions for evaluating the square root of 2.[20] They then analyze yet

[20] Besides books dedicated to continued fractions, there is a nice discussion of them in Elaydi (2005, Ch. 9), mentioned in Section 4.9, where also the continued fraction for a general square root of a positive numbers is given (page 408). The following can make for a nice student assignment involving proving the general result, and doing some programming and "analysis". In particular, for the square root of $g > 0$, the continued fraction is, when expressed as a sequence t_n, $t_1 = 2$, $t_{n+1} = 2 - a/t_n$, where $a = 1 - g$; this converging to

another iterative example for computing $\sqrt{2}$, namely that based on $q_2 = (2+q_1)/(1+q_1)$, for q_1 a positive (rational) number.

Lang being another book I refer to in this discussion, we note that there is no mention of Fibonacci numbers or continued fractions. □

This is the second printing of the second edition, and so one would not expect many typographical errors, and indeed there are not many. There is in fact an errata sheet, available online from the publisher Springer (easily found using a search engine and the obvious keywords). It contains only eight entries (six of which occur in a short span of pages within the solutions to exercises at the end of the book), but I have found others. For example (and as proof that I actually read some of these books!)

(a) page 35, line 5, n in "for $i = 2, 3, \ldots, n$" should be m.

(b) page 43, last line, the subscript j in the first occurrence of $\mathbf{A}_j$ should be an i.

(c) page 44, the argument that row equivalence is an equivalence relation: The part about transitivity does not seem correct, or at least I could not follow their argument: They do not link what they state to the general transitivity statement of equivalence relations. Worse, this is the first occurrence in the book of the concept of equivalence relation, and it is *not* defined and given a few dedicated sentences. So, I claim the uninitiated reader will find this passage frustrating and inadequate.

(d) page 45, line 2. The sizes of matrices F and A obviously precludes computing their matrix product. The error is in the definition of matrix F, which anyway should be square (see the definition of elementary matrix on their page 32), and should be $m \times m$. Then everything goes through.

(e) page 131, line 3. In the statement "By (D_3') and (D_4), this reduces to". I claim that it is more clear to the student if the following is written: "By (D_3), (D_3'), and (D_4), this reduces to". In their presentation, assuming (D_1), then (D_3') $\iff$ (D_3) via their Theorem 8.1, so what the authors have is not incorrect, but not well spelled out. They could also have written: "By (D_3'), Theorem 8.1, and (D_4), this reduces to".

(f) page 132 line 8, the exponent on (-1) should be $i+i$, and not $1+i$, so that, and crucially for the result to hold, $(-1)^{i+i} = (-1)^{2i} = 1$. In full, the line should read:

$$f_i(I_n) = (-1)^{i+i}\delta_{ii}D(I_{n-1}) = 1.$$

(g) page 177, line 10, "It is immediate from the above Corollary ..." should be (there are no above corollaries in this chapter): "It is immediate from Theorem 10.2 ...".

(h) page 220, solution to exercise 6.20, In "The fact that $a_1, \ldots, a_n$ are linearly independent", "$a_1, \ldots, a_n$" was supposed to be "$\nu_1, \ldots, \nu_n$".[21]

$1 + \sqrt{g}$. Programming this such that it iterates until the solution is within some fixed tolerance, and plotting the required number of iterations, say r, indicates that r increases monotonically with g. For $g = 2$ and a tolerance of 10^{-15}, i.e., approximate (usual) machine precision, we require $r = 21$. With $g = 200$, $r = 262$. As the tolerance gets smaller, the algorithm fails to converge (presumably due to round off error). For example, with the tolerance of 10^{-15}, the algorithm fails for $g \geq 227$. With a tolerance of (the still quite respectable) 10^{-12}, the algorithm works for $g = 30,000$, but fails if the tolerance is changed to 10^{-13}. To enable usage for all positive numbers (that can be represented in a computer), and also keep the required number of iterations as low as possible, the student can then be asked to make an algorithm to determine the smallest required g^* based on integer square roots, e.g., if $100 < g < 121$, then, letting $b = 100 = 10^2$, $g^* = g/100$, and we compute $\sqrt{b \times g/b} = 10\sqrt{g^*}$. Trivially, $(n+1)^2/n^2 \to 1$ as $n \to \infty$, so that g^* is bounded and approaches 1, and thus this method is valid for all positive reals, and ensures that calculation of $\sqrt{g^*}$ to machine precision is reliable and fast. However, the determination of base b would presumably be the bottleneck in the algorithm.

[21] Indeed, this corrected statement follows their definition on page 40, and this same formulation is used by other authors, e.g., in Lang's *Linear Algebra* (1987, p. 10). However, some books emphasize that the preferred expression is that the *set is* linearly (in)dependent, i.e., vectors themselves cannot be linearly independent, but rather only sets of them. This is the case for example in Meyer (2010, p. 181); and also Ricardo (2010, p. 36), where the latter author also states sometimes one will "speak somewhat loosely and say that vectors

The MAA review https://www.maa.org/press/maa-reviews/basic-linear-algebra is very positive, also, among other things, praising the good choice of exercises. We read:

> This book is clearly geared towards students of mathematics (rather, perhaps, than to users of linear algebra from other disciplines). The manner of exposition is terse yet very clear and elegant, and it is apparent the authors are algebraists.
>
> In the end, I suspect that this book's somewhat abstract and spare expository style may stretch the mathematical sophistication level of many American undergraduates who are taking a first course in linear algebra, but I would relish the opportunity to offer an honors course in linear algebra from it or to suggest this book for supplementary readings and projects for motivated students. I believe *Basic Linear Algebra* is a valuable reference for an inspirationally elegant and streamlined algebraic development of the foundational ideas of the subject.

The MAA review for the second book, https://www.maa.org/press/maa-reviews/further-linear-algebra, done by a different reviewer than the first book, is short, positive, and has nothing negative to say. We read:

> Many linear algebra books cover the theory of Normal Matrices only over the complex field. By contrast, in chapter 10 of this book the authors cover the theory of Normal Matrices over the real field. This is actually more interesting than the complex theory, since one can not apply the spectral theory which is valid only for Normal Matrices on the complex field.
>
> In chapter 11, the authors focus on computational aspects of linear algebra. Here they show the reader how to use the *LinearAlgebra* package of Maple to perform computations. There are several advanced Maple procedures written in this chapter to help the reader to perform linear algebra computations.
>
> In chapter 12 of this book the authors present brief biographies of great mathematicians who had made major contributions to the field of linear algebra, including Fourier, Parseval, and Hilbert.

As alluded to above, solutions to exercises are given at the end of the books, adding (I claim significantly) to their value for students (or instructors looking for sources of questions for homework or exams), and also adding evidence to my claim that the authors wrote books that (to the uninitiated) *look like* a never-ending sequence of cold theorem-proof combinations, while in fact, there is a lot of behind the scenes hand holding and guidance for the student starting his or her journey into higher mathematics.

For me, it is hard to *not* like this first book, despite some minor gripes mentioned above. And keep in mind there is the second volume, picking up where the first leaves off; see Section 4.1. It should be noted that the first volume does *not* cover some topics common for a first course, but these topics are addressed soon into the second volume. In particular, these are: norms, distance, inner products, complex inner product spaces, inequalities (Cauchy-Schwarz, triangle, Bessel's), orthonormal subsets and bases, Gram-Schmidt, direct sum of subspaces, projection, orthogonal complement, and the Fourier series representation of a function.

All things considered—notably, how many excellent books are now available; the pdf files of Blyth and Robertson are presumably not formally legal; the appearance of the book is "1980's style", i.e., no use of color, no use of text-in-margins, font is fully readable but a bit thick and plain, very few graphics—, this first book, or pair of books, by Blyth and Robertson would probably not be my first choice to use for a first-course or a two-course

$v_1, v_2, \ldots, v_k$ are linearly independent". Apparently, Blyth and Robertson use both formulations: On page 123, they refer indeed to a *set* of vectors and say it *is* independent. So, either one of the two authors prefers this formulation, or they both simply use them interchangeably.

sequence. In that case, however, they would still play a "rôle", and be very high on my list of suggested (or enforced) supplementary reading, due to their (mostly) clear and well-organized instruction, the very nice sets of exercises, endowed also with solutions, and as a vehicle to get students used to concise, rigorous, mathematical presentation.

LA-1 Book Choice 12: *Introduction to Linear Algebra*, Gilbert Strang, 2016, 5th edition.

I have heard from a colleague in an applied mathematics department that this is among the most commonly used books in the field—and the author has several, all variations on the same theme. (As an aside, I often hear from colleagues in *pure* mathematics departments that Axler's book is preferred. I think both choices are well-justified, and I discuss these two books separately below in the comments given in Section 3.3.)

Strang's book absolutely belongs in a list of the best books. Nevertheless, I am not in favor of choosing it. One reason, seemingly foolish, regards the quality of the pdf file: The "free one", found on the usual pirate web sites, and presumably not supported by the author, has some scanned equations, making it literally a headache to read. As we all know, students will be accessing this pdf file, as opposed to purchasing (an expensive no less, in the case of Strang's book) clunky hard copy. Recall the discussion in Section 2.1. Another reason is discussed in the comments in Section 3.3 below.

LA-1 Book Choice 13: *Linear Algebra*, by Elizabeth S. Meckes and Mark W. Meckes, 2018. This is yet another well-written book designed and very suitable for a first course. Like some others, it explicitly emphasizes proofs. It also introduces linear maps early on, and provides examples of abstract fields and vector spaces (a fact that did not overly impress the reviewer discussed below).

It is 378 pages to the end of the last chapter, and then contains appendices, hints and solutions to some exercises, an index (but no bibliography), resulting in about 425 pages. I begin with some arguable superficialities: The book is optically very nice, with use of color, boxes, and solutions to "inline" little exercises throughout the text given, *upside down*, at the bottom of the page, obviously to entice the reader to try it on his or her own before reading it. There are some attempts at humor, such as the footnote on page 185. To address the question that arises to anyone who knows how many books, not to mention how many *good* books, are available for a first course in linear algebra, the authors begin their preface stating

> "It takes some chutzpah to write a linear algebra book. With so many choices already available, one must ask (and our friends and colleagues did): what is new here?"

The authors then continue with some highlights of their book.

The problem is that the (presumably illegally available) pdf file of the book is not crystal clear — it looks like a scan, but the text can be marked (so perhaps it was converted via text recognition software) and thus searched. Either way, it is optically a headache, and 176 megabytes. If no better pdf file can be found, then students will be annoyed, either at this pdf file, or having to buy a physical book—an outright impertinence and audacious imposition, not to mention a *Zumutung* and *Frechheit* to ask of modern students.

There is a very positive and informative MAA review at https://www.maa.org/press/maa-reviews/linear-algebra-9, where many good things can be read about this book. This includes the availability of a detailed 194 page solutions manual available from the publisher CUP (and which I have obtained). The reviewer mentions "Topics not covered in the book include the minimal polynomial, the Jordan canonical form, and bilinear and quadratic forms. These omissions seem entirely reasonable in a book that is intended as a text for a first course in the subject." I agree with that assessment, though I think these topics could be taught in a first course, notably for master's students in subjects such as finance, computer science, and econometrics (where, for example, quadratic forms are encountered).

The reviewer is not pleased about a small handful of things, one of which I think is substantial, namely the authors' embracement of the idea to banish determinants from their typical early appearance and central role, to the end of the book. (This idea seems to have been popularized by Sheldon Axler: See Section 3.3 regarding his book.) While arguably a matter of opinion (see Edwards' thoughts on this, in the discussion of Axler's book), this objectively results in some poor pedagogical structure, e.g., as the reviewer states, "The decision to defer determinants until the very end of the book also results in a very late definition of the characteristic polynomial. So, for example, the fact that an $n \times n$ complex matrix has n eigenvalues (counting multiplicity) is not established until about a dozen pages from the end of the book."

LA-1 Book Choice 14: *Linear Algebra with Applications*, 2021, by W. Keith Nicholson, 663 pages. This is now an open source book with a Creative Commons License. A (positive, but not raving) MAA review (of the 2018 version) is `https://www.maa.org/press/maa-reviews/linear-algebra-with-applications-two-volumes`. In some ways, it is hard to rave at any such first-course linear algebra book, as the standard these days, reflected in virtually a dozen books, is so high, and many of these books are now "exchangeable" to a large extent. Nevertheless, this is a beautifully typeset book, with appropriate use of shading and color, and with very nice graphics.

The author tries to address two groups of readers with his book, namely those with lesser skills in mathematics and proofs, and one that is ready for them. He states in the preface, page ix:

> While the treatment is rigorous, proofs are presented at a level appropriate to the student and may be omitted with no loss of continuity. As a result, the book can be used to give a course that emphasizes computation and examples, or to give a more theoretical treatment (some longer proofs are deferred to the end of the Section).

and on page xi:

> Proofs are presented as clearly as possible (some at the end of the section), but they are optional and the instructor can choose how much he or she wants to prove. However the proofs are there, so this text is more rigorous than most.

The 2021 version, as compared to 2018, has a new section on the SVD. It is definitely for a first course, though it contains material that fits better in a second course, such as the SVD, polar decomposition, pseudoinverse, constrained optimization, and principle components analysis, but notably the last chapter on block triangular and Jordan canonical form. (In the preface, the author says "Chapters 1–4 contain a one-semester course for beginners, whereas Chapters 5–9 contain a second semester course".) That it includes the Jordan form indicates immediately that this is not a baby book. Further, I looked at chapter 9 (change of basis; because I like to inspect the notation used; see Section 3.5)[22] and see that the material shown, as well as some of the exercises, have some real "substance". Using the extent to which results on determinants are proved as our measure of mathematical sophistication, this book is "high": The major results

[22] As one of the only parts of the book I carefully looked at, it is perhaps a bad omen that there is a mistake in this definition. One can only hope this was a freak coincidence, and that it is not the case that many other typos are lurking in the book. I wrote an email to the author, but so far, have not obtained a response. Update! After having waited a few weeks in vain, I wrote to the publisher, and got a quick response, and eventually an email from the president of Lyryx Learning (and a professor himself) confirming I was correct, and that the new version of the book will correct this. Note in particular the benefit (at least to students) of open-source books!

are proved.[23] As the author states in the preface, longer proofs are deferred to the end of the section (and also that they are optional), which is a nice way of presenting the material if the intent is to have the student skip some of the rigor.

To help seal the deal on this very good, optically wonderful (and legally free!) book, selected answers to exercises are provided in the book, and a complete solution manual is available for instructors. From the book's web page https://lyryx.com/linear-algebra-applications/, it appears that teaching slides might be available.

LA-1 Book Choice 15: *Matrix Theory and Linear Algebra*, by Peter Selinger,[24] August 2020. I was not able to locate the book on the Amazon online bookstore, nor is there an MAA review of it.

The book (as a free and legal pdf file) can be found here, https://th1lib.org/book/5846287/a4affa, where also its magnificent cover picture can be enjoyed. The table of contents (and the length of the book, about 540 pages, including solutions to exercises) is very similar to Anthony and Harvey, who, interestingly, also have a lovely "nature" color picture on the cover. (Perhaps Selinger recognizes A&H as the obvious competitor?) Being an open text, one can obtain the LaTeX and modify things. The layout of the book is fine (and looks like a beamer package for latex, with some colors), and there are some good graphics.

My very brief inspection (out of necessity—how many of these books can I stand to look at anymore?) suggests that this is an excellent project. Given that the pdf of the book is legally free, I feel behooved to suggest that this book be used at least as secondary reading. Topics it contains that I have not found in comparable books include the (very basic) basics of Fourier analysis, with the usual function approximation graphs, whereas some other books (such as Lang) just mention this, and show the Fourier coefficient integrals (as projections).

LA-1 Book Choice 16: *Linear Algebra with Applications*, by Otto Bretscher, Fifth Edition, 2013. The book received a very positive review from MAA, https://www.maa.org/press/maa-reviews/linear-algebra-with-applications

> This book contains the standard material usually found in an introduction to linear algebra course in U.S. colleges and universities. It does have some novel features, but it does not overdo it.
>
> Chapter 1 also includes the first taste of historical comments that are liberally placed throughout the text. Many sections of the text also include exercises either with an historical bent or from actual original sources. This is one of the novel and very welcome features of this book.
>
> ... This is a nice compromise between the "linear algebra as the study of linear transformations" adherents and the "linear algebra as the study of matrices" followers. We all know that in finite dimensions these are one and the same but the choice of perspective can very much change the nature of an introductory linear algebra course.
>
> I was pleased, however, to see singular values and the singular value decomposition included, although again applications were wanting.

[23] For the adjugate formula, $\mathbf{A}(\text{adj }\mathbf{A}) = (\det \mathbf{A})\mathbf{I} = (\text{adj }\mathbf{A})\mathbf{A}$, Nicholson gives a proof for the 3×3 matrix case on page 160, and states that "[t]his argument works in general". As a compliment to the author, we see Nicholson's book (fourth edition, 2003) referred to for the proof of the adjugate result "and other facts that may appear in this chapter without proof" in Edgar G. Goodaire's *Linear Algebra: Pure and Applied*, (2014, p. 240).

[24] This is an open text, under the Creative Commons License, accredited to Peter Selinger, based on the original text by Lyryx Learning and Ken Kuttler, with contributions also from Ilijas Farah and Marie-Andrée B. Langlois.

Each section contains a very nice list of exercises, and these exercises range from basic to challenging. There are typically some easy (but realistic) applications, particularly early on in the book. These early application exercises can be used to help motivate the study of linear algebra beyond solving systems of linear equations.

The author uses a combination of the "definition/theorem/proof" style with a more "conversational" style.

The author includes historical commentary and problems. These are not an afterthought; history is included at various places throughout the book and I found the author's historical comments a nice jumping off point for further study. It is by no means an historical text, but the historical material is a very nice addition to a solid introduction to linear algebra.

In summary, this book covers the typical introduction to linear algebra course. It does not suffer from "textbook bloat," but then again a few of an instructor's pet topics might be omitted.

The book is optically very attractive, and a "perfect" pdf file is available. Further, an instructor's solutions manual is available (in the same place where a pdf of the book can be found), with solutions to all the problems in the text.

LA-1 Book Choice 17: *Linear Algebra: Theory, Intuition, Code*, by Mike Cohen, 2021, 580 pages. Jumping to the chase, this is an outstanding book, conditional on the intended audience, which is anything but mathematics majors. This is the lowest level book, in terms of technicality, that I include in this document, and in fact that I have ever seen on linear algebra: There are no theorems and proofs, but rather motivation, explanations, examples, and occasionally some basic exercises (with solutions). Impressively, there is a 30 page discussion on the SVD (though the "density of information per page" in this book, both in terms of spacing and font size, as well as in terms of pace and content of the presentation, is comparatively low). PCA and covariance matrices are covered.

According to my algorithm for deciding what books make the list given here, this one should not be on it, but rather mentioned as "extra softy" reading in Section 3.4. I opted to place it here because it is especially well done, and also because the author discusses (of course, very basically) the SVD and PCA. To be clear, I do *not* suggest usage of this text as the primary (or even secondary) book for a first course in linear algebra, but include it in this list to draw attention to it: It could be recommended to students to read on their own, for example, during the semester break, before the formal course starts. This is also quite feasible for students opposed to illegally accessing the pdf file of a book: The price, at the time of this writing, of the Kindle version of the book is USD $9.75, while the paperback is (without shipping costs) $26.00.

The author writes in a charming and disarming way, and without being idiotic, or insulting the reader's intelligence. There are nice comments along the way, side-notes in the margin, and attempts at humor. For example, the author makes fun of himself in a margin note on page 528, saying: "Important information hidden inside a long paragraph. Terrible textbook writing style!" in reference to the main text's discussion that "the eigenvectors of a symmetric matrix point along the ridges and valleys of the normalized quadratic form surface".

The entire writing style is non-conventional, with this becoming clear already from proverbial page 1: Instead of a book "Foreword", there is a "Forward", and it seems the author did not make a gaffe, but rather did it intentionally, as a play on words, because its entire contents is: "The past is immutable and the present is fleeting. Forward is the only direction."

Regarding content and its low level: The chapter on determinants, for example, is very modest, showing essentially only illustrations of the computation for matrices of sizes

2 and 3, as well as diagonal matrices. Similarly, the chapter on least squares is very rudimentary. The chapter on rank is vastly different than "traditional" expositions, in that the author motivates its importance (e.g., "Indeed, one of the main goals of regularization in statistics and machine learning is to increase numerical stability by ensuring that data matrices are full rank. So yeah, matrix rank is a big deal."), and discusses the ranks of sums and products, and also that of matrices $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$.

The price to pay for going low on technicality is that motivating some results without a full scaffolding can occasionally require some hand waving, or referencing results not yet discussed. This comes up on page 190 in the discussion of the rank of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}}$. We read:

> I'm going to present two explanations here. Unfortunately, both of these explanations rely on some concepts that I will introduce later in the book. So if you find these explanations confusing, then please ear-mark this page and come back to it later. I know it's a bit uncomfortable to rely on concepts before learning about them, but it often happens in math (and in life in general) that a purely monotonic progression is impossible.

I like his forthright approach to dealing with this (and it is indeed quite a feat to fully prevent this, *and* make an interesting book), and the reference to "life in general". Again referring to the chapter on rank, there is a short section on the rank of random matrices—admittedly a conceptually interesting thought experiment for the intended audience, especially because we all simulate matrices occasionally. (And possibly a nice device to segue to a basic discussion of probability and measure-zero events?—perhaps not for the intended audience of this book, but if anyone could do it successfully, it would be this Mr. Cohen).

A very nice feature of the book is that it includes codes throughout, namely both Matlab and Python (showing code for each language for each assignment). On a less positive note, some chapters do not have exercises at the end. For those that do, the number of questions is meager, and—perhaps obviously—the questions are the plug-and-chug type, as opposed to algebraic and requesting proofs of results. Solutions are obviously provided—in a book this student-friendly, this goes without saying. Most chapters have "code challenges" (programming exercises), and the solutions are also given (in both Matlab and Python). As a final small point on the optics of the pdf file: The text of margin notes appearing at the bottom of the page are sometimes truncated, such as on pages 217 and 524.

Further teaching projects by the author can be found at http://sincxpress.com/.

LA-1 Book Choice 18: *Introduction to Linear Algebra with Applications*, Jim DeFranza and Dan Gagliardi, 2009, 488 pages.

As a recent book with a major publisher, it of course covers all the expected topics, and with painstaking clarity and examples. The level is not sophisticated enough, as with some books, to declare it being more of a "linear mappings and transformations" or more of a "matrix algebra" book, nor does it take an overtly geometric or staunchly algebraic approach.

It does have a "theorem-proof structure", as opposed to being a verbal discourse designed to assuage the math-intimidated, and it covers topics such as isomorphisms (and, necessarily, the concepts of injective and surjective mappings) and similarity, as is the coordinate vector with respect to an ordered basis and the change of coordinates transition matrix. Inner product spaces are covered, obviously at a basic level, along with orthogonal complements, least squares approximation, quadratic forms, and diagonalization of symmetric matrices. There are numerous applications, 43 in fact (easy to ascertain because there is a nice table at the beginning of the book listing all of them), some of which are end-of-chapter exercises. Somewhat uniquely, and usefully and even impressively, there is a 30 page appendix that "contains background material on the algebra of sets, functions, techniques of proof, and mathematical induction. With this feature, the instructor is able to cover, as needed, topics that are typically included in a *Bridge Course* to higher mathematics." This appendix is a gentle and short preparation for a first course in real analysis, or also a first course in linear algebra using a higher mathematics level.

Using our gauge of the extent to which proofs of the major results associated with determinants are given to assess the overall math level (see Section 3.1), this book is (based on a rather coarse equivalence class partition) "medium level".

The book is optically excellent, with that "modern" method of colorful bars and shading, a big margin where sometimes graphics are found, etc.. The pdf file is perfect.

There is an instructor solutions manual containing detailed solutions to all exercises (also available in the same place where the pdf of the book can be found). The back of the book has short answers to the odd-numbered problems.

It appears to me that its closest relative, in terms of appearance, level, style, and contents, is Williams (2018), considered next. For example, DeFranza & Gagliardi discusses traffic flow as an example of the use of matrices, as well as electrical networks, these being found on pages 80 and 88, respectively. Williams' book covers these examples on pages 61 and 59, respectively. Williams (page 111) also shows how matrices can be used for encoding and decoding messages, this example not being in DeFranza & Gagliardi. Now turning to Leon (2015), we see on page 33 traffic networks, page 35 electric circuits, and page 120 using matrices (and their inverses) to encode (and decode) messages.

LA-1 Book Choice 19: *Linear Algebra with Applications*, by Gareth Williams, 9th Edition, 2018, 594 pages. The MAA review for the fourth edition (2004), https://www.maa.org/press/maa-reviews/linear-algebra-with-applications-1, is very positive. For the 9th edition, the available pdf file is perfect, and it is an optical treat, on par with the two books from Nathaniel Johnston (2021), and contains at the beginning of each chapter big color photos of "impressive things", e.g., usually buildings, but also a highway. Regarding content, any book in its 9th edition (and any author invited by the publisher to make a 9th edition) is most likely highly successful and highly optimized. It appears to be the case for this book. Along with the now-near-obligatory example of internet search engines, we also have numerous other interesting examples and discussions, such as, just to name one, section 6.2 on Non-Euclidean geometry and special relativity.

The presentation is certainly first rate, at least from the viewpoint of a modern young student who (i) is not going to acquiesce lightly if asked to read a—to him or her archaic looking—1980s style math book; and (ii) has an average level high-school-mathematics background and would do better with a book that in fact looks like a modern high-school book, not just in terms of pretty format, but also lots of basic simple examples with real (genuine that is) numbers. Having said that, the book should not be dismissed as being too simple: The mathematical sophistication is "medium", based on my skim of the section on determinants. All the major topics associated with a first course in linear algebra are covered (and a few more, as discussed next).

Like Robinson (2006), Williams includes a chapter on linear programming and the simplex method. There is also a section on numerics involved in solving linear systems of equations and a discussion of the condition number and roundoff errors; and even the power method for computing the largest eigenvalue and its corresponding eigenvector. Finally, among other things, there is a section on the SVD, including its use for computing the Moore-Penrose inverse.

In summary, what makes this book very attractive is its "modern" look and feel (i.e., font, color, spacing), very modest beginning, indeed suitable for high-school students, replete with clear, simple numeric examples, *but* slowly but surely moving on to all the essential topics, and in fact going slightly but clearly beyond. To accomplish this requires much more than 200 pages; nearly 600 to be more precise. Thus, a book such as *Linear Algebra*, from Klaus Jänich (1994), at 204 pages, (and which I admire, praise, and recommend below in Section 3.4) cannot compete with Williams with respect to certain criteria: Much has happened with regard to teaching and pedagogics within a quarter of a century.

LA-1 Book Choice 20: *Linear Algebra: Theory and Applications*, by Ward Cheney and David Kincaid, second edition, 2009, 740 pages. This book enters last on my list simply because it is being "judged from a distance": A pdf of it is not available (arguably an immediate buzz kill for students); and I base my judgement—high enough to include it in this list—on (i) my familiarity with another book from the authors, namely *Numerical Mathematics and Computing*, now in its 6th edition, 2008, that I used and remember from my undergraduate days (it must have been the 2nd edition) in a course on (obviously) numerical analysis; and (ii) the very strong evaluation it received in the MAA review https://www.maa.org/press/maa-reviews/linear-algebra-theory-and-applications. In particular, we read:

> Ward Cheney and David Kincaid have produced a number of excellent textbooks together. Their new linear algebra textbook, Linear Algebra: Theory and Applications, continues this tradition of excellence. True to its subtitle, the book contains all the theoretical content one would expect in an undergraduate linear algebra textbook as well as a rich assortment of applications. As the authors explain in the preface, one could select material from the book to teach a one-semester course emphasizing either theory or applications. Alternatively, one could cover the entire book in two semesters for a multi-faceted course in linear algebra covering algebraic theory, practical applications, mathematical software (i.e. MATLAB, Mathematica, and Maple), and an introduction to numerical linear algebra.
>
> Linear Algebra reads easily. It is written in a conversational style that draws the reader in. You get the feeling that the authors are not in a hurry. They provide a generous amount of expository prose and pause along the way to share witty quotes and give interesting historical footnotes. Even though the book proceeds at a leisurely pace, it covers a great deal of material. It covers the standard topics as well as several others not always included in an undergraduate course such as singular value decomposition, Gerschgorin's theorem, and iterative methods for solving linear equations.

3.3 Some Comments

1. **Inclusion and Style of Proofs** Some books are structured as "theorem proof", obviously at this level also interspersed (sometimes generously, sometimes less so) with explanations and examples, while other books downplay proofs (thinking, most surely correctly, that their typical audience, e.g., US college undergraduates outside of a mathematics department, abhor such presentations). Books in the latter group include Nicholson, and Strang, with the extreme being Cohen. I have preference for the former category, with "representative element in the equivalence class" being Blyth and Robertson.

   As well as Blyth and Robertson, the book by Anthony & Harvey (A&H), and that by Meyer, have clear, detailed proofs of theorems in the book, as opposed to asking the reader to do so ("Proof left for the reader" or given as one of the end of chapter exercises). This is mostly the case for the book by Ricardo—all proofs of major results are given in the text (and I particularly like his style, though the proofs in the other books are equally excellent), while the exercises in Ricardo ask for proofs of related, often easy results. I have studied these four books intensely during the development of this document, and so am biased a bit towards them, though some of the others mentioned in the LA-1 Book Choice list are undoubtedly comparable.

   In the book by Norman and Wolczuk, some theorems have the proof relegated to the exercises (without solutions in the book), e.g., page 300-301, theorems 4.7.3, 4.7.4, and 4.7.5, with 4.7.4 being: a linear mapping is one to one if and only if it is onto. (This is proven, for example, on page 106 of Blyth and Robertson.) There is a justifiable reason to state results without proof, and ask the reader to do it, namely to force (well, strongly entice) students, via homework, to work on the proofs, and thus better understand the material. (For an extreme book in this regard, see Erdman, 2018, *A Problems Based Course in Advanced Calculus*, mentioned below in Section 5.3.) Another school of thought (to which I belong, and is far more popular with students) is that it is better to just have all the proofs, in full detail, in the book, so the student can "read" the book. For students outside of mathematics, e.g., social science students, this helps respect their time amid all the heterogeneous course material they have to take, and also respects their—on average—lower intrinsic affinity for mathematics.

   Author Dan Wolczuk responded quickly and politely to my email, and sent me the solutions manual (which indeed contains the relevant proofs), as well as some further, very well done notes for teaching.

   In an attempt to finish this discussion (started in Section 2.3) regarding the extent to which students should be asked to construct their own (or fill in the details to sketched) proofs of important (or even less relevant) results, I briefly address the age-old question regarding if and to what extent proofs should be requested in exams. Harold M. Edwards, in the preface to his 1995 book *Linear Algebra*, perhaps says it best:

   > My answer to the perennial question 'Are we expected to know the proofs for the exam?' is 'no,' not because proofs are unimportant, but because a proof cannot be known, it must be understood, and understanding is very hard to test on an exam.

   An idea for exams (primarily for students outside of mathematics, notably in the social sciences) is to (among other types of questions) ask students to reproduce (obviously from memory, thus necessitating a closed-book exam) certain proofs—of which they were previously, i.e., well before the exam, informed. Thus, the student does not have to be a math genius, but rather be good enough to understand the material, and be motivated and disciplined enough to learn it so adequately that she can reproduce it. This obviously and undoubtedly has a strong learning effect, *because it is vastly simpler to memorize something that you understand.*

One could discuss this further: In an exam, with time pressure, giving the student a theorem that is not in the book (or possibly one that is, but amid a closed-book exam) and asking for a proof, certainly tests the student's understanding of the material: However, it also "tests" other things, such as their creativity and overall mathematical prowess, as well as their ability to think clearly and deliver under pressure. Some of these aspects surely play a role in the success of a mathematics professor, but are they really relevant for typical students, with above-average quantitative abilities compared to the general population, and who are pursuing quantitative subjects, but not planning on being research academics, notably in pure or applied mathematics?

I personally prefer to give exams such that the proverbial average student can, with solid investment of time, get a very high (perhaps not perfect) grade, *in a deterministic manner*, by which I mean the exam questions are close enough to those they had access to for studying, and harder questions (proofs) are indicated beforehand, so that even a less gifted but highly motivated and disciplined student could, in principle, just memorize the proof. What about rewarding the top students, such as to give them grades higher than (or, in the German system, lower than) what the "regular student who just memorized everything" got; or knowing who should receive a very strong letter of recommendation if asked? In my classes, top students get top grades. To get that killer, personalized, amazing letter of recommendation, I tell students to participate intelligently in class, and, crucially, do some outside reading that I suggest (usually additional material from the main book, and some content from another book), and convince me you know it, in an oral exam conducted by me and with my PhD students sitting in, and sometimes cross-firing. I get some takers—most do extremely well, by obvious self-selection, and they get their letter, and, deservingly so, often into top PhD programs or top industry (in my case) financial institutions.

2. **Strang's books** The books on linear algebra by Gilbert Strang are popular, excellent, and anything but dry: The author is known to write passionately about what appears to be his favorite subject. Hence, his book(s) deserve and get a separate entry for discussion points.

   On this page, `https://www.physicsforums.com/threads/difference-between-strangs-linear-algebras.482250/`
   there are numerous people against Strang's book(s), notably complaining that they are too chatty and not precise enough. One favors Meyer, and writes:

   > ...while remaining perfectly rigorous: Carl Meyer's [title of book], which is free of charge in PDF form. It covers more ground than Strang, it's user-friendly without being overly informal as I found Strang's book to be, and it comes with a full solutions manual.

   A reviewer at Amazon praises Meyer's book, and says:

   > Gilbert Strang's book is excellently suited for those who are completely new to the subject, this book is excellently suited to those who seek a little bit more.

   I agree. Meyer is sheer outstanding, for a variety of reasons, notably the ones mentioned here (though note that, while in the past, the pdf of Meyer's book was indeed legally free, it is no longer available, at least as of August 2021), along with that it covers everything a first book should, along with all of the most important topics for a second course, at least for applied mathematicians, statisticians, machine-learners, etc.. Its exercises are predominantly algebraic based, notably after the first chapter (as opposed to plug and chug), and a full and detailed solutions manual is available, along with a (short) errata list (though I have found others, and emailed with the author about them). Further, a new edition is expected soon, perhaps early 2022.

Strang's 2019 *Linear Algebra and Learning from Data*, from which I take the quote that begins Section 1, received a poor review from MAA, `https://www.maa.org/press/maa-reviews/linear-algebra-and-learning-from-data`, and some unhappy reviews at Amazon as well. This is not to say that the book has no value: On the contrary, it is bursting with useful information, but, in my opinion, the presentation needs a full overhaul.

As presumably made painfully clear in Section 3.2, there are now so many excellent linear algebra books to choose from; and there are many more good books, some of which are packaged below in Section 3.4 as possible supplementary reading. Further, Section 4.1 lists yet more linear algebra books, these being more suited for a second course, and thus encompassing anything "of an intermediate level" covered in Strang's book.

From the union all of these various constructed organizational categories of books I give, I do not believe that there is one single book that emerges as being blatantly superior to all the others. Strang's *Introduction to Linear Algebra* is surely good, but it is, especially now, as opposed to earlier in time, far from being the only game in town.

3. **Axler's book** Sheldon Axler's book is both impressive and unique, and hence also gets its own entry as a topic of discussion. His (amusing, if not outrageously, named) *Linear Algebra Done Right* is a highly praised and seemingly very popular book. It is definitely for a second course (at least outside of a mathematics department), not because of the sophistication of the material *per se*, but because (i) it concentrates on proofs, as opposed to basic concepts and mechanics and numeric examples; and (ii) Axler blatantly belongs to the "structural" group, as opposed to computational, these designations being explained below in Section 3.4.1 when I discuss Valenza's book *Linear Algebra: An Introduction to Abstract Mathematics.*

   The "done right" part comes from presenting determinants at the end of the book, instead of near the beginning, and not using them for proofs of things that can be proven without them. See Axler's article *Down with Determinants* in the American Mathematical Monthly 102 (1995), 139-154, available here: `https://www.maa.org/sites/default/files/pdf/awards/Axler-Ford-1996.pdf`. As an example of its influence, Ricardo (2010, p. 296) gives a proof that comes from Axler's book, namely that every square matrix has a (possibly complex) eigenvalue.

   An interesting comment on Axler's work is provided by well known mathematician Harold M. Edwards: From his *Essays in Constructive Mathematics*, 2005, Sec. 5.3, *Overview of 'Linear Algebra'*, we have a rebuttal of sorts of Axler's righteous stance, done with a touch of humor. Quoting,

   > Some years ago, Sheldon Axler published a book with the audacious title "Linear Algebra Done Right". I was probably more struck by the audacity of his title than most readers, because only a few years earlier I had published my own book called Linear Algebra, in which the subject had in fact been done right, but I had never thought to say so in the title.
   >
   > ...
   >
   > His choice of title is—I assume—intended as a joke, just as I am joking when I say that my linear algebra book had already done it right. But, in both cases, not really. And I expect that if you ask the first mathematician who comes along which of us is right, the reply will be that both are wrong, and the right way to do linear algebra is ... .

   The third edition of Axler's book is in color, formatted very handsomely, fixes the older typos, and now includes the dual and quotient spaces (topics we see in intermediate books such as some of those mentioned in Section 4). It is reviewed here, https://www.maa.org/press/maa-reviews/linear-algebra-done-right-0, while for the review for the second edition, given here, https://www.maa.org/press/maa-reviews/linear-algebra-done-right, we read:

> This book can be thought of as a very pure-math version of linear algebra, with no applications and hardly any work on matrices, determinants, or systems of linear equations. Instead it focuses on linear operators, primarily in finite-dimensional spaces but in many cases for general vector spaces.
>
> The book is not as bold as its title indicates; "done right" refers to the very technical device of avoiding determinants. But avoidance does have the useful effect of forcing you to think directly in terms of vectors and operators and not dive into a pile of calculations. Axler has come up with some very slick proofs of things that normally require a lot of grinding away, and that in itself makes the book interesting for mathematicians. The book is also very clearly written and fairly leisurely.

I own and have read (and enjoyed) the second edition. (The author is also responsive—I have emailed with him about an issue in his book, and also obtained a solutions manual to the exercises.) As the above discussion makes clear, the book is an outstanding pedagogic work of mathematical art, though arguably better suited for budding algebraists and undergraduate mathematicians, as opposed to people wishing (of course to understand the theory of linear mappings, but also) to do applications involving numerics and machine learning. There are numerous other excellent books now on linear algebra, notably the ones mentioned in Section 3.2, but also see the various entries in the next Section 3.4, as well as the intermediate and advanced books discussed in Section 4.1.

4. **A Useful Book Review** I cite and quote from several dozen book reviews in this document, and, having read so many of them, it appears that their quality and usefulness follow a (bivariate) bell curve. As an example of an excellent, exemplary review, with rather relevant information in general for evaluating first-course linear algebra books, along with good sample exercises to replace the mindless plug-and-chug exercises often found in such books, I recommend reading the one from Jeffrey L. Stuart, in the American Mathematical Monthly (2005) 112(3), pp. 281-288. He extensively covers not one, but rather the following four books:

   - John B. Fraleigh and Raymond A. Beauregard, *Linear Algebra*, Third Edition (1995);
   - David Lay, *Linear Algebra and its Applications*, Third Edition (2003);
   - Theodore Shifrin and Malcolm R. Adams, *Linear Algebra: A Geometric Approach*, First Edition (2002);
   - Gilbert Strang, *Introduction to Linear Algebra*, Third Edition (2003).

   Cutting to the chase, of the four books, none comes out ahead or behind. This should not surprise, given that they are all quite worthy presentations. The four are, however, nicely contrasted. Of them, only Lay's book, now in its fifth edition and authored by David C. Lay, Steven R. Lay, and Judi J. McDonald, does not appear in any of my lists. I mention it in the remark near the end of Section 3.1, where I indicate my reasons for not including what used to be a rather popular book. (See also the angry reviews at Amazon on the recent edition.) Stuart concludes by making a good point about having available answers in the book. He is not for or against, but rather discusses the following issue:

   > Students in linear algebra struggle with writing enough details. Too often their answers are naked numbers or simple phrases such as 'Independent,' lacking sentence structure or justification. Unfortunately, such answers echo

> the laconic style that has become standard in textbook solution sets, including those of Fraleigh-Beauregard, Shifrin-Adams, and Lay. In contrast, Strang has most non-numerical answers embedded in complete sentences, often with complete justifications. Finally, we note that the increasing availability of correct exercise solutions on the web means that instructors need to be ever more careful about giving credit to students for the work of others; skipping the problems whose answers are in the back of the book no longer suffices.

Regarding "Students in linear algebra struggle with writing enough details", I do not think it is limited to them, but rather that this is a more systemic problem: I see it in my own lectures in master's level probability and statistics. I jokingly but strongly encourage students to not only answer questions I pose, but to practice speaking in perfect, elegant, complete (English) sentences—not (just) because I am an uptight, old-school, grouchy crank, but because I am helping to prepare them for job interviews. (The students definitely "get it" and appreciate my concern of this issue, notably when they get a job, and email, greet, and thank me.) Also, it might appear to be students in linear algebra because of their age (it is a first course, and mandatory course, in "real math" for many) and, thus, their general maturity in terms of self expression (quite possibly in a foreign language for many of them no less) is still in development.

Regarding solutions being found on the web, Stuart was quite prescient about times to come after he wrote that in 2005: Nowadays, students can (and do) easily find a large number of entire books available for download on the internet (a practice that I am not against; see Section 2; and we cannot prevent anyway, so best we embrace it and work with and around it), along with web sites dedicated to book solutions. Other ideas for homework and exams might be needed; see my ideas given above in the first discussion point of this section, on Inclusion and Style of Proofs.

5. **Problems and Solutions books** There are some books not meant to be a textbook for an initial exposure to the material, but rather are sources of exercises and solutions. Section 5.5 gives a list of such books for real analysis. Here I mention just one that is suitable to accompany a first course in linear algebra, namely Fuzhen Zhang, *Linear Algebra: Challenging Problems for Students*, Second Edition, 2009. See also Section 4.7 for another (very impressive) book by Fuzhen Zhang on matrix algebra.

3.4 Further, Possibly Supplementary Books

> Dual spaces. As lovely as the ideas are, this topic is too abstract, and to my knowledge, unneeded at this level where almost all of the spaces are either finite dimensional or Hilbert spaces. Jerry Kazdan, 1966[25]

As discussed in Section 3.1, I envision the course to have a primary textbook, such as one from the list in Section 3.2, and then have the students access optional supplementary material from other books, for (i) additional material and topics; and/or (ii) enriched, particularly good explanations of one or more topics. Further supplementary books can be given with the intent of assisting the very math-shy, by providing them yet easier, more verbose, example-laden texts.

3.4.1 Older Generation

Let's hop in a time machine, back to the 1980s (or early 1990s), before Amazon, the internet, mobile phones, and when "pdf" only meant (to us at least) "probability density function", and head to the library to search for books. (Remember those days, and what a library is, and smells, like?) Here are some books we would find there, given chronologically. Many of these books can still be highly recommended to serve as supplementary reading for a first course, or a second course; if not mandatory reading for those students who either simply love books, and/or take their mathematics education very seriously.

1. Robert R. Stoll, *Linear Algebra and Matrix Theory*, 1952 (reprinted 1990, 2012), 272 pages. The pdf file is, surprisingly, excellent, with the text being extremely readable. The book starts modestly, but is more appropriate for a second course. (See the next entry, same author, for a first-course book.) From the back cover description,

   > Advanced undergraduate and first-year graduate students have long regarded this text as one of the best works available on matrix theory in the context of modern algebra. It is particularly well suited to bridge the gap between ordinary undergraduate mathematics and completely abstract mathematics such as one finds in a course in "modern" algebra.

   From the preface, we garner some useful information about the book:

   > Very little in the way of formal mathematical training is required. However, a certain amount of that elusive quality known as "mathematical maturity" is presupposed. For example, an understanding of "necessary and sufficient conditions" will be much more useful equipment than an understanding of "derivative." For a class whose membership is drawn from the social sciences, Chaps. 1 to 5, plus possibly a little time devoted to numerical methods in their special field of interest, should constitute a one-semester course.
   >
   > The original intention of the author was to write a book solely on the theory of matrices. In view of the widespread applications of this theory, as mentioned above, together with the scarcity of books on this subject that are suitable for texts, no apology is required for such a proposal. However, when it was realized that without much additional space it would be possible to discuss those topics in algebra which underlie most aspects of matrix theory and consistently interpret the results for matrices, the latter course was decided upon.
   >
   > Last, but not least, the author's wife deserves credit not only for typing the manuscript but also for her encouragement and understanding.

[25] From his set of notes https://www2.math.upenn.edu/~kazdan/425S11/math21-2011.pdf, from 1964 and 1966, entitled *Intermediate Calculus And Linear Algebra Part I*. This set of notes was (apparently, so I presume) the forerunner of *Calculus Two: Linear and Nonlinear Functions*, by Francis J. Flanigan and Jerry L. Kazdan in 1971, with 2nd edition in 1990; a book I (like and) mention below in Section 7.2.

There is an "index of special symbols" (table of notation), and a topic index.

The first 4 chapters comprise most of the usual material for a first course (but also includes quotient spaces, and does not cover eigen-things). Chapter 4 is on determinants, proving all the major results. Chapters 5 to 9 cover more advanced material, suitable for a more advanced course.

The word "eigenvalue" is not used, and the minimum polynomial is discussed in section 7.3. (It is however called the "minimum function", i.e., from page 171, "There exists a unique monic polynomial of least degree in $F[\lambda]$, called the *minimum function* of $\mathbf{A}$ and designated by $m(\lambda)$, such that $m(\mathbf{A}) = \mathbf{0}$.") In section 7.4 (page 177) we read "In the next section we shall have use for a polynomial in λ, $\det(\lambda I - A) = \ldots = \lambda^n + \cdots \pm \det A$, the so-called *characteristic function* of the matrix A." In section 7.5 (page 182), we meet eigenvalues and eigenvectors, called characteristic values and characteristic vectors, respectively.

This is presumably not an ideal primary book for either a first or a second course, but still worth knowing about, and could be enjoyed and studied "to see a typical presentation earlier in time", after the material is learned using a more modern book.

2. Robert R. Stoll and Edward T. Wong, *Linear Algebra*, 1968, 326 pages. The available pdf file is fully serviceable, easy to read, and searchable and markable (highlighting text with the mouse, and copying the text is supported). Annoyingly, at least in the pdf file, there is no table of contents.

   It is odd that, in the reasonably long list of references, and "suggestions for further reading", categorized no less into six possible directions, one of which is "Linear Analysis", there is no mention of the first author's (1952) book discussed above. (Just to be extra sure it is the same Mr. Robert R. Stoll, in both books, we see Stoll's affilation as the Department of Mathematics, Oberlin College, Ohio.) The pdf is searchable, and his name only occurs on the title pages. The "minimum function" is now called the minimal polynomial, though "eigen" is still "characteristic": Section 6.3 is entitled Characteristic Values and Vectors.

   The review by K. Singh (1970, Canadian Mathematical Bulletin, 13(3), pp. 404-5) is short, positive, but notes "there is a large number of misprints and a few minor errors." The review by Louis Shapiro (1970, American Mathematical Monthly, 77(7), p. 788) is mixed. We read:

   > This book has some very good features. Some of the examples and problems are very good, the material covered is substantial and well chosen, and there are some nontrivial applications of linear algebra in differential equations, quantum theory, chemistry, and economics. The authors also view linear algebra as a part of mathematics, not as an efficient way to solve linear equations. In the fall of 1969, I used it in a class consisting mostly of junior mathematics majors. I like the book as a reference and I have learned from it.
   >
   > I found it very difficult to teach from, however, and not a good text for this class. One fault is the misprints which range from barely noticeable, to silly, to down-right false (e.g., an $n \times n$ matrix is diagonal if and only if its minimal polynomial can be expressed as a product of distinct linear factors with scalars in the field at hand). Also no answers are included, the exposition is too terse in places, and there are no nontrivial applications. Unless the decomposition theorems are covered most of the applications are inaccessible, and the decomposition theorems come very near the end of the book. The arrangement of topics is sometimes puzzling. Matrix multiplication is introduced first as a method of changing bases. The first exercises on this come later when matrices have been connected with linear transformations.

> Rings are not mentioned until their heavy use in the development of the minimal polynomial.

Finally, in the review by D. C. Kay (1971, American Mathematical Monthly, 78(4), pp. 421-2) is also mixed. We read:

> A strong, basic text on linear algebra, the book begins with the usual axioms for general vector spaces and develops the basic concepts in the first 5 chapters. The heart of the book is in Chapters 6–8, which provides a well-motivated presentation of the characteristic minimum polynomials, and related theory, leading to a lucidly written section on extremal properties of characteristic values, and climaxing in the spectral decomposition theorem (p. 266). Finally, the results of Chapters 6–8 are put to effective use in solving certain problems in applied mathematics, a unique feature of the book, written with the help of carefully chosen experts in the fields of economics, chemistry, and physics. For the most part the exercises are reasonable, interesting, and excellently chosen; computation-type problems are included in practically every section.
>
> The book was written for the more mature undergraduate, in that abstract reasoning is used as the principal mode of communication; it is thus unsuitable for general use at the pre-advanced-calculus level (the authors say only that the book was intended for undergraduates). Regarding length, in classes at the University of Oklahoma most of us who have taught a semester course from this book are successful in covering only the essential topics from Chapters 1-5 and a sampling from Chapter 6.
>
> Although this reviewer concurs wholeheartedly with the choice of topics, the style in the early chapters is such that it requires meticulous study by even the mature student (for example, one finds a general proof of the existence of a basis—involving Zorn's lemma—as only the third topic covered). Further, certain ideas that should be eminently transparent seem to be obscured either by over-information or location with "optional" material (case in point: the solvability of an $n \times n$ system closes the discussion on annihilators). Conspicuously omitted is the very useful theorem concerning the existence of an orthogonal transformation which maps a given orthonormal basis $\{\alpha_1, \cdots, \alpha_n\}$ onto another $\{\beta_1, \cdots, \beta_n\}$ (the formula for the desired transformation is simply $\alpha A = \sum_{i=1}^{n} (\alpha \mid \alpha_i)\beta_i$; this result should replace the useless Theorem 7.2).
>
> A more serious fault of the early chapters, in view of the subject matter, is the ponderous treatment of matrices and determinants. The authors actually seem to belittle that area in the remark on p. 145: "For some, matrices have a life of their own, that is, an existence apart from representing linear transformations." Insisting on expressions such as $(a_{ij})(b_{ij})$ to represent matrix multiplication, the authors choose not to use standard notation, and matrix proofs are avoided throughout. Determinants are developed axiomatically, with an unduly involved proof used for the property $\det AB = \det A \det B$ (depending on the uniqueness of the determinant function as opposed to the use of elementary matrices to reduce the problem to diagonal matrices).
>
> It is the reviewer's opinion that, intentionally or otherwise, the authors place themselves in the category of those who, along with hosts of other writers on linear algebra, are guilty of "killing determinants" and, depending on one's point of view, this will undoubtedly serve to identify them as either heroes or villains.

3. Howard Whitley Eves (1911-2004), *Elementary Matrix Theory*, 1968, 325 pages. This book is outstanding, packed full of well-presented information, along with historical detail.[26] It did not place in my above list because (i) it concentrates on (as the title

[26] That the book contains historical information will not surprise some readers: Eves is the author of sev-

suggests) matrix algebra, whereas I think a first course should be on "linear algebra"; and (ii) the pdf file: The available pdf file is surprisingly clear and very readable, though text cannot be marked or searched, this being an annoyance for the modern reader, accustomed to this luxury.[27] I mention Eves' book several times above in the context of discussing that of Blyth and Robertson—they obviously had a copy of Eves' book on their shelf.

4. Charles G. Cullen, *Matrices and Linear Transformations*, Second Edition, 1972, 336 pages. The book covers the expected topics in chapters 1-4, and goes into more advanced material (suitable for a second course) in chapters 5 to 9. The pdf appears to be obtained from an Amazon Kindle book, which means much of the text is crystal clear, but stand-alone math equations are scanned, ugly, and smaller than the other text, making the book a headache to read. On Amazon, one can view the paperback version, and it looks very good. Given a price tag of $6.52 USD, I would include it in my hard-copy collection to supplement reading material for a second course in linear algebra.

5. Hans Schneider (1927-2014)[28] and George Phillip Barker, *Matrices and Linear Algebra*, Second Edition, 1973, 413 pages. This is another excellent book. Starting with essential superficialities, the available pdf file is, perhaps surprisingly, very clear and fully readable. The formatting of the book itself is what I will call the "classic" style of the 1980s and 1990s (there is no need to reserve the word "classic" for any other time in history, because all of us, notably students, won't be reading those books, and they are anyway not available as pdf files, outside of some famous ones available as scans), with simple thick black Roman font; as opposed to the modern style, with use of color, beautiful, clear fonts, extraneous embellishments such as colored lines under section headings, and often very crisp, well-done graphics.

   Regarding content, this book is in the same equivalence class as Lang's (1987) *Linear Algebra*, and Blyth and Robertson's (2002) first volume and the beginning part of their second volume. In particular, Schneider and Barker cover topics such as Jordan canonical form, functions of matrices, unitary equivalence, Hermitian matrices, and also, in the last chapter, differential equations. Answers are provided at the end of the book for some of the exercises. A very short, and very favorable, review is given by A. B. Farnell (1969) in the American Mathematical Monthly, 76(8), page 989. We read: "The text

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eral books, including the highly praised and enjoyable *Mathematical Reminiscences* (2001), but more importantly, the over 775 page and highly acclaimed *An Introduction to the History of Mathematics with Cultural Connections*, Sixth Edition, 1990, having first appeared in 1953. The MAA review of these books is worth reading, see https://www.maa.org/press/maa-reviews/mathematical-reminiscences and https://www.maa.org/press/periodicals/convergence/an-introduction-to-the-history-of-mathematics, respectively. His book *Elementary Matrix Theory* does not have an MAA review, but it does enjoy many 5-star reviews at Amazon, with one reader remarking "This is best mathematics book I've ever read." Another writes "I love this book. It contains a wealth of information on basic matrix theory that one almost never gets in the classroom or typical undergraduate texts." An Amazon reviewer for Eves' book *Mathematical Circles Volume III* writes "Had Dr. Eves as a teacher many years ago (think 1969). He enlivened class then with many of his great stories. If you enjoy mathematics, you will enjoy this book." Paying yet further tribute to Mr. Eves, this time regarding not his pedagogic skill and entertaining writings of mathematical and cultural history, but rather his mathematical prowess, he won the 1992 George Polya award for showing that "there exists a tetrahedron which has the same volume as a given sphere"; see the discussion in Alfred S. Posamentier and Ingmar Lehmann, *Pi: A Biography of the World's Most Mysterious Number* (2004, pp. 293-5) for further detail and, importantly, why this result was so celebrated.

[27] The ability to search within a pdf file is sometimes a function of the viewer. I have found some books, usually older ones, that cannot be searched with (the otherwise fantastic, because it allows inverse search when working with WinEdt) SumatraPDF, but can be using Microsoft Edge. The same applies to copying text, with Edge being more reliable in this regard. For Eves' book, and also that from Curtis mentioned below, no viewer I have was able to search in the text.

[28] According to the MAA *In Memoriam*, "Schneider is considered one of the one of the most influential mathematicians of the 20th century in the field of linear algebra and matrix analysis, as his contributions formed a basis for the algorithms leading to the development of Google." His own obituary ("Last Words"), and "A Personal History" (from his childhood in Vienna, fled in 1938, due to Nazi occupation, via his parents bribing a Czechoslovakian border guard) can be read here: https://people.math.wisc.edu/hans/

appears to be an excellent introduction to matrices and linear transformations." and "An attempt is made to motivate many of the theorems and definitions, and most of the theory is well-illustrated with concrete examples." There are also some rather pleased reviewers at Amazon. I also favorably mention their book when discussing Blyth and Robertson's presentation of permutations and determinants.

6. Charles W. Curtis (1926-), *Linear Algebra: An Introductory Approach*, 4th edition, 1984. This appears to be a real classic, very well written, and is advanced undergraduate / beginning graduate level. I would offhand say that it is roughly in the same equivalence class as those from Weintraub, Woerdeman, Liesen & Mehrmann, and Friedberg, Insel, and Spence. The pdf file, while very readable and clear, typical-1980s-Springer font, is not searchable. Curtis could serve as supplementary, somewhat more advanced reading, on a few choice topics, notably for a second course in linear algebra. Similar to Eves' book, it could also be used as a way of seeing how books have, or have not, evolved in terms of topics and presentation, over about half a century, at least if you consider starting from the time Curtis' first edition was written.

7. Serge Lang (1927-2005), *Linear Algebra*, 3rd edition, 1987. This is a book I know from my own undergraduate days. There is also the 2nd edition, 1986, with a different title, *Introduction to Linear Algebra* and which, rather uncharacteristically, is a different book than the 3rd edition. The *Introduction* book is, as might be expected, easier, but has about 60% identical, exact text overlap with its more senior (albeit still undergraduate) 3rd edition.

   Good pdf files (very readable, and with searchable text) are available in the usual places on the web for both books. Despite their age, they both have good, simple graphics. I (re-)read both of them as part of my "research" for this project, and they still impress me in terms of how well some topics (but not all) are explained. As one of numerous examples, Lang gives an elegant proof of Cauchy-Schwarz via projection instead of the standard, very easy, ubiquitous proof.[29] The reader of this document can search for "Lang" and see other occurrences of where I discuss (and often praise) his linear algebra book.

   Several other topics are treated very well in Lang, such as (in both books) an excellent chapter on the theory and proofs of determinants—far better than, say, the recent book on multivariate calculus by Shurman (2nd printing, 2019), *Calculus and Analysis in Euclidean Space*, which I found shockingly disappointing; see Section 7.1 for further discussion of Shurman's book. Section 7.6 of Lang's *Introduction to Linear Algebra* is entitled *Determinants as Area and Volume*, and it is very detailed and well written, and serves, for example, to prove some results stated without proof in Ricardo's book, namely that given on the bottom of page 278, the multilinear function aspect of determinants, e.g., $\det(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \det(\mathbf{x}, \mathbf{y}) + \det(\mathbf{x}, \mathbf{z})$. This section on area and volume is not

[29] The well-known, very easy proof is given (correctly so, to the aid of students) in most books, such as Anthony & Harvey. Possibly of interest, I came across the following during the construction of this document: The original proof by Cauchy is given in D. Garling's *A Course in Mathematical Analysis: Volume II: Metric and Topological Spaces, Functions of a Vector Variable*, page 304. (See also Exercise 11.2.1, page 650, of Thomson, Bruckner, and Bruckner, *Elementary Real Analysis*, 2008, though they do not mention Cauchy in the exercise.) The derivation attributed to Hermann Schwarz is given in Stephen Boyd and Lieven Vandenberghe, *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares* (2018, p. 57). Finally, a very clever proof is given in Jacob and Evans, *A Course in Analysis Volume l: Introductory Calculus, Analysis of Functions of One Variable*, 2016, pp. 205-7, due to (articles they cite by) Lin, M., *The AM-GM inequality and the CBS inequality are equivalent*, The Mathematical Intelligencer, 34(2), 2012; and Maligranda, L., *The AM-GM inequality is equivalent to the Bernoulli inequality*, The Mathematical Intelligencer 34(1), 2012. One proof is not enough in Hongwei Chen's *Excursions in Classical Analysis* (2010, Sec. 1.2), where he gives three proofs, the first of which also uses the AM-GM (Arithmetic Mean and Geometric Mean) inequality. Chen also cites (arguably the obvious choice, and highly recommended) excellent book from Michael Steele, *The Cauchy-Schwarz Master Class*, 2004. Steele's MAA review is unsurprisingly outstanding: `https://www.maa.org/press/maa-reviews/the-cauchy-schwarz-master-class-an-introduction-to-the-art-of-mathematical-inequalities`.

included in Lang's more senior book *Linear Algebra*, however the latter has a full (and I think very well done) derivation of the properties of determinants, notably the use of permutations, whereas permutations are not discussed in the introduction book.

As a counterpoint, Lang's treatment of a small handful of topics, such as change of basis, and dual spaces, in *Linear Algebra*, is not on par with the rest of the book. It seems as though they were rushed, in order to put more advanced material into the third edition and adorn it with a title devoid of the word "introduction".

Lang's linear algebra book(s) alone cannot compare to, say, Meyer for breadth, detail, and presentation quality; or Anthony and Harvey for simplicity, clarity, and hand-holding details, but I think they can be very useful complements for certain topics, and absolutely do *not* belong in the dustbin of history.

8. Gilbert Strang, *Linear Algebra and its Applications*, Third Edition, 1988. I mention Strang's 5th edition (2016) in Section 3.2. It appears safe to say that, in 1988, Strang's book was among the best. Some might argue that it, and its successors, still are the best choices. Either way, a student of linear algebra needs to be aware of his books and the influence they had on subsequent authors. As two of many examples, the bibliography of Robinson (2006) and the acknowledgements of Ricardo (2010) (both of which are discussed in this document) list Strang.

9. David B. Damiano and John B. Little, *A Course in Linear Algebra*, 1988, 434 pages. The pdf is "just good enough". There is a 2011 reprint, with errata sheets bound in, but I was not able to locate the pdf file. From the preface,

   > We have included some important mathematical applications of the topics that we cover, such as the application of the spectral theorem for symmetric real matrices to the geometry of conic sections and quadric surfaces, and the application of diagonalization and Jordan canonical form to the theory of systems of ordinary differential equations. We have not included many applications of linear algebra to problems in other disciplines, however, both because of the difficulty of presenting convincing, realistic applied problems at this level, and because of the needs of our audience. We prefer to give students a deep understanding of the mathematics that will be useful and to leave the discussion of the applications themselves to other courses.

   Indeed, chapter 6 is entitled "Jordan Canonical Form", and is about 42 pages. This makes the book valuable as an additional resource for this important and reasonably sophisticated topic; recall the discussion in Section 2.2. Note also that chapter 8 of Weintraub (2019), *Linear Algebra for the Young Mathematician* is praised for its presentation of Jordan canonical form; see Section 4.1.

   The MAA review is short, https://www.maa.org/press/maa-reviews/a-course-in-linear-algebra, stating:

   > This is a leisurely and mostly traditional introductory text in linear algebra. The book is aimed fairly low, and despite its thickness only covers the beginning of linear algebra. It is aimed at a one-semester course for sophomores, although there is enough material for a full year, and there are several ways of selecting material for a one-semester course. Conspicuously absent are any numerical methods and considerations. It does not cover any of the various decompositions (QR, LU, SVD), although it does have a good bit on Jordan canonical form. The discussion of eigenvectors is thorough and includes the spectral theorem for Euclidean spaces.

10. Thomas Banchoff and John Wermer, *Linear Algebra Through Geometry*, Second Edition, 1992, 303 pages (and with 92 illustrations—as would be expected in a book emphasizing geometry, and as Springer so proudly stated up front in their books).

11. Robert J. Valenza, *Linear Algebra: An Introduction to Abstract Mathematics*, 1993, 255 pages. Of the two viewpoints for linear algebra: (i) as a matrix-centric field for computation; or (ii) linear algebraic transformations, Valenza openly and explicitly states in the preface that he belongs to the latter. The author refers to these as *computational* and *structural*, respectfully. As he states, "What is the nature of linear algebra? One might give two antipodal and complementary replies; like wave and particle physics, both illuminate the truth". He then goes on to elaborate on the structural and computational approaches, and writes:

    > This text leans heavily, even dogmatically, toward the structural reply. In my experience in both pure and applied mathematics, the recognition that a given problem space is a vector space is often in itself of more value than any associated computation. Moreover, realistic computational problems are almost exclusively done by computer, and therefore incessant hand-drilling in matrix techniques is both redundant and maladroit. Finally, linear algebra as abstract, noncomputational mathematics is often one's first encounter with mathematics as understood and appreciated by mathematicians. I hope that the student will learn here that calculation is neither the principal mode, nor the principal goal, nor the principal joy of mathematics.

    This adherence to the structural approach is shared by more recent authors, notably Steven Weintraub, whose books I mention (and praise) below; and differs markedly from that of Harold M. Edwards, who is blatantly in the other camp, and discussed below. Based on my perusal, the book is about the same level as, and covers much of the same material and in a similar style, as Blyth and Robertson's (2002) first volume. Valenza however also covers dual spaces, Hermitian and unitary transformations, and the Jordan normal form, topics that appear in volume II of Blyth and Robertson. As such, Valenza is somewhat more advanced than the entries in the primary list of recommended books I give in Section 3.2. The only book reviews I was able to find are those from customers at Amazon. Of the five, four are 5-star, highly praising the book and the author. One review is 1-star, questions the validity of those top reviews, and has some complaints, notably about the inappropriateness of combining linear and abstract algebra.

12. Klaus Jänich, *Linear Algebra*, 1994, 204 pages. This book definitely does not suffer from "textbook bloat", though it is interspersed, if not filled, with plenty of graphics, as well as historical notes and "friendly" comments to the reader, e.g., on page 77,

    > "Why does one need quotient spaces? In a first course of linear algebra they are perhaps unnecessary. But in higher mathematics, particularly in algebra and topology, quotients of all sorts occur so frequently that I thought it worthwhile to introduce you to this notion."

    It is hard not to appreciate Jänich's style, and indeed, his book on topology (see Section 7.5), originally in German, and from 1980 no less, has been highly praised for its style. Jänich's *Linear Algebra* may not be optimal for a primary textbook, notably for students outside of pure mathematics, but it is very high on my list of books for recommended (or enforced) supplementary reading.

13. Harold M. Edwards, *Linear Algebra*, 1995. Edwards has several books on algebra, and his book on advanced calculus via differential forms, as well as his 2005 *Essays in Constructive Mathematics*, the latter making his stance clear. The summary text for his book on linear algebra says:

    > In his new undergraduate textbook, Harold M. Edwards proposes a radically new and thoroughly algorithmic approach to linear algebra. Originally inspired by the constructive philosophy of mathematics championed in the 19th

> century by Leopold Kronecker, the approach is well suited to students in the computer-dominated late 20th century.
>
> Each proof is an algorithm described in English that can be translated into the computer language the class is using and put to work solving problems and generating new examples, making the study of linear algebra a truly interactive experience.

From the preface of the book, we read:

> Although the title "Linear Algebra" describes the place of this book in the mathematics curriculum, a better description of its actual content would be "The Arithmetic of Matrices." The ability to compute with matrices—even better, the ability to imagine computations with matrices—is useful in any pursuit that involves mathematics, from the purest of number theory to the most applied of economics or engineering. The goal of this book is to help students acquire that ability.
>
> Accordingly, the emphasis throughout the book is on algorithms. A by-product of this emphasis is the complete disappearance of set theory, a disappearance that will greatly disturb teachers accustomed to the standard linear algebra course but will not, and should not, disturb students in the least.

This is a markedly different underlying paradigm as compared to that used in Valenza's aforementioned book, and Weintraub's two books, *A Guide to Advanced Linear Algebra*, 2011, and (even more so) *Linear Algebra for the Young Mathematician*, 2019; see the remarks below in Section 4.

While the book was designed for, and suitable for, a first course in linear algebra, the table of contents of Edwards' book already makes clear some differences to the standard curriculum. Some chapter titles include "Testing for Equivalence", "Matrices with Rational Number Entries", "Matrices with Polynomial Entries", "Similarity of Matrices", "The Spectral Theorem", and an appendix on Linear Programming. There are also some short supplements to some chapters, e.g., on page 58 we have a part entitled "Supplementary Unit: Finitely Generated Abelian Groups". There are many exercises throughout the book, many algebraic and interesting—and, very conveniently I believe, Edwards provides what appears to be a full set of solutions, in smaller print, single spaced, many of which are proofs and some of which take nearly the entire page.

This book is not ideal supplementary reading for our intended audience, but certainly belongs in a large-scale assessment of linear algebra books, as I attempt to do here, and also it should be on the reading list of students in mathematics.

14. John B. Fraleigh and Raymond A. Beauregard, *Linear Algebra*, Third Edition, 1995, 538+57 pages. One highlight of the book is chapter 10, on solving large linear systems, and thus covering computational algorithms and touching upon numerical analysis, roundoff error, timing, etc.. Chapter 9, while not unique to such books, is devoted to linear algebra within a complex field.

    Unfortunately, the available pdf file is not adequate: It is based on a scan, and not a good one, so reading the text is quite an annoyance on the eyes. On some pages, there is a whole thin piece of vertical text missing from the left or right side; other pages are "curved"; and on some pages, some text is blacked-out, e.g., on page 391, the box dedicated to detailing the "convenient procedure for finding a change-of-coordinate matrix" is unreadable.

3.4.2 Newer Generation

This list contains more recent books, reverse chronologically, and some are such that they could be used as the primary book, but did "not make the cut" to enter the above primary list for certain reasons that might be idiosyncratic with me, and thus appeal to others. The two most common reasons (but not the only reasons), based on my assessment, are either (i) the book is too concise—it is more like an overview rough set of notes, as opposed to a polished, impressive textbook; or (ii) too verbose and "dumbed down". Other common reasons (that an instructor may find petty, but students do not) are that (i) the formatting of the book, notably when they are legally free as pdf files, is not optically enticing; or (ii) the solutions to exercises are not provided. Another reason, used once, for Woerdeman (2021), is simply that the book is so new that the pdf is not yet available on the web. (That was the case in June 2021 for Bisgard's 2021, *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*, so I bought the pdf version of the book; but, as of August 2021, the pdf file is in fact available on the web.)

1. Hugo Woerdeman, *Linear Algebra: What You Need to Know*, 2021. I cannot find a free pdf on the web, but there is a preview, including the table of contents, at Amazon. I include this book also because the author has his attractive 2016 book, mentioned below in Section 4.1, *Advanced Linear Algebra*.

2. Anthony Roberts, *Linear Algebra for the 21st Century*, 2020, 688 pages. At the time of this writing, there is no pdf file for the book on the web; so I bought the book as a Kindle, at Amazon. As a relevant aside, some Kindle math books are horrendous in appearance, due to the equations. It appears that there is a newer Kindle format for such books that is equivalent in appearance to a pdf format, and that is the case with Roberts' book. To be extra clear: The Kindle version of Roberts is optically wonderful.

   The book is unique, and valuable, because the author explicitly uses the SVD as the vehicle for solving systems of linear equations. It also shows code and computations occasionally throughout, using Matlab / Octave, the latter being a free legal semi-clone of the former, and Matlab being, since the 1980s, the near ubiquitous choice for conducting and demonstrating calculations in linear and matrix algebra.

   I was not as impressed with it as I was hoping to be, but that could also be because I am so accustomed to the traditional presentation of the material. The MAA review https://www.maa.org/press/maa-reviews/linear-algebra-for-the-21st-century is positive, though notes some minor issues with the presentation. I personally would not use this as a first-course book (even as supplementary reading), but rather incorporate it perhaps into a second course, not because the mathematical level of the book is too high, but rather because (i) I prefer the current standard set of material for a first course, and (ii) because emphasis on the SVD is of great value, notably for our target audience of budding researchers in machine learning.

3. Jim Hefferon, *Linear Algebra*, 2020. The book could be adopted as the main text: it is legally free (open source license), has an available solutions manual, and—potentially decisive—has (very nice) slides for teaching. The reasons I do not place it in the list of strong recommendations above are: It is nowhere near as optically nice as most of the books from the list in Section 3.2, which are all visual and pedagogical works of art that are very hard to beat. I criticize also below his notation for the coordinate vector with respect to a particular basis, and I also don't enjoy the notation of a vector having an arrow atop it. These are perhaps minor quibbles (and, fascinatingly, could be changed—the LaTeXis open source), and the book has a nice selection of "topics", e.g., magic squares, Markov chains, and many others.

   The book received a quite strong review at MAA, helped by the fact that it is free and open source, https://www.maa.org/press/maa-reviews/linear-algebra-3. If

I were teaching a course, I would consider using some of Hefferon as additional reading, notably his applications ("topics", as he calls them).

4. Peter Saveliev, *Linear Algebra Illustrated*, 2020, 516 pages. This is, optically at least, quite a treat, full of—and I mean, on nearly every page—color graphics, and color used for marking theorems and other boxes, and also color used in the text itself sometimes as well to emphasize something. (The author has several books like this, including for topology.) The book appears aimed at high school students, and thus would not be on my recommended supplementary reading list for university master's students in the social sciences, but deserves mention due to the amazing pedagogic effort made by the author. Regretfully, of the two Amazon reviews, one is extremely negative, complaining about something I have not checked and confirmed, namely "So many typos, and they definitely are typos, and when they kept happening every second page, the writing was on the wall to get my refund."

5. Al Cuoco, Kevin Waterman, Bowen Kerins, Elena Kaczorowski, and Michelle Manes, *Linear Algebra and Geometry*, 2019, 557 pages. It is easy to quickly dismiss this (optically very pleasant) book as the epitome of "dumbed down", except when you realize (by reading the preface) that it is intended for high school students. In that light, the book is outstanding.

6. Bruce N. Cooperstein, *Elementary Linear Algebra*, 2019, 942 pages, available as a free pdf file from https://www.centerofmath.org/textbooks/ela/index.html (upon filling out a form, and with a very restricted number of countries, when specifying your affiliation address). The 2016 version is available on the usual pdf book web sites, and is entitled *Elementary Linear Algebra: An eTextbook*, 949 pages. This latter version is the one I have and use for subsequent discussion. Note that Cooperstein is also the author of *Advanced Linear Algebra*, 2nd edition, 2015.

   The coverage, in terms of essential topics, is very good, and executed very patiently. As the author boasts about, and correctly so, the book is unique in that it is filled with dynamic links (clickable entries), allowing the reader to constantly check the definition of any stated term. A typical chapter begins with the following outline (with each entry being a dynamic link):

   - Am I Ready for This Material
   - Readiness Quiz
   - New Concepts
   - Theory (Why It Works)
   - What You Can Now Do
   - Method (How To Do It)
   - Exercises
   - Challenge Exercises (Problems

   Adding such content takes quite some diligence, patience, and clear love of teaching, on behalf of, no less, a Distinguished Professor of Mathematics. Indeed, he is "involved in efforts to improve mathematics education at all levels", as we can read here, https://mdtp.ucsd.edu/about/workgroup/Professional%20Biographies/cooperstein-bruce-bio.html.

   Highlighting background colors are used to nicely flag definitions (yellow) and theorems (orange), while hyperlinks are blue. My scroll-through of the 942 pages indicates, however, that there are very few graphics, and the few that are there are very simple, e.g., Pythagoras, page 228; and orthogonal projection, page 242. (Being extra picky, I also do not like the "dot" the author uses, sitting on the bottom of the line, for the dot product; see, e.g., page 226.) There are answers to quiz questions, but not the section and chapter exercises, nor the "challenging exercises".

For a free book, and very well done in terms of content, paced delivery, and the useful hyperlinks, this book easily should be near the top of the list for supplementary reading, but I, perhaps rather idiosyncratically, do not think it is good enough to compete with the top handful of books in my primary list.

7. Przemyslaw Bogacki, *Linear Algebra: Concepts and Applications*, 2019. The 30 page treatment of the SVD along with interesting applications might be enough to warrant asking students to have a look at that chapter, and possibly more, from this book.

8. Dan Margalit and Joseph Rabinoff, *Interactive Linear Algebra*, 2019, 436 pages. This is a free electronic book, with a GNU Free Documentation License; see https://textbooks.math.gatech.edu/ila/ila.pdf. It was constructed as such, and there is also an interactive html version. While not my cup of tea, modern students might prefer this format for learning. Even the pdf has clickable links that lead to an online demonstration.

   The book is written in a "friendly fashion", such that there are very few theorem-proof pairs, but rather definitions, and lots of examples, and lots of (often color) graphics. The approach is more geometric than algebraic, and the level is obviously not high, and was in fact customized to a particular course (Math 1553 at Georgia Tech). Given the nature of the presentation (anything but terse theorem-proof), its use of clear, verbose explanations, its optical pleasantness via use of color and many graphics, its online interactive ability, and, finally, that it is legally free, it would make for very suitable self-study, possibly before taking, and in preparation for, a course in linear algebra, by those students not yet ready for a more hard-core presentation.

9. J. S. Chahal, *Fundamentals of Linear Algebra*, 2019, about 207 pages. This book is for a first course, containing a bit more than the standard topics, e.g., there is also the SVD, and chapter 10, "Selected Applications", which includes systems of differential equations, multivariate calculus, the special theory of relativity, cryptography, and "Solving Famous Problems from Greek Geometry". It is among the shortest books at this level (though see below for Lankham, Nachtergaele, and Schilling, as a more advanced book), and includes brief mention of several more advanced topics, e.g., quotient spaces and duality.

   It would not be my first (or second or third) choice as the main book for a first course, or even a second course, but it could be of value for supplementary reading, notably with the nice applications.

10. Thomas Shores, *Applied Linear Algebra and Matrix Analysis*, 2nd edition, 2018. It is a bit unconventional in its exposition, as compared to mainstream books, but has some unique features and topics, and worth a look.

11. Ravi Agarwal and Elena Flaut, *An Introduction to Linear Algebra*, 2017. From the preface, "Although several fabulous books on linear algebra have been written, the present rigorous and transparent introductory text can be used directly in class for students of applied sciences." I am not sure what that means, and the MAA review https://www.maa.org/press/maa-reviews/an-introduction-to-linear-algebra is also not very enthusiastic about the project. The book is too terse to be used as a first course, and, despite what the authors say about it being useful for beginning graduate students, the reviewer strongly questions this.

   I believe it could be useful as supplementary reading. The book is a bit more advanced than the base-level introductory books, covering a few more topics (e.g., dual space, SVD, Moore-Penrose, quadratic forms). The book is organized into 25 chapters, spanning only about 220 pages. Thus, it is terse, and can be used for extra reading after the student has worked through a more verbose, basic presentation. Hints or answers are provided in the book to the problems.

12. Arak M. Mathai and Hans J. Haubold, *Linear Algebra: A Course for Physicists and Engineers*, 2017, 450 pages. This might serve as background reading for math-phobic students: The book does not take on the theorem-proof structure, and gives many examples. It received a positive review in MAA, https://www.maa.org/press/maa-reviews/linear-algebra-a-course-for-physicists-and-engineers, stating for example

    > The book is written in a lovely style: it is easy to read, it is self-contained and assumes no mathematical knowledge beyond high school level. It also contains a huge number of examples showing how linear algebra can be used in other mathematical, physical and engineering domains and even in social science.

13. Stephen Andrilli and David Hecker, *Elementary Linear Algebra*, 5th edition, 2016, 750 pages. With such a "bloated" size, and meant for a soft introduction, there are many examples, and much text. I include this book because it might be wise to inform students about this and related books, if they feel the need for more hand-holding. An MAA review is reasonably positive, but does not shower the project with praise: https://www.maa.org/press/maa-reviews/elementary-linear-algebra-2.

14. Marcel B. Finan, *The Basics of Linear Algebra*, 2015, 307 pages, available as a free pdf file at https://faculty.atu.edu/mfinan/4003/LINENG.pdf; and the older but different version, *Fundamentals of Linear Algebra*, 2001, 196 pages, https://faculty.atu.edu/mfinan/algebra2.pdf.

    Both are very well written, covers the compulsory material, and is filled with examples (in the form of exercises and solutions). The 2001 book has the solutions immediately following each question, while the 2015 release has the more common format of having the solutions—briefly but adequately—at the end of the book to all the exercises. There is a separate document available for instructors that contains yet more detailed solutions to all the problems. The author was kind enough to respond immediately to my email request for this.

    Notably given its legally free status, solid presentation, and numerous worked problems, I would certainly recommend this book as supplementary reading. In fact, it nearly made my main list, but did not did not because (i) there are already so many other books, some outstanding, and also numerous very good ones whose electronic versions are legally free; and (ii) it is short on applications—as was intentional by the author and explained in the preface:

    > This book is addressed primarily to second and third your college students who have already had a course in calculus and analytic geometry. It is the result of lecture notes given by the author at The University of North Texas and the University of Texas at Austin. It has been designed for use either as a supplement of standard textbooks or as a textbook for a formal course in linear algebra. This book is not a "traditional" book in the sense that it does not include any applications to the material discussed. Its aim is solely to learn the basic theory of linear algebra within a semester period. Instructors may wish to incorporate material from various fields of applications into a course.

15. Edgar G. Goodaire, *Linear Algebra: Pure and Applied*, 2014, 716 pages.

    From the MAA review https://www.maa.org/press/maa-reviews/linear-algebra-pure-and-applied, we read:

> As indicated in the latter part of his title, Edgar Goodaire takes a "pure and applied" approach to learning Linear Algebra. Along with rigorous presentations of theorems and proofs, there are some more applied topics such as the pseudoinverse of a matrix in chapter 6 and the Singular Value Decomposition in chapter 7. Goodaire stresses he wants nothing in the book to be stated without justification. To help students with the proofs, he has included an appendix called "Show and Prove" and "Things I Must Remember" which were complied by his students over the years of writing the book.

Regarding "rigorous" and "nothing in the book to be stated without justification", see footnote 15 for a case in point of Goodaire including a result that we do not always see in such books; and footnote 23 for an example of something he does not prove (and cites a book from 2003 for the proof). The "Things I Must Remember" is basically a review of crucial results throughout the book (and with page references to where the topic was introduced), which I think is excellent, though could have been expanded upon a bit more. The reviewer also says:

> If there is enough time in the semester to get to orthogonality and the Spectral Theorem, then the final two chapters of the book are great for further and deeper study of Linear Algebra.

I also agree that these last two chapters contain highly relevant, somewhat more advanced material (QR factorization, pseudoinverses, the spectral theorem, SVD, etc.) that could be allocated to a second course.

For such a recent book, the available pdf file is "barely just good enough" to be read, and is far from the optical treat we usually get from new books, notably with a good choice of font. For example, I still cannot tell if a vector is designated with boldface or not, given the think print for all characters. Further, the presentation optically has no "bells and whistles", i.e., no use of color or margins. There are boxes around some text to indicate importance, and a few (simple, and black and white) graphics.

16. Géza Schay, *A Concise Introduction to Linear Algebra*, 2012. This is a very attractive book, and could have been placed above in the list of recommended books. For some topics, it is indeed a bit too concise, and thus I did not place it above. I like that it has Matlab exercises. A solutions manual is available from the publisher. As a hint that some creativity went into the writing of this book, the author shows an alternative proof (from a 1995 AMM journal) for the dimension theorem, on page 131. The last chapter is somewhat unique for books in this category and level, on numerical methods in linear algebra. Notably for applied researchers in machine learning, this could be seen as quite an advantage. One could however also argue that inclusion of introductory material on that topic is superfluous in light of (i) the already large number of basic topics that need to be covered in an introductory course; and (ii) the many books dedicated to the topic (see Section 4.5 below). I am of the opinion that this topic is of great importance, but best addressed in a subsequent course.

17. David Poole, *Linear Algebra: A Modern Introduction*, 3rd edition, 2011, 728 pages. The author goes to quite some length in the preface to make a strong case for the pedagogical features. The optical format and presentation of the book are similar to those of DeFranza and Gagliardi (2009), Williams (2018), Norman and Wolczuk (2020), and Johnston (2021). Poole is also among the largest—a great case in point of textbook bloat and emphasis on eye-candy and constant use of rudimentary, inane numerical examples reminiscent of a high-school trigonometry book. There are pictures and stories about mathematicians, plenty of examples, and quite a large number of interesting applications, at least in enough detail to indicate to a student the importance and relevance of linear algebra.

The book appears to cover all the essential topics for a first course, and, arguably does so in the recent pedagogical style of much hand holding, conversational style, extensive applications and examples, and possibly some "dumbing down". Quoting the author from the preface, "I have tried my best to write the book so that students not only will find it readable but also will *want* to read it."

The MAA review is not very positive: He remarks on the very high price of the book (notably in light of all the competition for this market niche of first-course books), but also is not enthusiastic about the style of the book. See https://www.maa.org/press/maa-reviews/linear-algebra-a-modern-introduction. Note that the pdf of the book is available on the web, though it is not open source, and thus it is not clear if instructors can legally ask students to formally access that pdf and read parts of the book.

18. Derek J. S. Robinson, *A Course in Linear Algebra with Applications*, Second Edition, 2006, 436 pages.

This book is intended for a first course, but is rather mature (meaning, brief and mathematical) in its exposition, and also goes beyond the minimal set of mandatory topics. From the preface to the second edition, we read, in addition to adding material on linear programming and the simplex method:

> Some further applications of linear algebra have been added, for example the use of Jordan normal form to solve systems of linear differential equations and a discussion of extremal values of quadratic forms.
>
> On the theoretical side, the concepts of coset and quotient space are thoroughly explained in Chapter 5. Cosets have useful interpretations as solution sets of systems of linear equations. In addition the Isomorphisms Theorems for vector spaces are developed in Chapter 6: these shed light on the relationship between subspaces and quotient spaces.

Chapter 9, "More Advanced Topics" covers (i) Eigenvalues and Eigenvectors of Symmetric and Hermitian Matrices; (ii) Quadratic Forms; (iii) Bilinear Forms; and (iv) Minimal Polynomials and Jordan Normal Form. My perusal of this chapter indicates that it is very well written, already justifying its purchase and recommendation to students.

I did read Chapter 3 closely, on determinants. He proves everything (of interest at this level), rather succinctly and elegantly: I still have a preference for Lang's (1987) presentation, but that of Robinson can serve as an excellent compliment to Lang, because it shows alternative proofs that ultimately further deepen one's understanding of the topic.

Robinson's book has further good features: The MAA review https://www.maa.org/press/maa-reviews/a-course-in-linear-algebra-with-applications is very positive, also praising the author's presentation of (one of the topics I inspect, namely) determinants and permutations. He writes:

> [T]he text devotes an entire chapter of about thirty pages to determinants. Robinson carefully brings in the relevant permutation theory, etc., and as a result ideas such as Cramer's Rule can become intuitive to the diligent reader.
>
> A chapter on eigenvalues and eigenvectors also introduces Markov processes and prepares the reader for two closing chapters touching on symmetric and Hermitian matrices, quadratic forms, bilinear forms, Jordan normal form, and linear programming through to the Simplex Algorithm. (The linear programming material forms the core of the additions to this edition from the first edition.)
>
> Overall, this book can serve well as a one semester introductory course in linear algebra, or a complete package for self-study for even the uninitiated.

With respect to the last statement I show from the previous review, I highly disagree: Robinson's book, admirable indeed, absolutely *cannot* serve as a self-study package for the "uninitiated" if this includes students whose intrinsic interest and skill in mathematics is low, as compared to their STEM-capable compatriots, e.g., a solid percentage of undergraduate university students in the social sciences. For example, on page 24, the student gets a crash course in rings and fields, for starters. We might thus expect to find a disgruntled reviewer at Amazon: Indeed, among the very small set of reviews, one verified purchaser writes (Dec. 2019) "Incomprehensible. I have used over a dozen books on linear algebra. This is the worse book. There are hardly any examples and void of [sic] amy explaination. This book is not worth the paper it is printed on. You are much better off with Anton, Lay or Larson."

Solutions to some of the exercises are provided at the end of the book. Annoyingly, it is just a small handful of them, and are typically the numeric ones, with just a cryptic numeric answer given. One can speculate the noble goals of the author to omit answers requiring proofs, but this further rules out using the book for self-study for most Homo Sapiens.

The available pdf file is very clear and fully readable, abeit far from an optical treat: The book itself has the "classic" (1980s, 1990s) presentation style, i.e., black and white, simple font, no use of margins, or any bells and whistles.

I found some typos (and a couple of "complaints"):

- page 81, line −4, "Theorem 3.3.5" should be "Theorem 3.3.6".
- page 83, in both equations showing determinants there needs to be an $= 0$ on the right side.
- page 121, line −4, the submission should be over j and not i.
- On page 127, line 3, regarding matrix $B = EA$, where E is square, full rank, and an elementary row operations matrix, we read: "The row times column rule of matrix multiplication shows that each row of B is a linear combination of the rows of A." This is of course true but I don't think it is at all obvious to the casual reader: A good place to see it is Henry Ricardo's book, page 188, in the theorem and proof given about matrix multiplication in terms of rows. It is also nicely spelled out in Meyer (2010, p. 98).
- page 129, line 5, the sentence beginning with "therefore", I am not sure this is so trivial to understand, or at least I don't fully understand it.

19. James B. Carrell, *Fundamentals of Linear Algebra*, 2005, 412 pages. The pdf file is freely available from https://secure.math.ubc.ca/~carrell/NB.pdf. From the introduction: "This textbook is meant to be a mathematically complete and rigorous introduction to abstract linear algebra for undergraduates, possibly even first year students, specializing in mathematics." The book definitely covers the main topics for a first course, and goes a bit beyond, with, e.g., quotient spaces, and the Jordan decomposition theorem, but also, and rather uniquely, "coding theory". As the author writes in the introduction,

> Chapter [6] is an introduction to linear coding theory. The basic results about error correcting codes are proven, and we treat perfect codes in some detail. As well as being a timely subject, the topic of linear coding theory illustrates as well as anything I know of how powerful and useful the results of elementary linear algebra really are.

While those aforementioned topics that go beyond a standard first-course are covered in the book, the SVD is not covered, nor generalized inverses, nor matrix exponentials, nor most of the topics typically associated with an advanced linear algebra book, nor any of the applied topics covered in, say, Olver and Shakiban (2018) (as appears in the list in Section 4.1). As such, the book is for a first course in undergraduate linear algebra, possibly an "honors" first course, with the book being indeed more towards the sophisticated end of the spectrum (and with the somewhat unique chapter on coding theory),[30] but it is not suitable (nor did the author say it is) for both a first and second course. There is a large number of exercises, but apparently no solutions available.

The lack of solutions (to the—admirably—large number of exercises), and (arguably superficially, but remember, the choice of book should please the students, not necessarily the instructor), the presentation looking like a (solid and extensive) set of basic LaTeX notes, and with a bit too much white space, as opposed to an optimized and well-formatted textbook, led me to not include this book in the main list. Compare, for example, to the highly polished, if not immaculate, works of art by Meyer, and Johnston, both of which also include solutions. The point of mentioning this is also to emphasize that *with respect to content and presentation*, Carrell is noteworthy, and might serve as excellent supplementary reading.

20. Gerald Farin and Dianne Hansford, *Practical Linear Algebra: A Geometry Toolbox*, 2005, 384 pages. The book is definitely unconventional in that it does not concentrate on algebraic proofs, but rather indicates most everything geometrically (in 2 and 3 dimensions). They provide good motivation for such an approach in terms of student's ability to learn and understand concepts. The MAA review https://www.maa.org/publications/maa-reviews/practical-linear-algebra-a-geometry-toolbox is lengthy and informative, and overall positive (though points out some issues and gaffes), concluding with:

    > I would rather use Strang's book or Bretscher's with the types of students I have been fortunate enough to encounter in linear algebra, but I liked this book more than I expected to. Its strengths are readability and concreteness. I think many students would acquire a good intuition for and working knowledge of linear algebra, at least in 2 and 3 dimensions, by following it.

    The reviews at Amazon are mixed, with some strongly praising the book as being "exceptional" and "perfect", while others are not so pleased, finding the book rather poor. One writes "This book may, perhaps, be useful as a reference book to someone who already understands the material and need not be guided through the processes, but as a learning aid, it is an absolute disaster."

[30]Other books that discuss error correcting codes are Bruce N. Cooperstein, *Elementary Linear Algebra: An eTextbook* (2016, Sec. 5.8); and Helene Shapiro, *Linear Algebra and Matrices: Topics for a Second Course* (2015, Ch. 18).

3.5 A Detailed Inspection of Notation

> Notation is extremely important in mathematics. Much care and attention should be paid to it—it is an intrinsic part of the mathematics. Good notation frees the mind to focus on the concepts involved while poor notation detracts.
>
> Gareth Williams (2018, p. 300)

The notation for most objects in linear algebra is the same across most books. As examples, most books use a lower case bold face letter, e.g., $\mathbf{v}$, to denote a vector, while a few others such as Norman and Wolczuk (2020); and Hefferon (2020), use the arrow on top notation, as is more common in physics. In the majority of books, the length, or norm, of a vector $\mathbf{v}$ is written as $\|\mathbf{v}\|$ though some, such as Schay (2012) and Roberts (2020), use $|\mathbf{v}|$. For the transpose of a matrix A, more common is A^T or some variation of that, notably with the transpose operator on the top right of matrix A, while Lang, in his two books on linear algebra, uses ${}^tA$, as does Weintraub (2011) and Andreescu (2014).

Arguably inconsequentially, books differ regarding their definition of vectors and their shapes. To give some examples, Anthony and Harvey (2012, p. 23) provide the useful clarification "when we simply use the term *vector*, we shall mean a column vector". They designate matrices, row vectors, and column vectors using parentheses (as opposed to brackets); see their page 24. Roberts (2020, p. 4) writes "a **vector** is an ordered n-tuple of real numbers $x_1, x_2, \ldots, x_n$ equivalently written either as a row in parentheses or as a column in brackets" (and followed by an example of such). Ricardo (2010, p. 2) says that "[a] **vector** is an ordered finite list of real numbers", and goes on to distinguish between a row vector, written in brackets [...], and a column vector, also written in brackets. Meyer (2010, p. 81) defines n-tuples of real numbers, and then defines row and column vectors (and uses parentheses to demarcate both of them). One takeaway from this inspection is that students should be flexible with designating (row or column) vectors, and matrices, with either parentheses, or brackets, with some books using parentheses, some using brackets, and some using both!

For the coordinates of a vector relative to a specific ordered basis, we see more variation in notation across books. Interestingly, the above quote by Williams was stated precisely in his section on coordinate vectors (and his notation is given in an entry below). To get a strong sense of the different notations used, and also make clear what is by far the most common one (and also which ones I think are poor choices), I will give the choice of notation for 50 books.

One possibility is to not use any particular notation at all, and just show a usual vector of scalars (an ordered n-tuple), which is obviously what the coordinate vector is. Here are three books that follow this convention:

1. *A Concise Text on Advanced Linear Algebra*, Yisong Yang, 2015, page 14. In particular, the author writes:

   > **Definition 1.8** Let $\{u_1, \ldots, u_n\}$ be a basis of the vector space U. Given $u \in U$ there are unique scalars $a_1, \ldots, a_n \in \mathbb{F}$ such that
   >
   > $$u = a_1 u_1 + \cdots + a_n u_n.$$
   >
   > These scalars, $a_1, \ldots, a_n$ are called the *coordinates*, and $(a_1, \ldots, a_n) \in \mathbb{F}^n$ the *coordinate vector*, of the vector u with respect to the basis $\{u_1, \ldots, u_n\}$.

2. *A Course in Linear Algebra with Applications*, Derek J. S. Robinson, 2006, page 120. The same approach as in the previous entry is taken (except that Robinson uses a column vector instead of a row vector).

3. *Matrices and Linear Algebra*, Hans Schneider and George Phillip Barker, Second Edition, 1973, page 135.

Based on my inspection of the subset of books I include in this document for linear algebra, the following notation appears to be the most popular: Randomly picking one element from the induced equivalence class, we read, from Friedberg, Insel, and Spence, 4th edition, page 80:

> Let $\beta = \{u_1, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$, let $a_1, \ldots, a_n$ be the unique scalars such that
>
> $$x = \sum_{i=1}^{n} a_i u_i.$$
>
> We define the **coordinate vector of x relative to β**, denoted $[x]_\beta$, by
>
> $$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

I now give a partial list of books using *exactly* this notation (of course possibly defined using different letters for the variables), and in particular, notice that the vector on the right hand side of the definition, $(a_1, \ldots, a_n)'$, does *not* have a subscript β indicating the base. This makes sense, as $(a_1, \ldots, a_n)'$ is just an n-tuple of numbers.

1. As just shown, *Linear Algebra*, by Friedberg, Insel, and Spence, 4th edition, page 80.

2. *Matrix Algebra and Applied Linear Algebra*, Carl Meyer, 2010, page 240.

3. *Linear Algebra for the 21st Century*, Anthony Roberts, 2020, p. 610.

4. *Linear Algebra*, Nair and Singh, 2018, page 76.

5. *A Second Course in Linear Algebra*, Garcia and Horn, 2017, page 40.

6. *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*, by James Bisgard, 2021, page 48.

7. *Applied Linear Algebra and Matrix Analysis*, Thomas Shores, 2nd edition, 2018, Definition 3.9 in Section 3.3.

8. *Linear Algebra*, by Elizabeth S. Meckes and Mark W. Meckes, 2018 page 185.

9. *An Introduction to Linear Algebra*, by Norman and Wolczuk, 3rd edition, 2020, page 264. I show an example from their book that will be useful below to indicate a possible source of confusion *when using a notation from a book discussed below.* In particular, on page 291 of Norman and Wolczuk, they give a change of basis example using vector (written as transpose) $x = [5\ 2]'$ and a non-standard basis $\mathcal{B}$, and write: $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_{\mathcal{B}}$. Observe how this is fully correct, i.e., with x so defined, trivially,

   $$[x]_{\mathcal{B}} = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_{\mathcal{B}}, \tag{1}$$

   and, for the basis $\mathcal{B}$ used in the example in their book, namely

   $$\mathcal{B} = \left\{ \left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right], \left[\begin{smallmatrix} -4 \\ 3 \end{smallmatrix}\right] \right\}, \tag{2}$$

   we have (see also (4) below)

   $$[x]_{\mathcal{B}} = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_{\mathcal{B}} = \tfrac{1}{25} \left[\begin{smallmatrix} 23 \\ -14 \end{smallmatrix}\right]. \tag{3}$$

10. *Advanced Linear Algebra*, by Hugo Woerdeman, 2016, page 46.

11. *Linear Algebra*, by Belkacem Said-Houari, 2017, page 235.

12. *A Guide to Advanced Linear Algebra*, by Steven Weintraub, 2011, page 42. Unsurprisingly, and not listed as a separate entry, it is also the case for Weintraub's *Linear Algebra for the Young Mathematician*, 2019, page 150, Definition 5.3.2.

13. *Advanced Linear Algebra*, 2nd edition, by Bruce Cooperstein, 2015, page 42. (Yes, also page 42, as with Weintraub's book!) Unsurprisingly, and not listed as a separate entry, it is also the case for Cooperstein's *Elementary Linear Algebra: An eTextbook*, 2016, page 532.

14. *Linear Algebra as an Introduction to Abstract Mathematics*, by Lankham, Nachtergaele, and Schilling, 2016, page 111.

15. *Linear Algebra: A Modern Introduction*, by David Poole, 3rd edition, 2011, page 214.

16. *Geometric Linear Algebra 1*, by I-Hsiung Lin, 2005, page 34.

17. *Fundamentals of Linear Algebra*, by Marcel B. Finan, 2001, page 128.

18. *Linear and Geometric Algebra*, by Alan Macdonald, 2010, page 28.

19. *Linear Algebra with Applications*, by Steven Leon, 2015, 9th edition, page 169.

20. *Linear Algebra Done Wrong*, by Sergei Treil, 2017, page 69.

21. *Linear Algebra*, by Li-Yong Shen, Haohao Wang, and Jerzy Wojdylo, 2019, page 14.

22. *Matrix Theory and Linear Algebra*, by Peter Selinger, 2020, page 198.

23. *Differential Equations and Linear Algebra*, by Goode and Annin, 4th edition, 2016, page 312.

24. *Linear Algebra with Applications*, by Otto Bretscher, 5th edition, 2013, page 172.

25. *Interactive Linear Algebra*, by Dan Margalit and Joseph Rabinoff, 2019, page 103.

26. *Introduction to Linear Algebra with Applications*, by Jim DeFranza and Dan Gagliardi, 2009, page 175.

27. *Linear Algebra*, by R. R. Stoll and E. T. Wong, 1968, page 39.

28. *A Course in Linear Algebra*, by David B. Damiano and John B. Little, 1988, page 77.

29. *Advanced Linear Algebra*, by Steven Roman, 3rd edition, 2007, page 51. We read:

    > If $\mathcal{B} = (v_1, \ldots, v_n)$ is an ordered basis for V, then for each $v \in V$ there is a unique ordered n-tuple $(r_1, \ldots, r_n)$ of scalars for which
    >
    > $$v = r_1 v_1 + \cdots + r_n v_n$$
    >
    > Accordingly, we can define the **coordinate map** $\phi_{\mathcal{B}} : V \to F^n$ by
    >
    > $$\phi_{\mathcal{B}}(v) = [v]_{\mathcal{B}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$
    >
    > where the column matrix $[v]_{\mathcal{B}}$ is known as the **coordinate matrix** of v with respect to the ordered basis $\mathcal{B}$. Clearly, knowing $[v]_{\mathcal{B}}$ is equivalent to knowing v (assuming knowledge of $\mathcal{B}$).

    The only difference to the books previously mentioned is that Roman additionally defines the coordinate map $\phi_{\mathcal{B}} : V \to F^n$ (where F is a field, e.g., $\mathbb{R}$, as defined on his page 19).

30. *Advanced Linear Algebra*, by Nicholas Loehr, 2014, page 132, except that the vector of coordinate values is not written as a column vector, but rather as an n-tuple, i.e., $(a_1, \ldots, a_n)$.

31. *Linear Algebra and Differential Equations using MATLAB*, Martin Golubitsky and Michael Dellnitz, originally 1999, currently June 12, 2020, from Creative Commons; page 511. Similar to Loehr, the coordinates are written as a row vector, or n-tuple, $(\alpha_1, \ldots, \alpha_n)$.

32. *Advanced Linear and Matrix Algebra*, by Nathaniel Johnston, 2021, page 25. Similar to Loehr, and Golubitsky & Dellnitz, the coordinates are written as $(c_1, \ldots, c_n)$.

I present next a subsequent list: Starting with entry 1, up to and including entry 4, these are books that use the aforementioned notation *and also* a further notation (that I claim can be potentially misleading—generally so, but notably for students, who are already battling with the new conceptual ideas), namely adding the subscript referring to the basis *also* onto the n-tuple of coefficients, *in addition to* having it on the bracketed vector of interest, e.g., $[x]_\beta$. List elements 5 through 15 use a different notation.

1. The book *A Modern Introduction to Linear Algebra*, by Henry Ricardo, 2009, page 53, also has the $[\mathbf{v}]_B$ notation, namely:

   > We will use the notation $[\mathbf{v}]_B$ to denote the unique coordinate vector corresponding to vector $\mathbf{v}$ relative to basis B: $[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. To be precise, we must have an *ordered basis* (i.e., we should think of our basis vectors in a particular order) so that the coordinate vector with respect to this basis is unambiguous.
   > If basis B for a space $\mathbb{R}^n$ is the standard basis $\mathcal{E}_n$ and $\mathbf{v} \in \mathbb{R}^n$, then $\mathbf{v}$ is its own coordinate vector relative to B: $\mathbf{v} = [\mathbf{v}]_{\mathcal{E}_n}$.

   However, we then read on page 54:

   > An equation of the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{B_1} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}_{B_2}$ will indicate that a vector, whose coordinates with respect to an ordered basis B_1 are x, y, and z, has coordinates u, v, and w with respect to the ordered basis B_2.

   A manual search (and thus quite possibly not complete) indicates that this latter notation is not used much. I found it only once, at the top of his page 418.

   This latter notation, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]_B$, is, *outside of the context in which the author states*, potentially misleading when, in the same book, the notation $[\mathbf{v}]_B$ is used. For example, let B be the basis for $\mathbb{R}^2$ as given in (2), and observe that, with respect to the usual "$[\mathbf{v}]_B$ notation", $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]_B$ means that the vector of interest (in standard coordinates) is $\mathbf{v} = [5\ 2]'$ (where prime means transpose), and $[\mathbf{v}]_B$ refers to the 2-tuple of coordinates with respect to basis B. That is, letting B be the basis in (2), $[\mathbf{v}]_B$ is the vector given at the end of (3). This can be contrasted to the aforementioned notation used on page 54 of Ricardo, namely, for $n = 2$ and (non-standard) basis B, $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]_B$, *which shows the coordinates themselves, as opposed to the linear combination of basis vectors, with those given coordinates.*

   To make the distinction very clear (and thus the potential source of confusion for the reader), let B be the basis given in (2), and observe that, in the usual "$[\mathbf{v}]_B$ notation" and letting $\mathbf{v} = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]$, we have

   $$[\mathbf{v}]_B = \left[\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right]\right]_B = \tfrac{1}{25}\left[\begin{smallmatrix} 23 \\ -14 \end{smallmatrix}\right], \quad \text{i.e.,} \quad \mathbf{v} = \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right] = \left(\tfrac{23}{25}\right)\left[\begin{smallmatrix} 3 \\ 4 \end{smallmatrix}\right] - \left(\tfrac{14}{25}\right)\left[\begin{smallmatrix} -4 \\ 3 \end{smallmatrix}\right]. \tag{4}$$

Contrast this to the "page 54 notation", which would imply that

$$\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_B = 5\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right] + 2\left[\begin{smallmatrix}-4\\3\end{smallmatrix}\right] \neq \begin{bmatrix}23/25\\-14/25\end{bmatrix}.$$

This gives rise to the notational question, is $[[\begin{smallmatrix}x\\y\end{smallmatrix}]]_B = [\begin{smallmatrix}x\\y\end{smallmatrix}]_B$? The previous example suggests not, and even if the author distinguishes between the two, is that really doing the reader, especially a student, a favor?

The next entry, showing how author Chahal deals with this, shares the same potential source of confusion as Ricardo, though, like Ricardo, its usage in context is, presumably, clear in his book. Still, this does not justify its usage, and I deem it to be a poor choice of notation.

2. In *Fundamentals of Linear Algebra*, by J. S. Chahal, 2019, pages 65-6, we find:

> Suppose V is a finite dimensional vector space of dimension n and $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an ordered basis of V. It is easy to verify that each vector $\mathbf{v}$ in V has a unique representation
>
> $$\mathbf{v} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n$$
>
> as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The coefficients $x_1, \ldots, x_n$ are called the *coordinates* (or *weights*) of $\mathbf{v}$ with respect to $\mathcal{B}$. We then denote $\mathbf{v}$ by
>
> $$\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}_{\mathcal{B}} \qquad \text{or} \qquad \mathbf{v}_{\mathcal{B}} = \mathbf{x} = \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}. \tag{5}$$

This has the same potential for confusion as the notation in Ricardo, easily seen by trying to answer the question: For a given (non-standard) basis $\mathcal{B}$, e.g., (2), what is $(\begin{smallmatrix}1\\1\end{smallmatrix})_{\mathcal{B}}$? It appears that the (partial) clarity to determine what vector $\mathbf{v}$ is in (5) comes from the lettering convention used, namely $\mathbf{v}$ and $\mathbf{x}$, with the former being the vector of interest, and the latter being the coordinates. It is clearly untenable to link notation to the particular letters used. In particular, with $\mathbf{x}$ so defined on the right of that equation, we have from the left side ("denote $\mathbf{v}$ by") $\mathbf{v} = \mathbf{x}_{\mathcal{B}}$, whereas the right side implies $\mathbf{x} = \mathbf{v}_{\mathcal{B}}$. Again, for a given (non-standard) basis $\mathcal{B}$, what is $(\begin{smallmatrix}1\\1\end{smallmatrix})_{\mathcal{B}}$? Let $\mathbf{w} = (\begin{smallmatrix}1\\1\end{smallmatrix})$; then what is $\mathbf{w}_{\mathcal{B}}$?

Based on Chahal's notation (5), the more common designator $[\mathbf{w}]_{\mathcal{B}}$ does not have meaning: That is fine (though deviating from popular notation is arguably a bad idea, for students, and for book sales, unless one has very good reason). However, what if brackets or parentheses are required, as in $(\mathbf{u} + \mathbf{w})_{\mathcal{B}}$ or $[\mathbf{u} + \mathbf{w}]_{\mathcal{B}}$? Both are subject to confusion, as noted already, especially the latter one, because of the more common notation used across a large number of books that the student might be familiar with (or read while using supplementary books).

The notation used in Ricardo, and in Chahal, can be compared to that used in Anthony & Harvey, as discussed next.

3. *Linear Algebra: Concepts and Methods*, by Martin Anthony and Michele Harvey, 2012, page 185. Their Definition 6.32 states:

> If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V and $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$, then the real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the *coordinates* of $\mathbf{v}$ with respect to the basis, S. We use the notation
>
> $$[\mathbf{v}]_S = \begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{bmatrix}_S \tag{6}$$
>
> to denote the *coordinate vector* of $\mathbf{v}$ in the basis S.

The notation used by the majority of books mentioned above is to designate the coordinates of vector $\mathbf{v}$ with respect to basis S as either $[\mathbf{v}]_S$, or as an n-tuple, e.g., $(\alpha_1, \ldots, \alpha_n)$ (possibly transposed so that it is also a vector in the common usage that a vector is a column), whereas the Anthony and Harvey notation on the right side of (6) implies, as an n-tuple expressed (as usual) as a column vector in parentheses,

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}_S \quad \text{“can be expressed as”} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}, \tag{7}$$

or “is really the same as”, or “reduces to”, or “can be written as”, or, just, “equals”.

I argue that the second expression in (6), adorning also the vector of *coordinates themselves*, also with subscript S, is, at best, redundant, and, worse, possibly confusing. It *might* be of use in precisely the context mentioned above on page 54 of Ricardo, but I do not see it as being necessary, and instead just a source of notational confusion for students.

Via their strict use of parentheses to denote an n-tuple, either as a row vector or column vector (Anthony and Harvey, 2012, p. 23), and the bracket notation to designate a coordinate vector, this notation formally works. Still, as I mentioned near the opening of this section, given the plethora of linear algebra books, and that some use parentheses to demarcate vectors, while others use brackets, it might be wise to not define a vector-type object that relies on differentiating between parentheses and brackets.

Let's look at part of their first example, on the same page:

> **Example 6.33** The sets $B = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $S = \{\mathbf{v}_1, \mathbf{v}_2\}$, where
>
> $$B = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad S = \left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$
>
> are each a basis of $\mathbb{R}^2$. The coordinates of the vector $\mathbf{v} = (2, -5)^{\mathrm{T}}$ in each basis are given by the coordinate vectors,
>
> $$[\mathbf{v}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}_B \qquad \text{and} \qquad [\mathbf{v}]_S = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S.$$
>
> In the standard basis, the coordinates of $\mathbf{v}$ are precisely the components of the vector $\mathbf{v}$, so we just write the standard coordinates as
>
> $$\mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$$

So, from (6), this means

$$\mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} = [\mathbf{v}]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}_B,$$

and we should also be able to write, using (7):

$$\text{For } \mathbf{v} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}, \quad [\mathbf{v}]_S = \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix}\right]_S = \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

and note that

$$\left[\begin{pmatrix} -1 \\ 3 \end{pmatrix}\right]_S \neq \begin{bmatrix} -1 \\ 3 \end{bmatrix}_S = \left[\begin{pmatrix} 2 \\ -5 \end{pmatrix}\right]_S \neq \begin{bmatrix} 2 \\ -5 \end{bmatrix}_S.$$

Thus, for the question what is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_S$, where S is given above in their Example 6.33, e.g., $S = \{\mathbf{v}_1, \mathbf{v}_2\}$; the answer is, from the right hand side of (6), vector $(1, 1)^{\mathrm{T}}$. It is *not* the case that (for non-standard basis S) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_S = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_S$, nor is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_S$ equal to $(1)\mathbf{v}_1 + (1)\mathbf{v}_2$.

Finally, in their notation, the question: “Let $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; what are $[\mathbf{w}]_S$ and $\mathbf{w}_S$?” is not, in my opinion well-defined, though, in our discussions, author Ms. Harvey says that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_S$ is in fact well-defined, and its meaning is that given on the right hand side of (6). As I conveyed to her, I do not like it, because I find it hard to differentiate it from:

$$\text{Let } \mathbf{v} = \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right). \text{ What is } [\mathbf{v}]_S?$$

To me, here, $[\mathbf{v}]_S$ looks a whole lot like the left hand side of (6), and, for a non-standard basis (that being the whole point of introducing any of this notation), the lhs and rhs resolve to different n-tuples. It all comes down to context within their book, and they presumably get it right, so, formally, there can never be confusion. That is, assuming the reader can decipher precisely which meaning is meant. But I think the whole point of using impeccably defined, unambiguous notation is to add clarity, as opposed to (as it is called in computer science) "operator overloading", and letting the reader (or the code compiler) have to assess the correct meaning from the context.

This is why I think there is a reason that at least 46 linear algebra books do not use this notation, even though I concur with Ms. Harvey that, *within their book*, their notation is, formally speaking, consistent. This is the case because, as she pointed out to me, nowhere in the book will the reader find something like $\left[\left(\begin{smallmatrix}2\\-5\end{smallmatrix}\right)\right]_S$.

I let the reader decide if their notation is prudent.

4. *Linear Algebra*, by Jim Hefferon, 4th edition, 2020, page 124, uses similar notation as Ricardo, and Anthony & Harvey, but with some differences. In particular, we read (leaving off the arrow atop the vectors):

   > **Definition** In a vector space with basis B the *representation of v with respect to B* is the column vector of the coefficients used to express v as a linear combination of the basis vectors:
   >
   > $$\operatorname{Rep}_B(v) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$$
   >
   > where $B = \langle \beta_1, \ldots, \beta_n \rangle$ and $v = c_1\beta_1 + c_2\beta_2 + c_n\beta_n$. The c's are the *coordinates of v with respect to B.*

   We notice that the author uses $\operatorname{Rep}_B(v)$ instead of $[v]_B$, and the author indeed mentions (on page 125) that the latter notation is more popular, but the former is "harder to misinterpret or overlook". This $\operatorname{Rep}_B(v)$ is fine; the problem is the subscript on the right side vector in the definition. Notice that the right vector in the definition is in usual parentheses, whereas Anthony and Harvey purposely used brackets. So, what is the distinction between $(c_1, c_2, c_3)'$ (where the prime indicates transpose, to save space), and $(c_1, c_2, c_3)'_B$? They are both just n-tuples; the subscript B is not useful on that vector, unless it is used as in Ricardo, page 54, as discussed above, when comparing the coordinates of a vector with respect to two different bases. Indeed, in the subsequent examples in Hefferon, in about half, he uses the B subscript (but not for comparison across different bases), while in half, he does not use it. This lack of consistency makes matters even worse for the student.

5. *Linear Algebra*, by Serge Lang, 3rd edition, 1987, page 88. The coordinates of vector v in basis B are denoted by $X_B(v)$.

6. *An Introduction to Linear Algebra*, Agarwal and Flaut, 2017, page 83. They use the same notation as that in Lang, namely $y_B(v)$.

7. *Linear Algebra with Applications*, W. Keith Nicholson, 2021, page 496. The author uses the same notation as Lang, and Agarwal and Flaut, namely $C_B(\mathbf{b})$, and expressing this as a column vector (in brackets, as used in the book for column vectors and matrices, but notably without the B subscript). A benefit of this notation is that (as pointed out on the same page), for any n-tuple column vector $\mathbf{w} = (w_1, \ldots, w_n)'$ and basis B for vector space V given by $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $C_B^{-1} : \mathbb{R}^n \to V$ is such that $C_B^{-1}(\mathbf{w}) = w_1\mathbf{b}_1 + \cdots w_n\mathbf{b}_n$.

8. *Linear Algebra: A Geometric Approach*, by Theodore Shifrin and Malcolm R. Adams, Second Edition, 2011, page 227. Same as the previous ones, namely $C_{\mathcal{B}}(\mathbf{v})$ for ordered basis $\mathcal{B}$.

9. *A First Course in Linear Algebra*, by Robert A. Beezer, 2015, page 501. Same, namely $\rho_B(\mathbf{w})$.

10. *Matrices and Linear Transformations*, by Charles G. Cullen, Second Edition, 1972, section 2.5, definition 2.12 (approximately page 94).[31] Same, namely $\mathrm{Crd}_\beta(\alpha)$. That is, for ordered basis $\beta = \{\beta_1, \ldots, \beta_m\}$ and vector $\alpha = a_1\beta_1 + \cdots + a_m\beta_m$, Cullen writes:

$$\mathrm{Crd}_\beta(\alpha) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix},$$

the point of writing this to emphasize that the right hand side column vector is *not* endowed with the subscript β.

11. *Linear Algebra: Pure and Applied*, by Edgar G. Goodaire, 2014, page 429. The notation used is similar to the very common "bracket notation" $[\mathbf{x}]_{\mathcal{V}}$, except the brackets are left off, i.e., $\mathbf{x}_{\mathcal{V}}$.

12. *Linear Algebra*, by John B. Fraleigh and Raymond A. Beauregard, Third Edition, 1995, page 389. Again; the usual notation but leaving the brackets off, e.g., $\mathbf{v}_B$.

13. *Linear Algebra with Applications*, by Gareth Williams, 2018, 9th edition, page 299. Again; the same "subscripted but without the brackets" notation is used, as in Goodaire, and in Fraleigh and Beauregard.

14. *Linear Algebra and Geometry*, by Al Cuoco, Kevin Waterman, Bowen Kerins, Elena Kaczorowski, and Michelle Manes, 2019, page 372. Here we have something new: The authors use "subscripted, without the brackets, but underlined", i.e., $\underline{v}_{\mathcal{B}}$.

15. *A Concise Introduction to Linear Algebra*, by Géza Schay, 2012. The preface, page ix, explains his notation, and even criticizes the $[\mathbf{x}]_A$ "bracket notation". His choice of notation is introduced on his page 148. It is very similar to that of Goodaire. In particular, for $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ an ordered basis for X, the ordered n-tuple $\mathbf{x}_{\mathbf{A}} = (x_{A1}, x_{A2}, \ldots, x_{An})^T$ (T is transpose) is called the coordinate vector of $\mathbf{x}$ relative to A, and is a vector in $\mathbb{R}^n$. Its components x_{Ai} are called the coordinates of $\mathbf{x}$ relative to A.

[31] Page numbering was suppressed in the available pdf file, presumably because it was generated from the Amazon Kindle version of the book.

4 Course Proposal 2: "A Second Course in (Undergraduate) Linear Algebra"

> There are a large number of applications of the singular value decomposition.
> G. H. Golub and C. Reinsch (1970, p. 407).

> [T]he practical and theoretical importance of the SVD is hard to overestimate.
> Gene Golub and Charles Van Loan, *Matrix computations* (2013, Sec. 2.4).

> The SVD constitutes one of science's superheroes in the fight against monstrous data, and it arises in seemingly every scientific discipline.
>
> The SVD is an omnipresent factorization in a plethora of application areas.
> Carla D. Martin and Mason A. Porter (2012, p. 839, p. 848)

> There is a colossal number of examples of SVD's usefulness.
> James Bisgard (2021, p. xii)

> The singular value decomposition (SVD) is the central computation method used in Linear Algebra for the 21st Century. Peter Olszewski (2021)[32]

> The [SVD] is sometimes called the *jewel in the crown of linear algebra*.
> Anthony Roberts (2020, p. vii)

In my department, we have two master's programs (both international, both taught in English), namely (what we now call) the "regular" master's program in finance, and, separately, the quantitative finance master's program (https://www.msfinance.uzh.ch/en). A second, "intermediate" course in linear algebra (formally still undergraduate material, whereby adjective "undergraduate" comes from usage when describing the academic progression of a student in a mathematics department) would be designed with those latter finance students in mind, but intended also for students in the broader faculty, such as those involved with optimization (within the business department), econometrics (in both the economics and finance departments), and/or computer science (all of these subjects being primary examples of where the concepts and methods from machine learning are making deep and permanent inroads).

While the list of topics for a second course is not as well defined as for a first course, most "books for a second course" aimed at mathematics students have quite some commonalities in terms of topics (e.g., Jordan canonical form, more emphasis on linear transformations and operators, SVD, Moore-Penrose inverse, the Perron–Frobenius theorem, etc.), and with the presentation having a notable increase in "mathematical maturity".

For students intending to understand and further develop machine learning methodology, it is necessary (but of course not sufficient) in such a course to have extensive coverage of the SVD. The above sequence of quotes should make this clear, notably how they moved over time from "large" to "plethora" to "colossal" when describing the number of applications. This fact is also brought out in four new books, namely Bisgard (2021), Roberts (2020), Bogacki (2019), and a forthcoming second edition from Meyer. For the latter, Meyer's already lengthy and detailed section on the SVD in his book *Matrix Algebra and Applied Linear Algebra*, 2010, has been made yet lengthier, and planned for the second edition of his book. This is not secret information—his new SVD chapter is already available, for free download, from his web page.

If a textbook is chosen that does not have much detail on the SVD, then I strongly suggest supplementing it with chapters from one or more of the above mentioned books, and also the article by Dan Kalman, 2002, *A Singularly Valuable Decomposition: The SVD of a Matrix*, available here, https://www-users.cse.umn.edu/~lerman/math5467/svd.pdf; or the 1996 published version, in The College Mathematics Journal, 27(1), pp. 2-23; http:

[32] This is from his book review on Anthony Roberts' (2020) *Linear Algebra for the 21st Century*, https://www.maa.org/press/maa-reviews/linear-algebra-for-the-21st-century, which I discuss in Section 3.4.

//dankalman.net/AUhome/pdffiles/svd.pdf. See also https://sites.math.washington.edu/~morrow/336_13/papers/reed.pdf, which notably contains some very useful Matlab code when working with graphics.

This being a second course, and quite possibly populated with students having completed their undergraduate studies from other universities (and who, obviously thus did not take the first course from your department, and whereby the first and second courses are, naturally, well-coordinated), it suggests itself to (forgive me if I am stating the obvious): (i) inform them well before the first lecture date that they should familiarize themselves with and/or review the appropriate first-course material; (ii) spend one, maybe two, lectures, reviewing the relevant material; and (iii) have an (earlier announced, not surprise) exam in the second week on first-course material, to help encourage and incentivize students to indeed adequately review it.

There are several options for a second undergraduate course in linear algebra. We begin in Section 4.1 by giving a list of relevant books for what is probably the most popular—traditionally at least—content for a second course, certainly for a mathematics department. The material taught in such a course is (obviously, by construction) mandatory for students intending to venture into further, deeper mathematics courses, but also essentially mandatory for students outside of pure mathematics, intending to apply the methodology. As mentioned above, my added two cents here is to insist that the SVD gets appropriate coverage—which as we will see below, is not the case for several otherwise excellent texts.

All the other subsections of this Section 4 are possible alternative options for a second course, and all of which would also be (and what I recommend) very suitable for a *third* course in linear algebra, now more focused on a specific direction, e.g., numerical analysis, statistical inference, differential equations, generalized inverses and the SVD (see Section 4.4), or a yet more sophisticated presentation of linear algebra for which real analysis is required (e.g., Bisgard's 2021 book *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*).

I now outline the remaining sections after 4.1. Section 4.2 gives a short list of "special books" that do not otherwise fit into my partition. Section 4.3 mentions several books that are explicitly designed to be used in a third course, in preparation for, or already at the level of, a beginning graduate-level study of linear algebra in a mathematics department. Section 4.4 lists books that are more specific in their coverage (and dedicated to topics such as generalized inverses, the Jordan form, projection matrices, and the SVD), and could be suitable either as the material for a speciality course on those topics, or serve as supplementary reading for a more mainstream, broad-coverage second course. Sections 4.5 and 4.6 are for numerical linear algebra and general numerical analysis, respectively, these being academic subjects clearly of great interest in and of themselves, but possibly of significant relevance to students (obviously wishing to specialize in this branch of applied mathematics, and those who are) specializing in this specific branch of computer science. Section 4.7 covers the major books in matrix algebra, this being notably distinct from general linear algebra, with concentration—obviously—on algebraic aspects of matrices. These tend to be oriented towards applications in statistics and econometrics, and these, in turn, should no doubt be of interest to (some, depending on their area of speciality) students of machine learning, not to mention students in statistics and econometrics proper. Section 4.8 details a niche of books covering the interface of linear algebra and linear statistical models. Its importance needs little motivation, given that linear regression is the near universal starting point of algorithms and methods of machine learning (as well as the starting point of econometrics). Finally, Sections 4.9 and 4.10 address the subjects of difference and differential equations, respectively.

4.1 List of Suggested Second-Course Linear Algebra Books

Analogous to Section 3.2 giving book recommendations for a first course in linear algebra, I provide here some options; sixteen at last count—far more than the reader was probably hoping to see, again owing to the fact that numerous excellent books are available at this intermediate level. I omit mention of some classic older books, such as Georgi Shilov (1961) (translation from Russian) *An Introduction to the Theory of Linear Spaces*; Paul R. Halmos (1958) *Finite-Dimensional Vector Spaces*; and F. R. Gantmacher (1977) (translation from Russian) *The Theory of Matrices*, 2 Volumes.

Abbreviation "LA-2" stands for: A second course in Linear Algebra.

LA-2 Book Choice 1: Carl Meyer, *Matrix Algebra and Applied Linear Algebra*, 2010. Namely, use the remainder of the book, what was not covered in the first course (see its entry in Section 3.2, where I also discuss and praise Meyer's book), which amounts to a further roughly 400 pages of material.

LA-2 Book Choice 2: Nathaniel Johnston, *Advanced Linear and Matrix Algebra*, 2021. As the second of two volumes on linear algebra, both intended for undergraduates, with the first having been mentioned and praised in Section 3.2, use of this book would clearly be highly desirable, notably, but not only, if the first book was used in a first course. As mentioned earlier regarding the first book, Johnston's second book is, similarly, optically fantastic, has excellent graphics, use of color in the text and graphics, contains many interesting topics, all seemingly very well done, as well as useful appendices to get the reader up to speed. Skimming in the second book makes me want to stop what I am doing and read the whole thing. Answers to most of the exercises are in the back of the book. If this book is not chosen as the primary one, it certainly should enter, and be high on, a supplementary reading list.

LA-2 Book Choice 3: Thomas Scott Blyth and Edmund Frederick Robertson, *Further Linear Algebra* (first edition), 2002, 230 pages. I can repeat for this book almost verbatim the first sentence directly above, in the context of Johnston's book. B&R, like Johnston, also give answers to the exercises in the book.

Excerpts from the MAA book review of Blyth and Robertson's second volume were given already for this book in Section 3.2. The second volume starts off with the unnumbered chapter entitled "The story so far", and, as anticipated, gives a 10 page review of the essential contents of the first book, allowing the reader to refresh, but also allowing the authors to conveniently refer to specific results. Jumping towards the end of the book, chapter 12 is called "... but who were they?", containing, as again anticipated, short biographies of the major players. This highly welcomed feature on history is of course not unique to their book, and Eves (1968) is a perfect case in point of "been there, done that", albeit with the historical content spread throughout the book.

Worth a passing mention, B&R's books of course cannot compete in terms of optical presentation with Johnston. (In fact, virtually no book can compete; though Saveliev gets the award for most graphics and use of color, in his *Linear Algebra Illustrated*, 2020.) The pdf file is, however, fully serviceable.

Based on my careful reading and strong impression of the first volume, and brief skimming of the second book, I expect (at least parts of) this book to be near the top of the short list of primary reading books for a second course. It has extensive coverage of Jordan-things (decomposition, form, basis) in chapter 5, while some parts of the book after that appear relatively advanced.

LA-2 Book Choice 4: James Bisgard *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*, 2021, 217 pages. Use of this book presupposes that the student has had an exposure to a first course in real analysis, which might be prohibitive for the target audience. On a positive note, as the title suggests, it emphasizes the SVD, this being one of our goals in setting up courses for students pursuing applications and machine learning. From the preface, we read:

> It is assumed that readers have had a standard course in linear algebra and are familiar with the ideas of vector spaces (over R), subspaces,bases, dimension, linear independence, matrices as linear transformations, rank of a linear transformation, and nullity of a linear transformation. We also assume that students are familiar with determinants, as well as eigenvalues and how to calculate them. Some familiarity with linear algebra software is useful, but not essential.
>
> In addition, it is assumed that readers have had a course in basic analysis.

As an example of the necessity of having had a first course in real analysis, on page 31, we are given the definitions of: "convergence, Cauchy sequences, completeness, balls, boundedness, open or closed, sequential compactness." As such, this book might present itself as a good choice for a *third* course in linear algebra, and after the student has also had a first course in real analysis (see Section 5).

LA-2 Book Choice 5: Hugo Woerdeman, *Advanced Linear Algebra*, 2016, 238 pages. The contents appear to me to be closer to intermediate level than advanced, despite the title. At just over 200 pages of not very dense writing, it is not so much material. Further appealing is that the author provides full solutions to the exercises in the book, and note that many of the exercises are algebraic in nature, as opposed to numeric plug and chug. I mention Woerdeman's book again (positively) in the discussion of the next book, by Weintraub. There is no MAA review of it.

LA-2 Book Choice 6: Steven Weintraub, *Linear Algebra for the Young Mathematician*, 2019. Crucial is the informative (and seemingly competent) MAA review `https://www.maa.org/press/maa-reviews/linear-algebra-for-the-young-mathematician`. It showers great praise on the author's chapter 8, on the Jordan canonical form, making me want to use the book just for that chapter. The rest of the review however, is not so favorable: The reviewer makes clear that this is absolutely not to be used as a first book in linear algebra, despite Weintraub saying that it "presupposes no prior knowledge of the subject". For example, the formula for the determinant of a 2×2 matrix never appears, nor does the statement $Av = \lambda v$ in the context of eigenvalues. The reviewer states "The chapter on determinants contains exactly two examples, both of which compute the determinants of 4×4 matrices". Precisely as was said in the MAA review about his 2011 book (discussed below), Weintraub's view of linear algebra is that of linear mappings, as opposed to matrix algebra. Thus, this book is (as the title suggests) more of interest to (budding) pure mathematicians than it is to applied researchers and numerical analysts. The reviewer concludes by saying (with JCF being Jordan canonical form):

> I enjoyed this book. It contains as clear an exposition of the JCF as I've seen anywhere, many applications which illustrate how transparent certain facts from calculus and differential equations are when viewed in the context of linear algebra, and is very well-written. I do not, however, have a clear sense of who its true audience is. As I believe my review makes clear, I do not believe that the book is appropriate for readers lacking a previous exposure to linear algebra. On the other hand, it is not quite appropriate for a second linear algebra course or a graduate course either, as it omits many of the topics that the instructor of such a course would want to mention. Because the book largely ignores the computational side of linear algebra and focuses almost exclusively on linear transformations between vector spaces rather than on matrices, I find it hard to believe that readers whose interests are not firmly in pure mathematicians will find much value here. Perhaps the best I can say is that this book is likely to be a useful reference for readers desiring a

> rigorous, well-written, vector space oriented treatment of the standard topics covered in a strong undergraduate linear algebra course.

I like Weintraub's (2019) notation (introduced on page 154) for the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{C}$, namely $[P]_{\mathcal{C}\leftarrow\mathcal{B}}$. Woerdeman (2016, p. 62), uses precisely the same notation, *and even the same letters*, which might induce us to assume that this was not by chance. Perhaps not, but note Weintraub's (2011) book, *A Guide to Advanced Linear Algebra*, uses (unsurprisingly) that same notation. In particular, on page 44 of Weintraub (2011), we are introduced to the notation for a linear map from V to W, such that V and W have bases $\mathcal{B}$ and $\mathcal{C}$, respectively; namely via matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$. (Subsequently on page 48 the author discusses his choice, mentioning other notations used, and how they can be confusing.) Thus, if there is any causality, it runs from Weintraub to Woerdeman, or so it seems just based on these books. Either way, perhaps there are further notational and conceptual links between these books, possibly suggesting their paired use.

As an aside, and as is perhaps obvious from the title, Weintraub's (2011) book, *A Guide to Advanced Linear Algebra*, is more advanced than Weintraub's (2019) *Linear Algebra for the Young Mathematician*. The latter "presupposes no prior knowledge of the subject", but both the above reviewer, and I, think it can only be used as a second course, and not as a first course. So, perhaps we can conclude that Weintraub is a bit biased towards understatement in his assessment of prerequisites—in which case, let's see what he says about his 2011 book. From the preface:

> We presume you are already familiar with elementary linear algebra, and that you know how to multiply matrices, solve linear systems, etc. We do not treat elementary material here, though in places we return to elementary material from a more advanced standpoint to show you what it really means. However, we do not presume you are already a mature mathematician, and in places we explain what (we feel) is the "right" way to understand the material. The author feels that one of the main duties of a teacher is to provide a viewpoint on the subject, and we take pains to do that here.

Both of his books are optically fantastic (and the available pdf files are perfect). There is even a joke on page 22 of the 2011 book. The coverage of the SVD in Weintraub (2011) is short and theoretical, as opposed to emphasizing its importance for applications (though not as short as that given in Lankham, Nachtergaele, and Schilling). The MAA review https://www.maa.org/press/maa-reviews/a-guide-to-advanced-linear-algebra is mixed, drawing attention to some shortcomings, such as no exercises, the chapter on Lie groups is not ideal, and too little emphasis on certain useful and common matrix results. Of the two camps for viewing linear algebra: (i) matrices and computation; or (ii) linear algebraic transformations, the author is clearly of the latter school of thought.

LA-2 Book Choice 7: Peter Petersen *Linear Algebra*, 2012, 387 pages. The book is published under Springer's UTM (Undergraduate Texts in Mathematics) series, giving a strong suggestion of the intended audience. It however would not be suitable for a first course, notably for students outside of a mathematics department. From the preface, we read:

> This book covers the aspects of linear algebra that are included in most advanced undergraduate texts. All the usual topics from complex vector spaces, complex inner products the spectral theorem for normal operators, dual spaces, quotient spaces, the minimal polynomial, the Jordan canonical form, and the Frobenius (or rational) canonical form are explained. A chapter on determinants has been included as the last chapter, but they are not used in the text as a whole. A different approach to linear algebra that does not use determinants can be found in [Axler].

> The expected prerequisites for this book would be a lower division course in matrix algebra. A good reference for this material is [Bretscher].
>
> In the context of other books on linear algebra it is my feeling that this text is about on a par in difficulty with books such as [Axler, Curtis, Halmos, Hoffman-Kunze, Lang]. If you want to consider more challenging texts, I would suggest looking at the graduate level books [Greub, Roman, Serre].
>
> This book has been used to teach a bridge course on linear algebra [and] its purpose was to ensure that incoming graduate students had really learned all of the linear algebra that we expect them to know when starting graduate school.

The books referred to as Axler, Bretscher, Curtis, Lang, and Roman, are (except for possibly being an older edition) the ones mentioned in this document. (Some of these authors have more than one book mentioned in this document, so, to be clear, Lang refers to *Linear Algebra*, 3rd edition, 1987; and Roman refers to *Advanced Linear Algebra*, 3rd edition, 2007, mentioned below in Section 4.3.) The author speaks of having used the book as a bridge course between undergraduate and beginning graduate studies, and based on the contents, this seems accurate.

Another "bridge course" book, similar in content and level to Petersen, is Shen, Wang, and Wojdylo (2019), discussed below, along with the reason why it did not get included in this primary list.

Usefully, Petersen also has a 57 page supplementary document; it is available on the author's web page `https://www.math.ucla.edu/~petersen/`.

LA-2 Book Choice 8: Peter J. Olver and Chehrzad Shakiban, *Applied Linear Algebra*, Second Edition, 2018. At over 630 dense A4 pages, there is a lot of material. Included are sections on fixed points, PCA and wavelets, DFT, FFT, as well as numerical computation of eigenvalues. From the preface, we read:

> This text is designed for three potential audiences:
>
> - A beginning, in-depth course covering the fundamentals of linear algebra and its applications for highly motivated and mathematically mature students.
> - A second undergraduate course in linear algebra, with an emphasis on those methods and concepts that are important in applications.
> - A beginning graduate-level course in linear mathematics for students in engineering, physical science, computer science, numerical analysis, statistics, and even mathematical biology, finance, economics, social sciences, and elsewhere, as well as master's students in applied mathematics.
>
> Although most students reading this book will have already encountered some basic linear algebra — matrices, vectors, systems of linear equations, basic solution techniques, etc. — the text makes no such assumptions.

The book is definitely not suited for a first course, as described in Section 3 above, but seems to have very high potential for use as a second course, given its coverage of the aforementioned (and more) topics.

Not wishing to step on anyone's turf, the preface also refers to Strang's books *Introduction to Applied Mathematics* (1986) and *Linear Algebra and Its Applications*, Third Edition (1988), saying:

> Our inspirational source was and continues to be the visionary texts of Gilbert Strang. Based on students' reactions, our goal has been to present a more linearly ordered and less ambitious development of the subject, while retaining the excitement and interconnectedness of theory and applications that is evident in Strang's works.

Subsequently, in the Acknowledgments, we read:

> First, let us express our profound gratitude to Gil Strang for his continued encouragement from the very beginning of this undertaking. Readers familiar with his groundbreaking texts and remarkable insight can readily find his influence throughout our book.

The web site for the book, http://www.math.umn.edu/~olver/ala2.html, lists (among other things such as errata) Matlab programs, making the usage of this book even more appealing.

LA-2 Book Choice 9: Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence, *Linear Algebra*. The 5th edition is 2019, and the associated "free PDF file" is a scan, which is rough on the eyes to read. However, the 4th edition, 2003, does not differ much from the 5th edition, and is available as a perfect pdf file.

From the preface of their fourth edition, we read: "Although the only formal prerequisite for this book is a one-year course in calculus, it requires the mathematical sophistication of typical junior and senior mathematics majors. This book is especially suited for a second course in linear algebra that emphasizes abstract vector spaces, although it can be used in a first course with a strong theoretical emphasis."

The book covers topics the basics, as well as the usual intermediate topics, e.g., SVD, generalized inverses, Jordan canonical form, etc., and in that sense is one large book suited for two courses, similar to Meyer, or the two books by Johnston, and Blyth and Robertson. My brief inspection suggests that the entire book is "intermediate", and thus well suited for a second course, but less suited to use for both a first and second course. A solutions manual for instructors (albeit not with all solutions) is available.

LA-2 Book Choice 10: Joerg Liesen and Volker Mehrmann, *Linear Algebra*, 2015. This is another intermediate linear algebra book, definitely not graduate level (it appears in the SUMS: Springer Undergraduate Mathematics Series), though it is more abstract and terse than the first-course introductory books, and covers more sophisticated material. The book shows Matlab codes for calculations, a nice plus I think, even if done to show actual numerics for some theoretical result.

LA-2 Book Choice 11: M. Thamban Nair and Arindama Singh, *Linear Algebra*, 2018. I would put this in the same equivalence class as Liesen & Mehrmann and several other books listed here. It is clearly more suited for mathematics majors as opposed to math-timid social science undergraduates. It covers topics such as the Jordan canonical form, Schur decomposition, etc., as well as quotient and dual spaces, Riesz representation theorem, etc.. In the preface, the authors discuss "Special Features" of the book, saying

> There are places where the approach has become non-conventional. For example, the rank theorem is proved even before elementary operations are introduced; the relation between ascent, geometric multiplicity, and algebraic multiplicity are derived in the main text, and information on the dimensions of generalized eigenspaces is used to construct the Jordan form. Instead of proving results on matrices, first a result of the linear transformation is proved, and then it is interpreted for matrices as a particular case.

As such, and keeping our intended audience in mind, it might be wise to use this interesting book in conjunction with another, more mainstream one.

Some excerpts from the MAA review
https://www.maa.org/press/maa-reviews/linear-algebra-2 include:

> Designed to be an undergraduate text, this book is not for the faint of heart—it is a book for mathematics, science, and engineering majors providing students with a rigorous approach to the subject.

> The proofs presented in this text can be considered non-conventional.
>
> [It] is a very well written book. It is clear that Nair and Singh put a lot of work into the text to make the concepts elaborate and lively at the same time.
>
> I would highly consider reviewing this book for a linear algebra course you would be teaching in a future semester.

LA-2 Book Choice 12: Sergei Treil, (hilariously named, and most surely as a joke on Axler) *Linear Algebra Done Wrong*. Crucially, the electronic version is legally free, under a Creative Commons License. From the author's description of the book (his emphasis):

> It [is] supposed to be a *first* linear algebra course for mathematically advanced students. It is intended for a student who, while not yet very familiar with abstract reasoning, is willing to study more rigorous mathematics that is presented in a "cookbook style" calculus type course.

My superficial inspection indicates it to be solidly intermediate (albeit covering basic, first-course notions), similar to Friedberg/Insel/Spence. Some random readings suggest it is very well written, with clear effort from the author to explain advanced topics (e.g., dual spaces).

LA-2 Book Choice 13: Helene Shapiro, *Linear Algebra and Matrices: Topics for a Second Course*, 2015. This is an attractive looking book, and the MAA review is very positive, https://www.maa.org/press/maa-reviews/linear-algebra-and-matrices-topics-for-a-second-course. Solutions to exercises are not provided (a feature the reviewer finds good...). The author says in the preface that the book is suited for a second course, though a brief skim indicates that the level of this book is considerably higher. From the text on Amazon, "This book is intended for those who are familiar with the linear algebra covered in a typical first course and are interested in learning more advanced results."

LA-2 Book Choice 14: Charu Aggarwal, *Linear Algebra and Optimization for Machine Learning*, 2020, about 480 pages. While the title, preface, and table of contents of Aggarwal, taken together, seems to imply "the perfect book" for what we are looking for (with more than enough material, and some more advanced topics covered), some limited skimming of it suggests that some topics are not explained as well as they should be. More time reading this book is required to solidify or refute this current prior.

LA-2 Book Choice 15: Stephan Ramon Garcia and Roger A. Horn, 2017, *A Second Course in Linear Algebra*. Author Roger Horn is known for his book with Charles Johnson, *Matrix Analysis*, 2nd edition. Thus, as might be expected, this book emphasizes material relevant for application and computation (as opposed to mathematical structure of linear transformations, more suited for algebraists). The MAA review https://www.maa.org/press/maa-reviews/matrix-analysis is overall very positive. We read:

> Many results from across the matrix analysis literature are available in this book, including many that are not readily accessible elsewhere. The text has a specialized feel to it and is best approached with a solid background in linear algebra.
>
> The book would be a stretch for many undergraduates. It is dense and demands a fair amount of mathematical maturity and persistence. There are very few routine exercises and the hints provided in an appendix are rather sketchy.

The review by Peter Shiu in *The Mathematical Gazette* (2018), 102(554) discusses the contents in some detail (including that the SVD is used, instead of the Moore-Penrose approach, for least squares). He mentions that the book "is a fascinating 'second course' in linear algebra for mathematics students, with some material being more suitable for graduate students.” He concludes with “This is an impressive book for a second course in linear algebra, and there is much to learn from the excellent exposition. It is also a good reference text on matrix theory”.

LA-2 Book Choice 16: Jean Gallier and Jocelyn Quaintance, 2020, *Linear Algebra and Optimization: Volume 1: Linear Algebra for Computer Vision, Robotics, and Machine Learning*. (There is online the pdf of presumably an earlier version, entitled *Fundamentals of Linear Algebra and Optimization*, 2015.)

This presumably sets the student up nicely to read their subsequent and somewhat advanced *Fundamentals of Optimization Theory: With Applications to Machine Learning*, 2020. A perfect pdf file of this book can be found on the web.

While both titles are very attractive for the needs of master's level students eager to get their hands dirty with the mathematics required to delve deep into the underpinnings of machine learning, this is definitely advanced undergraduate or even graduate level material, and so has an infimum of “intermediate level”.

4.2 Some Possible Supplementary Books

Unlike its counterpart Section 3.4, this supplementary list is very short, listing only four books, all of which are “special” and unique in some regard.

1. Marc Peter Deisenroth, A. Aldo Faisal, and Cheng Soon Ong, *Mathematics for Machine Learning*, 2020, about 355 pages.

   This is an extraordinarily attractive book, optically (font and appearance, but also including some nice color graphics), and, crucially, also with respect to content. It begins with three chapters on linear algebra, then single chapters on: vector calculus, probability and distributions, continuous optimization. Then part II, on machine learning, with chapters When Models Meet Data, Linear Regression, Dimensionality Reduction with PCA, Density Estimation with Gaussian Mixture Models, and Classification with SVM.

   This book obviously mixes topics, doing a little bit of each, which could be a blessing or a curse, depending on what the instructor wants. At this point, I would suggest using this book as secondary reading in a variety of courses, such as linear algebra, statistical inference, linear models, and something like “introduction to machine learning”.

   A solutions manual is available—I obtained it from the publisher.

2. Li-Yong Shen, Haohao Wang, and Jerzy Wojdylo, *Linear Algebra*, 2019, 165 pages.

   This book can serve as very good supplementary material, and it is somewhat unique in that it contains some computer code and output. It covers useful and typical material for a second course, though the presentation is relatively short, and thus cannot have the depth or breadth as other books.

   I claim it is also not appropriate as a primary course book because of the (estimated) number of typos. There is one already on page 1, and I amassed eight typos up to page 7. On page 20, the authors do a great thing, missing from most (see Section 3.1 for one exception, Robinson, 2006) of the set of books that I inspected closely, namely: for a linear system of two equations in two unknowns, using rudimentary manipulations, they show that the solution for x_1 and x_2, when they exist, are given precisely by (what we recognize to be) Cramer's rule (and which they of course prove later in the book). This is a great demonstration to help the reader answer to herself “I fully understand the proof of Cramer's rule, but how would you have thought (or how did Cramer think) of it?” The problem is, a column of numbers in one of the 2×2 matrix determinant expressions is wrong, no doubt from a copy-paste error. Impressively, Robinson (2006, p. 58) also has this—and gets it right.

The available pdf file is "perfect", and the book is optically fantastic. It appears the authors need a proofreader, and/or, given three authors, to cross-check each other's writing and not work so hastily.

From their preface:

> The prerequisite for this book is a standard first year undergraduate course in linear algebra.
>
> This book should be thought of as an introduction to more advanced texts and research topics. The novelty of this book, we hope, is that the material presented here is a unique combination of the essential theory of linear algebra and computational methods in a variety of applications.

We further read:

> There are many good introductory texts on linear algebra, and the intention of this book is to be a supplement to those texts, or to serve as a text for senior undergraduate students or first year graduate students, whose interests are computational mathematics, science, engineering, and computer science. The presentation of the material combines definitions and proofs with an emphasis on computational applications. We provide examples that illustrate the use of software packages such as Mathematica, Maple, and Sage.

(They also have some Matlab as well.)

3. Vasile Pop and Ovidiu Furdui, *Square Matrices of Order 2: Theory, Applications, and Problems*, 2017, 370 pages. The title and length of the book implies an entire treatise dedicated to results for 2×2 matrices. It is a large and attractive collection of definitions, propositions, remarks, properties, theorems, examples, corollaries, proofs, and lemmas, and, of course, scores of exercises and solutions, for 2×2 matrices. Impressively (or absurdly), the book is for *everyone*, as indicated in the preface, from which we read:

> The book is geared towards high school students who wish to study the basics of matrix theory, as well as undergraduate students who want to learn the first steps of linear algebra and matrix theory, a fundamental topic that is taught nowadays in all universities of the world.
>
> We also address this book to first- and second-year graduate students who wish to learn more about an application of a certain technique on matrices, to doctoral students who are preparing for their prelim exams in linear algebra as well as to anyone who is willing to explore the strategies in this book and savor a splendid collection of problems involving matrices of order 2.
>
> We also address this book to professionals and nonprofessionals who can find, in a single volume, everything one needs to know, from the ABCs to the most advanced topics of matrices of order 2 and their connection to mathematical analysis and linear algebra.
>
> The book is a must-have for students who take a linear algebra class and for those who prepare for mathematical competitions like Putnam, Seemous, and International Mathematical Competition for University Students.
>
> The book can be used by our colleagues who teach elementary matrix theory in high school, by instructors who teach linear algebra in college or university, and by those who prepare students for mathematical competitions.

4. Dan Stefanica, *A Linear Algebra Primer for Financial Engineering: Covariance Matrices, Eigenvectors, OLS, and more*, 2014, 340 pages.

   This could be appropriate for (my setting, namely) a finance department, though I have not looked at this book yet. There is a solutions manual book available, from the author, also for purchase. Note that Stefanica is the author of the 2011 book *A Primer For The Mathematics Of Financial Engineering*, second edition, which I do own, and can confirm is good. Essentially all reviews at Amazon are positive (except for one person, who received the wrong book and gave it one star), According to one such positive reviewer, "the book doesn't teach much Linear Algebra... what it does is show how to apply linear algebra to Financial Engineering... but you have to know Linear Algebra (at the college undergrad linear algebra level) to make much use of the book." With this in mind, Stefanica could serve as mandatory supplementary reading, accompanying a dedicated linear algebra textbook.

4.3 Some Books for Advanced Linear Algebra

Here I list some books that are at a somewhat- or blatantly-higher than intermediate level linear algebra. I refer to these as "Advanced Linear Algebra", and that is also the name of many of them. I am not competent in (or even familiar with) much of the material at this level, so my list serves primarily to simply collect the relevant books, and comment on them rather superficially, relying now very heavily on the MAA book reviews. In reverse chronological order, we have:

1. Michael E. Taylor, *Linear Algebra*, 2020. Taylor has several books, including a sequence of two books for univariate and multivariate analysis, a book on measure theory, a graduate level book on complex analysis, and a sequence of three books on partial differential equations. This new book, on linear algebra, is published by AMS (American Mathematical Society), in their Pure and Applied Undergraduate Texts (or "Sally") series, but the book, I believe, is not for typical undergraduates: It covers, for example, principle ideal domains (PIDs), a topic found only in graduate level (linear) algebra books. The review at MAA, https://www.maa.org/press/maa-reviews/linear-algebra-11 writes:

   > The book starts more or less from scratch with vector spaces over the real and complex numbers, and proceeds at a rapid clip to discuss (all in the first chapter) subspaces, linear independence and dependence and the existence of bases (in the finite-dimensional case only), linear transformations, determinants and row and column reduction. The fact that all of this is covered in about 45 pages should give some indication of the succinctness of Taylor's exposition. The next chapter discusses eigentheory, and again accomplishes a lot in a small number of pages: in about 20 pages, the chapter goes from the definitions of eigenvalues and eigenvectors to the Jordan Canonical Form.

   Taylor then covers some advanced topics found within abstract algebra or advanced linear algebra.

2. Isaiah Lankham, Bruno Nachtergaele, and Anne Schilling, *Linear Algebra as an Introduction to Abstract Mathematics*, 2016.

   This is another intermediate level linear algebra book, meaning, definitely more abstract and terse than those for a first course (for example, the terms Monomorphism, Epimorphism, Isomorphism, Endomorphism, and Automorphism appear in the book), but it is not graduate level or "advanced". It is short, at 190 pages (with relatively spread out text no less): The SVD gets one page.

   According to the (rather detailed) MAA review
   https://www.maa.org/press/maa-reviews/linear-algebra-as-an-introduction-to-abstract-mathematics,

   > [this] is a well-written book, but the succinctness of exposition and failure to discuss proof techniques would likely lead me to find a different text for Iowa State's proof-based linear algebra course. However, this book might fare better as a potential text for an honors course in linear algebra.

For such a short book, it is surprising to find an appendix with some historical information, e.g., the origins of the equal sign and the infinity symbol.

A free legal pdf file of the book is available from one of the author's web sites.

3. Nicholas Loehr, *Advanced Linear Algebra*, 2014. Unlike Lankham et al., this book is 590 pages of dense A4 pages! The (at the time of this writing) five reviews on Amazon are all the highest rating. Just as the aforementioned book by Lankham, Nachtergaele and Schilling is too short, this book is perhaps too long, and a brief inspection indicates that the writing is "mature", much of it intermediate level, but some clearly higher; and, obviously, the amount of material covered is substantially greater than many books.

4. Cooperstein, *Advanced Linear Algebra*, 2nd edition, 2015. This is not as optically appealing as Weintraub's 2011 similarly named book, *A Guide to Advanced Linear Algebra*, though it has the same theorem-proof style (and without the jokes). My casual inspection indicates that it is at about the same level of abstraction and difficulty. Note that Cooperstein was the source of the quotes opening Section 2, as well as the author of a very patient, detailed, colorful, and free introduction to linear algebra.

   There is an appendix with answers to selected exercises and hints to selected problems. From the preface, we read:

   > Though the approach is structural and general, which will be new to many students at this level, it is undertaken gradually, starting with familiar concepts and building slowly from simpler to deeper results. For example, the whole structure theory of a linear operator on a finite dimensional vector space is developed from a collection of some very simple results: mainly properties of the division of polynomials familiar to a sophisticated high school student as well as the fact that in a vector space of dimension n any sequence of more than n vectors is linearly dependent (the Exchange Theorem).
   >
   > The material you will find here is at the core of linear algebra and what a beginning graduate student would be expected to know when taking her first course in group or field theory or functional analysis.

5. Yisong Yang, *A Concise Text on Advanced Linear Algebra*, 2015. An MAA (not overly positive) review is here, https://www.maa.org/press/maa-reviews/a-concise-text-on-advanced-linear-algebra. Of arguably great value is that a solutions manual is available to the exercises in the book.

6. Titu Andreescu, *Essential Linear Algebra with Applications: A Problem-Solving Approach*, 2014, about 500 pages. My first impression is that this book is not as "inviting" as several others, though as an MAA review points out, https://www.maa.org/press/maa-reviews/essential-linear-algebra-with-applications,

   > Many of problems have the flavor of the Mathematics Olympiad, with solutions that require intelligent and surprising tricks. On the other hand, the author sometimes gives some applications of the given subjects to other areas of mathematics.
   >
   > ...
   >
   > I think the book is very suitable for students who are going to prepare for competitions and Olympiads. Moreover, the solved problems make it a very good problem book, with nice ideas, for anybody working with linear algebra.

7. Thomas W. Körner, *Vectors, Pure and Applied: A General Introduction to Linear Algebra*, 2013. The author writes with an occasional attempt to be, for lack of a better word, entertaining. For example, chapter 4 is entitled "The secret life of determinants". This book is far from an introduction, and I believe falls under the "advanced" category, being also more sophisticated than the intermediate books I list above. The MAA

review is positive, https://www.maa.org/press/maa-reviews/vectors-pure-and-applied-a-general-introduction-to-linear-algebra, writing "Körner as a writer is crisp, clear and witty. His books are a joy to read. This one is a solid introduction to linear algebra with plenty of material beyond that for enrichment and further study." Note the author also has a book on real analysis (see Section 5.4), as well as several other books.

8. Jonathan S. Golan, *The Linear Algebra a Beginning Graduate Student Ought to Know*, 3rd edition, 2012. Skimming the book indicates that (it is very optically appealing, with numerous pictures of mathematicians and historical commentary, and) there are *many* exercises, page upon page. According to the preface, some are challenging. Seemingly there are no solutions available.

9. Peter D. Lax, *Linear Algebra and Its Applications*, 2nd edition, 2007. The pdf (or djvu) file available for free is optically very poor, detracting from an already terse, difficult book (though not as terse as Katznelson and Katznelson; see below). Given the apparent popularity of the book, and the high status of the author, we take a look at a review: From one in the MAA at https://www.maa.org/publications/maa-reviews/linear-algebra-and-its-applications-0, we read very positively:

   > This book is meant as a text for a serious second course on Linear Algebra. The intended audience is beginning graduate students and advanced undergraduate students. The presentation, by one of the master's of the subject and Abel Prize winner in 2005, is clear and concise, but not dry. The book starts with the basics and quickly moves on to topics such as matrix analysis, convexity, duality, normed linear spaces, and numerical solution of linear systems and eigenvalue problems. Sixteen short appendices discuss interesting special topics and complement in detail the eighteen main chapters, making the book suitable as a reference too.
   >
   > In all, an informative and useful book, distinguished by its blend of theory and applications, which fulfills its goals admirably.

10. Yitzhak Katznelson and Yonatan R. Katznelson, *A (Terse) Introduction to Linear Algebra*, 2008. At least the authors were honest in the title. This book is far too terse.

11. Steven Roman, *Advanced Linear Algebra*, 3rd edition, 2007, published as a Springer Graduate Texts in Mathematics. The several reviews on Amazon are extremely strong, though it must be noted that this is definitely a graduate level mathematics book, and its title is very appropriate.

12. Harry Dym, *Linear Algebra in Action*, 2006. (2nd Edition, 2013, but pdf not available). This book starts reasonably modestly, but then becomes "graduate level" after about the first third. It contains a large number of useful results, e.g., the Courant-Fischer theorem. There is a very positive review by Partington, 2008, Bulletin of the London Mathematical Society, 40, pp. 721-24; see https://booksc.org/book/43422328/d777a0.

4.4 Specific Topics from a Second Course in Linear Algebra

There is a set of books with emphasis on a set of specific topics in intermediate linear algebra. These might be consulted for secondary reading in a second course in linear algebra, or possibly used as the primary text. Such books include:

1. Adi Ben-Israel and Thomas N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd edition, 2003. From the preface of the first edition in 1974, we read:

   > This book is intended to provide a survey of generalized inverses from a unified point of view, illustrating the theory with applications in many areas. It

> contains more than 450 exercises at different levels of difficulty, many of which are solved in detail. This feature makes it suitable either for reference and self–study or for use as a classroom text. It can be used profitably by graduate students or advanced undergraduates, only an elementary knowledge of linear algebra being assumed.

2. Robert Piziak and P. L. Odell, *Matrix Theory: From Generalized Inverses to Jordan Form*, 2007. From the preface,

> This text is designed for a second course in matrix theory and linear algebra accessible to advanced undergraduates and beginning graduate students. Many concepts from an introductory linear algebra class are revisited and pursued to a deeper level. Also, material designed to prepare the student to read more advanced treatises and journals in this area is developed. A key feature of the book is the idea of "generalized inverse" of a matrix, especially the Moore-Penrose inverse. The concept of "full rank factorization" is used repeatedly throughout the book. The approach is always "constructive" in the mathematician's sense.
>
> The important ideas needed to prepare the reader to tackle the literature in matrix theory included in this book are the Henderson and Searle formulas, Schur complements, the Sherman-Morrison-Woodbury formula, the LU factorization, the adjugate, the characteristic and minimal polynomial, the Frame algorithm and the Cayley-Hamilton theorem, Sylvester's rank formula, the fundamental subspaces of a matrix, direct sums and idempotents, index and the Core-Nilpotent factorization, nilpotent matrices, Hermite echelon form, full rank factorization, the Moore-Penrose inverse and other generalized inverses, norms, inner products and the QR factorization, orthogonal projections, the spectral theorem, Schur's triangularization theorem, the singular value decomposition, Jordan canonical form, Smith normal form, and tensor products.

3. Haruo Yanai, Kei Takeuchi, and Yoshio Takane, *Projection matrices, Generalized Inverse Matrices, and Singular Value Decomposition*, 2011. From the preface,

> Some of the topics in this book may already have been treated by existing textbooks in linear algebra, but many others have been developed only recently, and we believe that the book will be useful for many researchers, practitioners, and students in applied mathematics, statistics, engineering, behaviormetrics, and other fields.
>
> This book requires some basic knowledge of linear algebra, a summary of which is provided in Chapter 1. This, together with some determination on the part of the reader, should be sufficient to understand the rest of the book. The book should also serve as a useful reference on projectors, generalized inverses, and SVD.

4.5 Numerical Linear Algebra

> The field of numerical linear algebra is more beautiful, and more fundamental, than its rather dull name may suggest. More beautiful, because it is full of powerful ideas that are quite unlike those normally emphasized in a linear algebra course in a mathematics department. (At the end of the semester, students invariably comment that there is more to this subject than they ever imagined.) More fundamental, because, thanks to a trick of history, "numerical" linear algebra is really applied linear algebra. It is here that one finds the essential ideas that every mathematical scientist needs to work effectively with vectors and matrices. In fact, our subject is more than just vectors and matrices, for virtually everything we do carries over to functions and operators. Numerical linear algebra is really functional analysis, but with the emphasis always on practical algorithmic ideas rather than mathematical technicalities. Trefethen and Bau (1997, p. ix)

An entire other category of "post first course in LA" is one that is oriented more towards numerics and algorithms. We begin with—what is now—an older book, though highly praised and worth knowing about, and the source of the previous quote. After that comes a list of more recent books, given in reverse chronological order.

1. *Numerical Linear Algebra*, Lloyd N. Trefethen and David Bau, III, 1997, 361 pages. The above quote indicates a certain "excitement" from the authors, and indeed, the book has received numerous glowing reviews, from numerous purchasers on Amazon, and academic book reviewers. An example of the latter is the detailed and informative SIAM Review, 40(3), 1998, pp. 735-739, by Ricardo D. Fierro, who, like several Amazon reviewers, emphasizes how "each lecture in the textbook is pleasantly written in a conversational style." He also writes that the book "seems to be intended for students who have already completed an undergraduate course in numerical linear algebra." This agrees with what we read in the book's preface, namely that the "alumni of this course, now numbering in the hundreds, have been graduate students in all fields of engineering and the physical sciences", from MIT and Cornell.

   Matlab is used throughout the book. The pdf file of the book is "good enough" to clearly read everything, though it is not as optically brilliant as the pdf of the newer books.

2. *Numerical Linear Algebra and Matrix Factorizations*, Tom Lyche, 2020, 362 pages. See the MAA review, `https://www.maa.org/press/maa-reviews/numerical-linear-algebra-and-matrix-factorizations`.
   For example, we read "The author's approach to numerical linear algebra has a strong theoretical flavor. Essentially all the results have detailed proofs. The book demands a good deal of mathematical maturity from the students and a broader awareness of topics that the stated background does not include."

   The author uses Matlab and shows code throughout. There is a separate book available, *Exercises in Numerical Linear Algebra and Matrix Factorizations*, 284 pages, with all the solutions to exercises, making the package quite attractive. From the MAA review of this solutions manual book, we read:

   > As the book progresses to more advanced topics, the exercises continue to include a range of exercises from very straightforward to quite challenging.
   >
   > More than 200 exercises are presented. Some of them use MATLAB code for solutions (with versions in Python provided online). The quality of solutions is very high. The authors take pains to write the solutions in some detail and they serve as effective teaching tools.

3. *Linear Algebra for Computational Sciences and Engineering*, Ferrante Neri, 2nd edition, 2019, abour 560 pages. From the description on Amazon,

> This book presents the main concepts of linear algebra from the viewpoint of applied scientists such as computer scientists and engineers, without compromising on mathematical rigor. Based on the idea that computational scientists and engineers need, in both research and professional life, an understanding of theoretical concepts of mathematics in order to be able to propose research advances and innovative solutions, every concept is thoroughly introduced and is accompanied by its informal interpretation. Furthermore, most of the theorems included are first rigorously proved and then shown in practice by a numerical example. When appropriate, topics are presented also by means of pseudocodes, thus highlighting the computer implementation of algebraic theory.

Solutions to exercises are provided at the end of the book.

4. *Numerical Linear Algebra: A Concise Introduction with MATLAB and Julia*, Folkmar Bornemann, 2018, translated from German. This is a much more modest project, at about 150 pages, and intended for undergraduates. Matlab code is embedded throughout the book, a feature that I have in my own books, and thus admire. There is an appendix on Matlab, but, more interestingly, and as alluded to in the title, one also on (the extremely promising newcomer) Julia, and translations of programs, though Julia's syntax is purposely very similar to Matlab.

5. *Numerical Linear Algebra: An Introduction*, Holger Wendland, 2018, about 395 pages. With publisher CUP, one expects a strong book, and this indeed seems the case. From the preface,

   > All algorithms are stated in a clean pseudo-code; no programming language is preferred. This, I hope, allows readers to use the programming language of their choice and hence yields the greatest flexibility. Finally, NLA is obviously closely related to Linear Algebra. However, this is not a book on Linear Algebra, and I expect readers to have a solid knowledge of Linear Algebra. Though I will review some of the material, particularly to introduce the notation, terms like linear space, linear mapping, determinant etc. should be well-known.

   From the MAA review https://www.maa.org/press/maa-reviews/numerical-linear-algebra-an-introduction,

   > Wendland's book provides the reader with rigorous and clean proofs throughout the text. There are a lot of new concepts being presented that can spark the interest of a student who wishes to take Numerical Linear Algebra and can also serve as an excellent resource for an independent study.

   A course based on this book (or any in this section in fact) would have to come after a second (but not more) course on linear algebra.

6. *Numerical Linear Algebra: Theory and Applications*, Larisa Beilina, Evgenii Karchevskii, and Mikhail Karchevskii, 2017, about 445 pages. The first 220 or so pages are really on intermediate linear algebra, while the rest is on numerics. From the preface,

   > This book is suitable for use as course material in a one- or two-semester course on numerical linear algebra, matrix computations, or large sparse matrices at the advanced undergraduate or graduate level. We recommend using the material of Chapters 1–7 for courses in the theoretical aspects of linear algebra...
   >
   > Compared with other books on the same subject, this book presents a combination of extended material on the rigorous theory of linear algebra together

> with numerical aspects and implementation of algorithms of linear algebra in MATLAB.

Matlab programs for the book are available on its webpage.

7. *Numerical Linear Algebra with Applications: Using Matlab*, William Ford, 2015, 602 pages. As the title suggests, Matlab is used extensively throughout. The print on each page is very small, meaning (i) it is annoying to read, even on a tablet; and (ii) there is a lot of material in this book, including about 120 pages of review of basic linear algebra, in order for the book to be self-contained. From the preface,

   > This book is novel, in that there is no assumption the student has had a course in linear algebra. Engineering students who have completed the usual mathematics sequence, including ordinary differential equations, are well prepared. The prerequisites for a computer science student should include at least two semesters of calculus and a course in discrete mathematics. Chapters 1-6 supply an introduction to the the basics of linear algebra. A thorough knowledge of these chapters prepares the student very well for the remainder of the book. If the student has had a course in applied or theoretical linear algebra, these chapters can be used for a quick review.
   >
   > Throughout the book, proofs are provided for most of the major results. In proofs, the author has made an effort to be clear, to the point of including more detail than normally provided in similar books.
   >
   > ⋮
   >
   > The book covers many of the most important topics in numerical linear algebra, but is not intended to be encyclopedic. However, there are many references to material not covered in the book. Also, it is the author's hope that the material is more accessible as a first course than existing books, and that the first six chapters provide material sufficient for the book to be used without a previous course in applied linear algebra. The book is also is very useful for self-study and can serve as a reference for engineers and scientists. It can also serve as an entry point to more advanced books, such as James Demmel's book or the exhaustive presentation of the topic by Golub and Van Loan.

   The MAA review `https://www.maa.org/press/maa-reviews/numerical-linear-algebra-with-applications`, is overall positive, but remarks that it appears written more for engineers than for mathematicians.

8. *Computational Methods of Linear Algebra*, Granville Sewell, 3rd edition, 2014. The chapters are: Systems of Linear Equations, Linear Least Squares Problems, The Eigenvalue Problem, Linear Programming, The Fast Fourier Transform, and Linear Algebra on Supercomputers. Matlab (and Fortran) are used. An MAA review of the second edition, from 2005, is very positive, `https://www.maa.org/press/maa-reviews/computational-methods-of-linear-algebra-0`.

4.6 Numeric Analysis (General, Including Linear Algebra)

This category is obviously related to the previous one, and just sets the scope a bit higher to include other topics in numerical analysis, e.g., root finding, numerical integration, solving differential equations, possibly the FFT, etc., as well as an obligatory (and important) section or chapter on rounding errors and finite precision floating point arithmetic in the computer. All these books also cover fundamental topics associated with numerical linear algebra, e.g., systems of linear equations, eigenvalues and vectors, and least squares type problems.

The set of such topics for this niche of books is reasonably well defined, as seen by noting the large amount of commonality (from the table of contents) of many books pitched at students (and instructors) in this area. In particular, they are aimed at undergraduates who have had basic linear algebra and calculus (and some computer programming).

We distinguish between a course on numerical analysis as just described, and one aimed at a mathematically more sophisticated audience, namely students having had at least a first course in real analysis, and ideally exposure also to multivariate real analysis and normed linear spaces. Books appropriate for such a "next course" in numerical analysis are, as expected and desired, much more of the theorem-proof type, with several of which mentioned below in Section 8.1. We continue now with a selection of books suitable for the former grouping, given alphabetically.

1. *A Friendly Introduction to Numerical Analysis*, Brian Bradie, 2006, 984 pages. The introductory chapter provides a very gentle and useful discussion of the basics of, and important aspects associated with, coding an algorithm (using simple examples, such as Newton's Method for computing a square root), and also the nature of floating point arithmetic.

   No MAA review available. The pdf file is readable enough to use, but is probably a photocopy, and is not markable or searchable.

2. *Numerical Analysis*, Richard Burden and J. Douglas Faires (1941–2012), Ninth Edition, 2011, 863 pages. The MAA review https://www.maa.org/press/maa-reviews/numerical-analysis-1 is very positive, also mentioning the prerequisite math knowledge required:

   > It assumes no background beyond a good first course in calculus. Some familiarity with differential equations and linear algebra would be helpful, but the authors provide adequate introductory material in those areas. The book has sufficient material for two or three courses over a full year — actually even more than that...
   >
   > Several changes have been made to this ninth edition. Perhaps the most significant one has been an extensive expansion of the treatment of numerical linear algebra. The authors have added a section on the Singular Value Decomposition (SVD), and rewritten pieces of earlier chapters to include necessary material on symmetric and orthogonal matrices. They have also incorporated many new examples and exercises. This is an important and valuable change since the SVD is now much more widely used across a variety of disciplines. It is, however, a bit of a concern for students who use this book without the benefit of an earlier linear algebra course. Without a somewhat deeper understanding of linear algebra, the SVD (and even the simpler matrix factorizations the authors discuss) can seem like sophisticated numerical magic. The problem isn't the lack of proofs so much as the absence of an intuitive foundation.
   >
   > There is much to admire about this book. It is so comprehensive that it can serve both as a good introduction and as a reference for many topics. There is almost nothing in numerical analysis that it doesn't cover.
   >
   > Overall, this is a polished and well-tested textbook. It has an appealing look and feel.

   The book is optically very nice, and the available pdf is perfect.

3. *Numerical Mathematics and Computing*, Ward Cheney and David Kincaid, sixth edition, 2008, 784 pages. I remember using this book (2nd edition) from my undergraduate days. These authors also have a second book on the topic pitched at a mathematically more sophisticated audience, which we list in Section 8.1. The MAA review is very positive, also giving details on the contents of each chapter: https://www.maa.org/press/maa-reviews/numerical-mathematics-and-computing.

> Cheney and Kincaid's Numerical Mathematics and Computing, Sixth Edition, is an excellent textbook for those students who are minimally comfortable with Calculus, basic linear algebra, and a computer programming language. The authors provide a very good review of linear algebra concepts in the appendix. For those who have forgotten their linear algebra, the appendix is a terrific reference. This text can certainly be used for self-study. For an instructor of a course on numerical analysis this is an ideal book to use since the material is self-contained.
>
> The text's exposition is very friendly to a wide variety of learning styles. There is an abundance of numerical examples and supporting pseudocode for those who learn by example. Each section is replete with problems of all levels of difficulty for those who like to simply tackle problems with only a cursory reading of the section. Finally, all concepts are explained in a rigorous manner with theorems and proofs given as necessary.
>
> ...
>
> The last few chapters in this text are meant to be surface-scratching overviews only of very deep and well-researched topics. A student wishing to take more advanced courses on the topics covered in Chapters 13, 15, 16, and 17, can certainly use this text as strong introductory preparation. Certainly the first 11 chapters would be sufficient for a standard one-semester course.
>
> All in all, [it] is a very well written and very well thought out text.

The book is optically excellent, and the pdf file appears perfect (also such that clicking on an entry in the table of contents brings you to that section), but text (outside of the preface) cannot be copied.

4. *An Introduction to Numerical Methods and Analysis*, James F. Epperson, 2nd edition, 2013, 614 pages. This 2nd edition now uses Matlab throughout. The author states in the preface that "a large number of" typos have been corrected from the first edition, and instead of adding new material, the exposition and presentation has been improved.

   From the preface,

   > This book is intended for introductory and advanced courses in numerical methods and numerical analysis, for students majoring in mathematics, sciences, and engineering. The book is appropriate for both single-term survey courses or year-long sequences, where students have a basic understanding of at least single-variable calculus and a programming language. (The usual first courses in linear algebra and differential equations are required for the last four chapters.)
   >
   > To provide maximum teaching flexibility, each chapter and each section begins with the basic, elementary material and gradually builds up to the more advanced material. This same approach is followed with the underlying theory of the methods. Accordingly, one can use the text for a "methods" course that eschews mathematical analysis, simply by not covering the sections that focus on the theoretical material.
   >
   > ...
   >
   > The style of exposition is intended to be more lively and "student friendly" than the average mathematics text. This does not mean that there are no theorems stated and proved correctly, but it does mean that the text is not slavish about it. There is a reason for this: The book is meant to be read by the students.

   It is enjoyable to see the interesting Section 1.7, "A brief history of computing". There is no MAA review. To help assess and compare the books, I read the sections on Newton's method for univariate root solving. The presentation in this book is particularly good, with (arguably superfluous) discussions of how to use Newton's method to conduct division —which was not possible on older computers, with only multiplication (and addition and subtraction) allowed; as well as a detailed analysis (including starting values) for computing square roots.

The pdf file is easily good enough to use, and such that the text is markable (but not the preface) and searchable.

5. *A Concise Introduction to Numerical Analysis*, Anita C. Faul, 2016, 292 pages. True to its title, this book is noticeably much shorter than some others. Matlab is used throughout, and the programs are optically well done, e.g., indented, commented, and Matlab keywords are in bold (achievable, also in color, with packages available for LaTeX). There is no MAA review. From the (short) preface,

   > [The] book makes the difficult balance between being mathematically comprehensive, but also not overwhelming with mathematical detail. In some places where further detail was felt to be out of scope of this book, the reader is referred to further reading.

   The web site mentioned in the preface to access solutions to the odd numbered exercises is no longer useful, but the material can still be found, namely at: `https://www.routledge.com/A-Concise-Introduction-to-Numerical-Analysis/Faul/p/book/9780367658564` (and then clicking on: Support Material).

   The author does not explicitly state the prerequisites, but it is clear from the table of contents giving the topics of chapter 2 that a first course in linear algebra is essential, with a second course desirable. The conciseness of the text and the level of mathematics used in the book indicates that the student would be advised to have had a first course in real analysis. The pdf file is perfect.

   With the other books in mind, this book might best be recommended to serve as supplementary reading. It appears about "midway" in terms of mathematical sophistication between a first course, as discussed here, and a second course, invoking a higher level of mathematics, as discussed in Section 8.1.

6. *Numerical Analysis: An Introduction*, Timo Heister, Leo G. Rebholz, and Fei Xue, 2019, 193 pages. This book is far shorter than its competitors. From the preface:

   > The major topics for a first course in scientific computing are covered, and we emphasize fundamental ideas, simple codes, and mathematical proofs. We do not get bogged down with low level details of the fifty different methods there are to solve a particular problem. Instead,we try to give an undergraduate math major an idea of what scientific computing and numerical analysis is all about, along with exposure to the fundamental ideas of the proofs.
   >
   > We assume a knowledge of calculus, differential equations, and linear algebra, and also that students have some programming experience...

   The authors use Matlab throughout the book. The pdf file is perfect.

7. *Explorations in Numerical Analysis*, James V. Lambers and Amber C. Sumner, 2018, 674 pages. The authors use Matlab. However, there is a 2021 "Python Edition". The MAA review `https://www.maa.org/press/maa-reviews/explorations-in-numerical-analysis` is very positive. We read for example:

   > This appealing introduction to numerical analysis is unusual in the sense that it is truly introductory. It begins with a lengthy chapter that tells students what numerical analysis is about, what its major components are, and why

> understanding error is so important. The book is designed to support a two-semester course, but it is structured to be able to support alternative courses.
>
> The authors do not explicitly identify prerequisites, but they effectively assume competence with multidimensional calculus and basic linear algebra, as well as acquaintance with differential equations. No programming background is expected, but the authors strongly promote a hands-on approach. They provide an extensive introduction to MATLAB and expect students to use it throughout.
>
> All the usual topics of basic numerical analysis are considered here, but the treatment is different in several respects. The chapter called "Understanding Error" is unusually detailed. It goes from questions of error analysis, sources of error, and error measurement to discussions of conditioning, stability, and convergence. This is followed by a careful discussion of the computer arithmetic. The presentation is clear and carefully paced. There is a danger in putting a large amount of material like this early in the book before students have been exposed to any substantial numerical algorithms, but the authors make it work. Good examples and exercises help.
>
> Numerical linear algebra is handled particularly well...
>
> The authors pace the exposition very carefully; they begin with direct solutions of linear equations, focus first on diagonal systems, and gradually work from there. This is characteristic of the book, and maybe a bit of a concern for strong students who may find the pace too slow in some sections. Nonetheless, there is plenty of material to challenge even strong students.
>
> The book has mostly standard treatments of nonlinear equations, optimization and differential equations. The chapters on polynomial approximation and approximation of functions are more elaborate and inclusive than in comparable texts.
>
> This book is definitely worth considering for anyone looking for a good introductory text.

A perfect pdf version dated 2015 is available on the web.

8. *Numerical Analysis*, Timothy Sauer, Third Edition, 2018, 655 pages. From the preface, "elementary calculus and matrix algebra" are the prerequisites. The MAA review for the second edition (2012, 646 pages), `https://www.maa.org/press/maa-reviews/numerical-analysis-0`, is positive, writing

> This is a very clearly-written modern introduction to many areas of numerical analysis. It provides a good balance of power and simplicity, with enough detail that the student can do something useful with what he learns, but not complicated enough to be overwhelming for lower-level undergraduates. The prerequisites are not stated, but the book assumes some familiarity with calculus and with matrices.
>
> [The] book has substantial chapters on the newer fields of random numbers and Monte Carlo methods, Fast Fourier Transform, and data compression. The book attempts to make the subject a harmonious whole rather than a bag of tricks by including frequent sidebars explaining the connection to the five key concepts of numerical analysis: convergence, complexity, conditioning, compression, and orthogonality.
>
> Many theorems are stated but few are proven, so this is a technique book rather than a proof book, but with very good explanations.

Matlab is used throughout. The pdf file is perfect, and the document is particularly attractive in terms of format and layout.

4.7 Matrix Algebra

Another category of "post first course in LA" is one that is oriented towards matrix algebra, such as Eves (1968). Books associated with this topic often are geared for graduate students in statistics. Given our underlying interest in machine learning (notably, but not only, in finance), this focus on higher statistical inference seems appropriate. See also the next entry, on linear algebra and linear statistical models.

As mentioned, the material is often with a view towards statistics, with some, such as Gentle, also addressing computation and numerics. Some of these books review basic linear algebra concepts. Some of the books (Magnus and Neudecker; Gentle, Harville, Horn and Johnson, Schott) are very popular, and "must haves" for the shelf of a researcher in linear statistical models. They are listed in chronological order.

1. *Matrices with Applications in Statistics*, Second Edition, Franklin A. Graybill, 1983. I used this book as a graduate student (at CSU, where Graybill had just retired). It is more of a reference book, but very extensive in coverage.

2. *Matrices for Statistics*, Second Edition, M. J. R. Healy, 2000.

3. *Matrix Algebra from a Statistician's Perspective*, David A. Harville, 2008. The 2008 version is the paperback version of the 1997 original, corrects typos, and "the typography has been improved". Harville has an accompanying entire (large) book on the exercises and solutions.

4. *Matrix theory: Basic Results and Techniques*, Second Edition, Fuzhen Zhang, 2011. The book review by Tin-Yau Tam (2012), Linear Algebra and its Applications, 437(6), positively reflects on some choices of proof used by the author. We further read:

   > The book is for a second Linear Algebra course that is appealing to audiences who are interested in statistics, physics, computer sciences and engineering as well as mathematics. The prerequisites for reading the book are elementary linear algebra and some calculus. People working in linear algebra and its applications will find the book enjoyable and useful.
   >
   > Technique and elegance are often in the author's mind.
   >
   > If the reader is a disciple of "learning mathematics by doing mathematics", then he or she will find the more than a thousand well-selected problems useful. Some problems are challenging.

   The book indeed is filled with exercises, but does not contain solutions, unlike his book *Linear Algebra: Challenging Problems for Students*, 2009, mentioned in Section 3.3. However, a highly detailed, 325 page solutions manual (and errata list) is available from the author (which he kindly sent to me), making this book a yet further attractive choice, one that I certainly plan on reading and learning from.

5. *Matrix Analysis*, Second Edition, Roger A. Horn and Charles R. Johnson, 2013. The MAA review https://www.maa.org/press/maa-reviews/matrix-analysis states:

   > There is, for example, no discussion of the calculus of matrices or of matrix functions like trace or determinant. Also, the matrix exponential is not discussed. Instead, this work stays much closer to the algebraic side. The major topics are eigenvalues, canonical forms, matrix factorizations, special matrices (Hermitian, positive definite or semidefinite, positive and negative), similarity, vector and matrix norms, and unitary equivalence. Many results from across the matrix analysis literature are available in this book, including many that are not readily accessible elsewhere. The text has a specialized feel to it and is best approached with a solid background in linear algebra.

> The book would be a stretch for many undergraduates. It is dense and demands a fair amount of mathematical maturity and persistence. There are very few routine exercises and the hints provided in an appendix are rather sketchy.

There is an errata list available.

6. *Generalized Vectorization, Cross-Products, and Matrix Calculus*, Darrell A. Turkington, 2013.

7. *Matrix Theory*, Xingzhi Zhan, 2013. A very positive MAA review is https://www.maa.org/press/maa-reviews/matrix-theory.

8. *Matrix Analysis for Statistics*, Third Edition, James R. Schott, 2016.

9. *Matrix Algebra: Theory, Computations and Applications in Statistics*, Second Edition, James E. Gentle, 2017. Quoting, out of amusement, from the preface, "Python is currently one of the coolest languages, but I personally don't like the language for most of the stuff I do." The MAA review https://www.maa.org/press/maa-reviews/matrix-algebra-theory-computations-and-applications-in-statistics is positive, and discusses the contents. We read:

   > This is a hard book; there is no other way I can put it. Gentle has put in a lot of time and effort to writing this book with careful attention to details. It is over 600 pages long, so at first I though there would be a lot of unnecessary elements. But it is all needed to make sure the student has a firm and solid understanding of matrix algebra on the graduate level. I would recommend this book for all those who teach graduate level matrix algebra or, if you dare, to those undergraduate students who wish to have an independent study.

10. *Matrix Differential Calculus with Applications in Statistics and Econometrics*, 3rd edition, Jan Magnus and Heinz Neudecker, 2019.

4.8 Linear Algebra and Linear Statistical Models

Here is yet another category of "post first course in LA", namely one that is oriented towards *both* linear algebra *and* linear statistical models. That means, the first few chapters are dedicated to (intermediate topics in) linear algebra, and then the focus is moved to linear models in statistics, e.g., multivariate normality, the distribution of quadratic forms, etc.. Clearly, a further prerequisite is basic probability theory, and also ideally a first exposure to statistical inference.

1. *Linear Algebra and Linear Models*, Third Edition, by R. B. Bapat, 2012, 175 pages (first and second editions having been 1993 and 2000, respectively). From the preface *of the first edition*,

   > The main purpose of the present monograph is to provide a rigorous introduction to the basic aspects of the theory of linear estimation and hypothesis testing. The necessary prerequisites in matrices, multivariate normal distribution, and distribution of quadratic forms are developed along the way. The monograph is primarily aimed at advanced undergraduate and first-year master's students taking courses in linear algebra, linear models, multivariate analysis, and design of experiments.
   >
   > The list of facts in matrix theory that are elementary, elegant, but not covered here is almost endless.
   >
   > We put a great deal of emphasis on the generalized inverse and its applications.

The statement "The list of facts in matrix theory that are elementary, elegant, but not covered here is almost endless" is rather interesting, making clear the importance of having had a previous course in *matrix* algebra (as opposed to, but of course in addition to, *linear* algebra, the latter being more the study of linear mappings). The relevance of generalized inverses is also mentioned, thus emphasizing the importance of covering it in, say, a second course in linear algebra.

From the preface *of the third edition*, we learn that the book now in fact covers more of the matrix algebra required:

> In this edition the material has been completely reorganized. The linear algebra part is dealt with in the first six chapters. These chapters constitute a first course in linear algebra, suitable for statistics students, or for those looking for a matrix approach to linear algebra.

While I disagree that the first six chapters would be appropriate for a typical first course in linear algebra, I do agree it is a great way to review (and go a bit beyond) some concepts already addressed in a first course, and also see more of a matrix approach, as is relevant for the subsequent chapters. Chapters 7 to 9 have considerable overlap with the numerous textbooks available for statistical analysis of linear models, including chapters 1 to 3 in my own book, Paolella, *Linear Models and Time-Series Analysis: Regression, ANOVA, ARMA and GARCH*, 2019, 880 pages. I could easily imagine using this book in conjunction with the first three chapters of my book for a course on "Linear Algebra and Linear Statistical Models".

There is also a 22 page section on hints and solutions to exercises, helping to make the book suitable for self study.

2. *Linear Algebra and Matrix Analysis for Statistics*, by Sudipto Banerjee and Anindya Roy, 2014. The MAA review `https://www.maa.org/press/maa-reviews/linear-algebra-and-matrix-analysis-for-statistics` is very positive, stating

   > This would be a reasonable candidate for use in a standard linear algebra course, even at institutions with no statistics majors. The word "statistics" in the title only indicates that preference has been given to topics used in statistics. Just how they are used receives scant attention, and students (and many teachers) using the text might well be unaware it has any special orientation toward statistics.
   >
   > The presentation is pretty much straight theorem-proof. The proofs are very detailed and the authors bind the argument together with clear text that flows beautifully.

   The reviews at Amazon are not as kind. Several complain about the large number of typos. At least two complain that there is little connection to statistics (a point discussed by the MAA reviewer).

3. *Matrix Algebra Useful for Statistics*, Second Edition, by Shayle R. Searle (1928–2013) and André I. Khuri, 2017, about 470 pages. The last three chapters are dedicated to SAS, Matlab and R, respectively, and about 60 pages of the book are solutions to exercises; the inclusion of such material being an idea I greatly appreciate and clearly abide by also in my books and their supplementary material. This is a larger book, page-wise, and also in scope, than Bapat's book above, also covering matrix calculus. Shayle Searle is a household name for linear models people, having several well known graduate level books on linear models going back quite a while. The first edition of this book goes back to 1982 (solo by Searle), and now updated (quite considerably) by André I. Khuri. This would be among my first choices to use in such a course.

4. *Linear Models and the Relevant Distributions and Matrix Algebra*, David A. Harville, 2018. About 500 pages. I know and admire the author's 2008 book *Matrix Algebra from a Statistician's Perspective*, with an accompanying entire book on the exercises and their solutions. As such, I have high expectations of this book. Much of it seems classic, Gaussian-based distribution theory (which I cover extensively in two of my own textbooks) though from the preface, we read:

   > The discussion herein of multiple comparisons is not confined to the traditional methods, which serve to control the FWER (familywise error rate). It includes discussion of less conservative methods of the kinds proposed by Benjamini and Hochberg and by Lehmann and Romano.

   Further:

   > The book could serve as the text for a graduate-level course on linear statistical models witha secondary purpose of providing instruction in matrix algebra.

   Usefully, the author states that a solutions manual is available for instructors who have adopted the book. Harville also has a section in the preface called "Credentials", which I found superfluous, knowing, from his other books, his acute competence.

5. *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares*, Stephen Boyd and Lieven Vandenberghe, 2018, 474 pages. The pdf version of the book is legally available for free (via an agreement with the publisher, CUP), and can be found, along with lecture slides, lecture videos, and further material, such as a 180 (192) page document dedicated to using the Julia (Python) language for linear algebra, at `https://web.stanford.edu/~boyd/vmls/`.

   As if that is not impressive already, about half of it is dedicated to aspects of least squares (including classification, constrained least squares, and nonlinear constrained least squares). As the authors say in the preface,

   > The book covers less mathematics than a typical text on applied linear algebra. We use only one theoretical concept from linear algebra, linear independence, and only one computational tool, the QR factorization.

   There is a sizeable number of reviews at Amazon, mostly very positive, though with some noting, praising, or complaining about, that the book is not written in a rigorous mathematical style, or that it is not so complete. This is in fact alluded to by the statement of the authors in the preface, as just mentioned. For example, one Amazon reviewer writes "I think it's important to highlight that the book is not rigorous, not even on Strang's Linear algebra level". Another writes "It is the exact opposite of the usual dry math for math's sake that people are forced to ingest in school; as such, I would recommend it to anyone who thinks he/she doesn't like math."

   Regarding the heavy use of the QR factorization, this is no doubt a high point of the book, and is adequate in a least squares setting when the design matrix is of full column rank. But what about the rank deficient case, in which case, the SVD would take prominence? Indeed, as one Amazon reviewer remarks (translating from French), "It is a pity that the work does not deal with decomposition into singular values that also provides solutions to least squares problems and is one of the important tools in the processing of data matrices." See also the quotes that open Section 4 regarding the relevance of the SVD.

   The MAA review `https://www.maa.org/press/maa-reviews/introduction-to-applied-linear-algebra-vectors-matrices-and-least-squares` is very positive, and I think very accurate also regarding how the book can be used:

> You'll find many topics in this book that are commonly covered in a first course in linear algebra, including basic operations on matrices and vectors, systems of linear equations, linear independence and bases, orthogonality, the Gram-Schmidt process, least squares problems, and the normal equations. However, you might be surprised by the standard topics that are excluded from this book, including complex vectors and matrices, vector spaces, the null space and range of a matrix, eigenvalues and eigenvectors, diagonalization, and systems of linear differential equations. The authors have clearly chosen to include only those topics from a conventional course that are necessary to support the data science applications in the book.
>
> Although this book is clearly not suitable for a mainstream introduction to linear algebra course, it could be used either as the textbook for a first course in applied linear algebra for data science or (using the first half of the book to review linear algebra basics) the textbook for a course in linear algebra for data science that builds on a prior introduction to linear algebra.
>
> This is a very well written textbook that features significant mathematics, algorithms, and applications. I recommend it highly.

The solutions manual is available to instructors. It is 212 pages, and highly detailed.

4.9 Difference Equations

These last two categories, in Sections 4.9 and 4.10, are not formally linear algebra proper, but rather (arguably very important) applications of it, namely in the study of difference and differential equations.

For the former, the following list gives a set of books, all of which look excellent, in terms of content and also in terms of optically beautiful layout (and availability of perfect pdf files). I have since years a hardcopy of Elaydi's book, and concur with the MAA and Amazon reviews that it is excellent.

1. Paul Cull, Mary Flahive, Robby Robson, *Difference Equations From Rabbits to Chaos*, 2005. This is my favorite of the books I list (and it is less ambitious than Elaydi, and also has an orientation to computation and algorithms). From the preface,

   > As a text, it is meant for undergraduate majors in one of the mathematical sciences, presumably in their junior or senior year. We've written it for the student who likes to compute and is comfortable with mathematical proof, but the book can be profitably read by students who approach the subject from either a computational or theoretical point of view.
   >
   > ...
   >
   > Along the way we use linear algebra, develop formal power series, solve some combinatorial problems, visit Perron–Frobenius theory, use graph theory, discuss pseudorandom number generation and integer factorization, and use the FFT to multiply polynomials quickly.
   >
   > ...
   >
   > One of the aims of this book is to show students that linear algebra is a powerful and coherent subject whose ideas have diverse applications, and we hope Appendix C is a helpful review.

   The topics in their appendix C are: Vector Spaces and Subspaces, Linear Independence and Basis, Linear Transformations, Eigenvectors, Characteristic and Minimal Polynomials.

   The MAA review `https://www.maa.org/press/maa-reviews/difference-equations-from-rabbits-to-chaos` is brief and positive, ending with:

> This text, although requiring some computational and/or theoretical maturity, seems more suitable to an undergraduate course than such recent books as Difference Equations: An Introduction with Applications (Second Edition) by W. G. Kelley and A. C. Peterson (San Diego: Academic Press, 2000) or An Introduction to Difference Equations (Third Edition) by S. Elaydi (New York: Springer, 2005). The text under review does not go as deeply into the subject as these other books, but it provides an accessible introduction to the material and a firm foundation for applications in various scientific fields.

2. Saber Elaydi, *An Introduction to Difference Equations*, Third Edition, 2005. In the preface, the author writes "The main prerequisites for most of the material in this book are calculus and linear algebra. However, some topics in later chapters may require some rudiments of advanced calculus and complex analysis." Indeed, chapter 3 makes use of matrix similarity, diagonalizability, repeated eigenvalues and the Jordan Canonical form — these clearly showing the value of the aforementioned material.

   The later chapters are best read by students having had a first course in real analysis, and at least some exposure to the very basics of complex numbers (and somewhat more, namely complex integration and residuals, in section 6.2.3), as well as the Laplace transform (used in section 6.7).

   The MAA review https://www.maa.org/press/maa-reviews/an-introduction-to-difference-equations is (as I expected) extraordinarily positive:

   > Additionally the book is full of good exercises at all levels replete with hints and answers...
   >
   > It is impossible not to admire Elaydi's achievement in putting together a textbook of such quality.
   >
   > *An Introduction to Difference Equations* is a terrific book almost every page of which contains marvelous things.

3. Ronald E. Mickens, *Difference Equations: Theory, Applications and Advanced Topics*, Third Edition, 2015.

4. Michael A. Radin, *Difference Equations for Scientists and Engineering*, 2019.

4.10 Differential Equations

A few of the books listed above on linear algebra also include chapters on differential equations. The two topics are (both of enormous importance, in and of themselves, and) highly intertwined—this also being reflected in the existence of several books that embody both topics in their titles, such as the following.

1. C. Henry Edwards, David E. Penney, and David T. Calvis, *Differential Equations and Linear Algebra*, 4th edition, 2018. As the title suggests, the book specializes in linear algebra for use in differential equations. The book is optically very nice, in the Pearson style (far removed from standard latex) that, to some (including me) can look like a high school book, but there are nice graphics.

   The 3rd edition (with just C. Henry Edwards and David E. Penney, 2010) received a strong review from MAA, https://www.maa.org/press/maa-reviews/differential-equations-and-linear-algebra.

2. Goode and Annin, *Differential Equations and Linear Algebra*, fourth edition, 2016, 864 pages. The book has nearly 400 pages dedicated just to linear algebra, and might be worth consideration as the textbook for a first course in linear algebra, notably if the goal is to also cover some aspects of solving differential equations. The MAA review of Goode and Annin is very strong, the only negative aspect reported being the

price—irrelevant, at least from the student perspective, when the book is freely available as a perfect pdf file on the web. See https://www.maa.org/press/maa-reviews/differential-equations-and-linear-algebra-0.

3. Gilbert Strang, *Differential Equations and Linear Algebra*, 2015.

4. Boelkins, Goldberg, and Potter, *Differential Equations with Linear Algebra*, 2009, reviewed by MAA here, https://www.maa.org/press/maa-reviews/differential-equations-with-linear-algebra.

There are presumably more such books; I did not search further. I do however also include a book that does *not* emphasize or assume knowledge of linear algebra, namely:

5. Henry J. Ricardo, *A Modern Introduction to Differential Equations*, Third Edition, 2020, 556 pages. The MAA review https://www.maa.org/press/maa-reviews/a-modern-introduction-to-differential-equations-0 is quite positive, noting:

   > This is an attractive book, designed for readability and well suited for an introductory course. It has a broad collection of worked-out examples and exercises that span application areas in biology, chemistry and economics as well as physics and engineering.

   Note that Ricardo is the author of *A Modern Introduction to Linear Algebra* (2010), which I discuss in Section 3.2, so it is perhaps ironic that he does not emphasize linear algebra in this book on differential equations. We learn why in the MAA review of the book's second edition, https://www.maa.org/press/maa-reviews/a-modern-introduction-to-differential-equations:

   > The reason for that is that at many universities, students are required to take this class before they take linear algebra, and not the other way around.

5 Course Proposal 3: "A First Course in Undergraduate Real Analysis"

> Real analysis stands as a beacon of stability in the otherwise unpredictable evolution of the mathematics curriculum. Amid the various pedagogical revolutions in calculus, computing, statistics, and data analysis, nearly every undergraduate program continues to require at least one semester of real analysis.
>
> Stephen Abbott, from the preface of his *Understanding Analysis* (2016).

This topic addresses (mostly or entirely) single variable calculus, albeit "from the ground up", starting with basic properties of (the usual sets of numbers, but notably) the real line, in particular, the concept of completeness. The emphasis is on rigorous proofs of the major basic results associated with sequences, series, continuity, the Newton quotient derivative, and Riemann integration. Such a course is, certainly in the Anglo-Saxon world and beyond (noting also the selection of book translations, such as from Italy and Russia), popular and rather well-defined. As such, there is an associated well-populated "book niche", though as we will see below in Section 5.4, there are numerous books that deviate (somewhat, or markedly) from the usual collection of topics and presentation style.

5.1 Introductory Remarks on Textbooks in Real Analysis

Before elaborating on the (literally now) dozens of books available to choose from, let's get something out of the way first: Certainly for the intended introductory level, but even for a more advanced course in analysis (see Section 7), the small set of "classic" books—the ones that still adorn the bookshelves of soon-retiring math professors and that they had to suffer through as students—will not be included. As an example regarding books considered appropriate for advanced undergraduates and beginning graduate students in mathematics,[33] I was happy to find the following quote in the MAA review of Royden's classic *Real Analysis*, 3rd edition, 1988 (the first edition of which having appeared in 1963, and sold for $9.00), https://www.maa.org/press/maa-reviews/real-analysis-1:

> With the possible exception of Rudin's Real and Complex Analysis, this may be the single most assigned text for the all-important graduate real analysis course. And frankly, I have no idea how anyone can learn real analysis from this thing.
>
> ...
>
> Scholars are sadly no more enlightened than the rest of human society—which is probably why after four decades and the death of its author, this book keeps selling, even at such an outrageous price. It is used as a test and a filter for graduate students. This is sad, since there are so many superior texts on this beautiful and critical subject—Angus Taylor's masterwork, General Theory of Functions and Integration (in Dover paperback, no less!), Hienz Baur's [*sic*; I presume Heinz Bauer's] very modern and probability geared Measure and Integration, the old fashioned but wonderfully clear presentation in R. Wheeden and A. Zygmund's classic Measure and Integral, for example. More recently, Stein and Shakarchi have produced a beautiful—and much less expensive—Real Analysis text in the Princeton Lectures in Analysis series.

At the time, the main competitor of Royden was the same-named book by McShane and Botts, 1959, which can be found now reprinted from Dover publications, 2013. Sometimes, it

[33]The suitability of Royden's book for "a good masters degree program or an honors undergraduate program" is mentioned at the end of Waterman's (overall positive) review (SIAM, Vol. 7, Issue 3, 1965, pp. 436-7) of Royden's first edition in 1963. The reviewer's opening statement is "This book is a well-organized and thoroughly readable introduction to a number of subjects". He goes on to say (his emphasis) "However, a student who spent a year studying from this text would find that he had obtained an *introduction* to many areas of mathematics, but in none of them would he have obtained the level usually reached in a first year course".

is better to just leave a classic presentation alone, instead of making new editions. The fourth edition of Royden exists, 2010, posthumously with co-author Patrick Fitzpatrick, whose 2009 book *Advanced Calculus* we will meet below. One might have thought that Fitzpatrick would "upgrade" the presentation (perhaps to be more suited to modern needs, notably graduate students outside of pure mathematics who have reason and desire to learn this material for their applied mathematics pursuits), though that appears to not be the case, at least not substantially so. Further, the Amazon reviews of this Royden and Fitzpatrick 2010 release indicate that the new book has so many errors and typos, there is literally a 19 page errata list (and the errata list also has errors). So much for Mr. Royden resting in peace.

I also do not include Tom M. Apostol's (1923-2016) book, first edition 1957, second edition 1974, which can be seen as having been a competitor of Rudin's books. Apostol's book is reviewed (and compared to Rudin) here, https://www.maa.org/press/maa-reviews/mathematical-analysis.

I have collected (mostly introductory—reflecting my level of knowledge) real analysis books since I was an undergraduate, simply because I love the subject and enjoy reading it, and own hard copies of over a dozen such books. A bit of internet inspection shows that there has been an explosion in the number of books in this category in the last two decades. This is also the case for books at the next level, i.e., "advanced calculus" (often synonymous with, or at least having a large overlap with, multivariate real analysis), and the next levels, namely measure theory and functional analysis; as well as complex and Fourier analysis. This is highly fortunate for both students and instructors, with more market participants, and thus the ensuing increase in competition and quality, though as remarked earlier in Section 2.2, can result in some frustration when it comes to actually selecting a single book to focus on and study. Similar to the book market for beginning and intermediate linear algebra, there are literally too many (good) choices. Nevertheless, I attempt to come up with some categories, structure, and recommendations.

Naturally, I cannot possibly inspect, or even know about, all the books available—there are more than most people would ever imagine. In addition to books that I simply am ignorant of, there are books that are not included because my assessment—based mainly on reviews by others (published, and/or by purchasers on Amazon)—is that they cannot compare to their competitor counterparts I give in my—already arguably oversized—recommendation lists. Finally, there are several books explicitly written to be more gentle, more verbose introductions (as many of their titles suggest) appealing to beginning students coming out of basic calculus and intimidated by, and/or not ready for, the level of abstraction and never-ending onslaught of theorem-proof pairs inherent in a first (let alone subsequent) course in real analysis. I do not include any of them in my recommendation list below in Section 5.2 *not* because they are poorly written, or so dumbed down that they are a waste of students' time, but because I find the books that I *do* recommend (not each and every one, but certainly enough of them) to be already verbose enough, and such that *all of them* (obviously, in my opinion) are pedagogically excellent for the student audience in mind.

Here is my list of (what I categorize to be) "rather verbose and/or basic" real analysis books. Students can and perhaps should be made aware of these, as optional extra reading, but I do not personally recommend them to be used *as the primary book*.[34]

1. *Yet Another Introduction to Analysis* by Victor Bryant, 1990. (Another of Bryant's books does in fact enter in one of my lists; see Section 7.4.)

2. *A Friendly Introduction to Analysis* 2nd Edition, Witold Kosmala, 2004.

[34]For the reader amused by the titles of these books, and others, such as *The Way of Analysis* by Robert Strichartz (2000), one might contemplate writing a similar book, entitled *The Tao of Real Analysis*. However, be aware of the books (two of which appear in the list in Section 5.2) by heralded mathematics genius Terence Tao. Perhaps Terry should consider this for his next book? Besides the wikipedia page, an enjoyable read about Tao is https://www.nytimes.com/2015/07/26/magazine/the-singular-mind-of-terry-tao.html.

3. *A First Course in Mathematical Analysis*, David Brannan, 2006, 459 pages. The MAA review (done by Henry Ricardo) `https://www.maa.org/press/maa-reviews/a-first-course-in-mathematical-analysis` is quite strong, indicating how friendly the book is (including use of "I give you an ϵ, and you have to counter with a δ game"), but also some bonus material. We read:

> Four appendices provide useful prerequisite information and solutions to problems (as opposed to 'Exercises', whose answers are not given). In between, most of the standard material of Advanced Calculus is treated in a way that even the 'friendliest' of current texts can only aspire to emulate. There are many helpful diagrams and marginal notes.
>
> Among the gems scattered throughout, we find nicely motivated discussions of π and e, a detailed treatment of the 'Blancmange' function (Takagi fractal curve) — continuous everywhere but differentiable nowhere — a proof of Stirling's formula, some theorems on the location of zeros of polynomials, and a proof that π is irrational. (Although the Index promises a discussion of the irrationality of e on p. 175, there doesn't seem to be a proof anywhere in the book.)

I mention Brannan's book also in Section 2.2, as an example of a book that provides excellent detail on (the one particular example I chose to inspect in detail, namely) Euler's constant.

One might argue: It borders on criminal to *not* make students of such a course aware of this book, and likewise the others in this short list, in particular students outside of pure and applied mathematics, many of whom enter such a course grudgingly, and surely with great apprehension and trepidation.

4. *Elementary Analysis: The Theory of Calculus*, 2nd edition, Kenneth Ross (born 1936), 2013, 409 pages. The first edition appeared all the way back in 1980 (with 351 pages), and received a very nice MAA review `https://www.maa.org/press/maa-reviews/elementary-analysis-the-theory-of-calculus-0`, from which we read:

> This book occupies a niche between a calculus course and a full-blown real analysis course. Its charm is that it gives very thorough and leisurely explanations, in a discursive style: You just read along about some interesting properties of the real numbers and then find them codified as a definition or theorem, rather than being confronted with a mass of definitions, theorems, and proofs.
>
> The book assumes the student has already been through calculus, but without the proofs. It presents most of the important ideas of real analysis without requiring any great conceptual leaps from a calculus course. It is very carefully positioned to lie between a non-rigorous calculus course and a real analysis course such as might be taught from Rudin's Principles of Mathematical Analysis or Apostol's Mathematical Analysis.
>
> The book is mathematically not very ambitious, and at first glance it may look like there's not much here. I think the book should be viewed as a text for a bridge or transition course that happens to be about analysis, rather than an analysis course per se.
>
> Certainly it has much less material than would normally be found in an analysis text. There's not much topology, no construction of the real numbers (there's a brief sketch of Dedekind cuts), no measure theory or Lebesgue integral, and no function spaces. The book depends almost totally on completeness of the reals for its proofs, although the Bolzano-Weierstrass theorem is introduced to back up uniform continuity, which is needed for integrability.
>
> There are several Very Good Features:

- Lots of counterexamples. Most calculus books get the proof of the chain rule wrong, and Ross not only gives a correct proof but gives an example where the common mis-proof fails. There are also examples of failures of L'Hospital's rule and of non-integrable functions.
- Introduces limits of sequences first, and only then goes on to continuity
- Lengthy discussion of the Riemann-Stieltjes integral, which is very handy in mixed discrete-continuous problems and number theory and which most texts don't cover at all

The book has a number of optional sections, which tend to be dead ends in this book but are interesting in themselves and in a more advanced course would have many consequences. Most of the topology material is in this category. There's a complete proof of the Weierstrass approximation theorem (using Bernstein polynomials), which doesn't go anywhere in this book but is certainly a startling result.

In an MAA review from 2008 by Ross on (obviously) a different book,[35] he takes the opportunity to mention his own book, saying:

> This may seem self-serving, but I really think there's a better approach to Riemann-Stieltjes integrals than the standard one presented in most books. See Chapter 7 of the book under review and compare with Section 35 in my book, *Elementary Analysis: The Theory of Calculus*. My approach avoids anomalies in the standard approach without losing anything useful.

One takeaway from these reviews is: Ross' book is, besides a gentle transition from calculus to real analysis, a very good source for development of the Riemann-Stieltjes integral.

5. *Real Analysis: A Long-Form Mathematics Textbook*, 2nd Edition, Jay Cummings, 2019. The book is 445 pages, and costs about 30 USD new, making it among the cheapest books. The Amazon evaluations are (like for all these books) mixed, but among the best, being praised for its explanations. I was not able to find a pdf of the book on the web.

6. *Real Analysis With Proof Strategies*, Daniel W. Cunningham, 2020. The book is discussed (and positively evaluated) in the MAA review https://www.maa.org/press/maa-reviews/real-analysis-with-proof-strategies.

The next list consists of books that are not about analysis *per se*, but rather can be used before taking (or maybe concurrently with) a first course in real analysis. They specialize in teaching the student about proof writing, and were designed to assist the transition from "rote learning" of basic mathematics to understanding and appreciating higher mathematics. The first three were jointly (and all favorably) reviewed by Silverman, as mentioned in Section 1.

1. *An Accompaniment to Higher Mathematics*, George R. Exner, 1996.

2. *Mathematical Thinking: Problem Solving and Proofs*, John P. D'Angelo and Douglas B. West, 1997.

3. *Journey into Mathematics: An Introduction to Proofs*, Joseph Rotman, 1998.

4. *How to Think About Analysis*, Lara Alcock, 2014.

[35] In particular, on Leonard Richardson's 2008 *Advanced Calculus: An Introduction to Linear Analysis*, https://www.maa.org/press/maa-reviews/advanced-calculus-an-introduction-to-linear-analysis, which Ross speaks highly of, though very briefly, in his review. There are numerous Amazon reviews from customers and, regretfully, the vast majority are anything but positive. This is the reason Richardon's book does not get a mention in Section 7.1, where it would otherwise have fit very nicely.

5.2 List of Suggested Primary Books

I turn now to my list of recommended books. It was supposed to be a short list, like a short-list of three to five candidates for a professorship, but there are simply so many excellent books that it became far longer than planned. Omitting a good book would have been an arbitrary decision, most likely as a function of irrelevant factors, and is neither fair to the author, nor to students or instructors who are relying on my document to provide some guidance. The ordering is not indicative of preference, and was chosen to augment a couple of subsequent comments. Keep in mind that, among this elite collection, there is some “exchangeability”, and no single book is (or can be, given the nature and amount of the material) superior in all regards to the others.

Abbreviation “URA” stands for Undergraduate Real Analysis:

URA Book Choice 1: Peter D. Lax and Maria Shea Terrell, *Calculus with Applications*, 2014, 2nd edition, 515 pages. This is somewhere between a calculus and analysis book, with applications of calculus (as the title suggests) but also proofs of some results. Such a book could be used in conjunction with a dedicated real analysis book, such as the other books listed here. A benefit of this book is (in addition to being very well written) that it has a chapters on complex functions and differential equations, and even probability theory. The presence of such topics is not so shocking, given the history of the book, a bit of which is discussed in the MAA review https://www.maa.org/press/maa-reviews/calculus-with-applications. It originally appeared in 1976, under the title *Calculus with applications and computing Vol. 1*, co-authored by Peter D. Lax, Samuel Burstein and Anneli Lax. It was supposed to be a “radical” calculus textbook, and it also included FORTRAN code! The reviewer concludes with:

> My criticisms and suggestions aside, this is an altogether excellent text. It is filled with beautiful ideas that are elegantly explained and chock-full with problems that will enchant both the experienced teacher and the curious novice. I recommend it strongly and look forward to an even better third edition!

URA Book Choice 2: Niels Jacob and Kristian P. Evans, *A Course in Analysis Volume l: Introductory Calculus, Analysis of Functions of One Variable*, 2016, 744 pages. Observe how it is literally two books in one; the first one being somewhere between Calculus and a first course in real analysis, and the second being definitive but still basic analysis, and both halves having been done very well. My “initial inspection” turned into me reading the book with a fine-toothed comb, and enjoying it immensely. (I have emailed with author Kristian Evans, pointing out a collection of further errata over and above those on the list he kindly sent me. He was very cordial and helpful—useful if one were to adopt their outstanding book series.)

The MAA review https://www.maa.org/press/maa-reviews/a-course-in-analysis-volume-i-introductory-calculus-analysis-of-functions-of-one-real-variable, covering this and the next volume in their series, is clearly positive. I mention volume II in Section 7.1, and give an excerpt from the MAA review.

URA Book Choice 3: Amol Sasane, *The How and Why of One Variable Calculus*, 2015, 464 pages. This appears to be a very well written book, with nice graphics and detailed explanations. It could have perhaps been allocated to the list I give above of “gentle analysis books”, but I include it here because my small inspection of it indicates it to be a pedagogic work of art, and also because the author has a subsequent (and attractive) book of interest to budding machine learners who want to significantly push their frontier of mathematical knowledge; see Section 8.2.

URA Book Choice 4: Claudio Canuto and Anita Tabacco, *Mathematical Analysis I*, 2008, 430 pages. This book can also be considered to cover both calculus and real analysis, similar to the previous books. The pdf file is good: Text can be marked and copied, but visually it is okay but not perfect. There is a volume II, 2010, 520 pages, and that one happens to have a perfect pdf file on the web.

URA Book Choice 5: Sudhir R. Ghorpade and Balmohan V. Limaye, *A Course in Calculus and Real Analysis*, 2018, 2nd edition, 538 pages. Despite the title, this is definitely an analysis book, though has some applied elements that one finds in some calculus books. This book may not be an optimal choice for, say, an honors calculus course, and certainly not for a typical beginning college calculus one. However, based on my careful reading of the first half of the book, it has emerged as one that I would highly recommend for a first course in analysis, the reasons for which I detail below.

Let's first look at what others think, starting with a review that is somewhat negative. The MAA review https://www.maa.org/press/maa-reviews/a-course-in-calculus-and-real-analysis-2nd-ed has a few good things to say, though elaborates extensively on a bad thing, namely the style of presentation being too rushed:

> The book straddles the world of basic calculus and the world of real analysis and it includes a wide range of topics, each of which is presented clearly and rigorously and it also includes some interesting historical background on these topics. I was particularly interested to see that Chapter 10 includes the Arzela bounded convergence theorem for Riemann integration and some nice applications of this theorem.
>
> I am not convinced that the writing style of this book is optimized for use as a primary textbook for introductory courses in real analysis.
>
> [It] covers many of its topics quite tersely and it frequently says too much at once.

The reviewer then goes on with several examples, and says there are plenty more. He then finishes with:

> I'll sum up by saying that everything works, but it is all fired from the hip. It comes too fast to be friendly enough for many students. My feeling is that none of the criticisms that I have made detract from the value of the book as a reference source but, as I have said, I would not want to use the book as a primary textbook in one of my courses.

I understand why the reviewer wrote this, when seen in the light of other books, such as those I mention in Section 5.1, that (admirably, given the increasing demand for this material by non-math-majors) progress more slowly and provide low-level details. However, the book is far from "fired from the hip": I claim it is impeccably organized, and also differentiates itself from other books in some important regards, as indicated for example in the extensive review by Wildberger, discussed below.

The relevance of theorems are motivated before presenting them, and proofs of theorems carefully refer to previous ones used (and with dynamic links no less in the pdf document). There is definitely enough detail, including well-done color graphics: The authors clearly did not hastily package some terse set of summary notes to accompany a lecture, but rather orchestrated a very coherent and polished presentation that I claim should be highly appealing to students. On three occasions within the first 100 pages, and in full "adversarial mode", I found myself desiring a bit more detail, and so I reached out to the authors. Sudhir Ghorpade kindly replied, and fleshed out what I was missing. In all three cases, their proof was correct.

I did find some typos over and above those given in the errata list, and the authors have updated it to reflect these (and kindly mention me); see http://www.math.iitb.ac.in/~srg/acicara/errata-2e-12Oct21.pdf.

I now turn to the review of the first edition of the book, by Norman Wildberger, given in the Australian Mathematical Society Gazette, 35(3), 2008, http://www.math.iitb.ac.in/~srg/acicara/Gazette_Review.pdf. It is a long, detailed review, overflowing with praise. Here are some quotes:

> It develops this subject carefully from a foundation of high-school algebra, with interesting improvements and insights rarely found in other books. Its intended audience includes mathematics majors who have already taken some calculus, and now wish to understand the subject more carefully and deeply, as well as those who teach calculus at any level.
>
> The extensive attention to detail shows, and an honest comparison with the calculus notes generally used in Australian universities would be a humbling exercise.
>
> The book augments the discussion of polar coordinates by giving precise definitions of an angle φ in various contexts. An angle is not some God-given notion bestowed on each of us at birth.
>
> Defining $\sin x$, $\cos x$ and $\tan x$ is less familiar, but a crucial point for calculus. Most texts are sadly lacking, pretending that these functions are somehow part of the background 'ether' of mathematical understanding, and so exempt from requiring proper definitions. The [approach] taken by Ghorpade and Limaye is to start with an inverse circular function. There are several good reasons to justify this choice. Historically the inverse circular functions were understood analytically before the circular functions themselves; Newton obtained the power series for $\sin x$ by first finding the power series for $\arcsin x$ and then inverting it, and indeed the $\arcsin x$ series was discovered several centuries earlier by Indian mathematicians in Kerala.
>
> This book is a tour de force, and a necessary addition to the library of anyone involved in teaching calculus, or studying it seriously.

Wildberger does however say in his first paragraph:

> Because of the high standard, only very motivated and capable students can expect to learn the subject for the first time using this text, which is comparable to Spivak's *Calculus*, or perhaps Rudin's *Principles of Mathematical Analysis*.

This statement indeed dovetails with the observation given in the above MAA review, though the latter indicates it as a critique of being too terse, while Wildberger is giving an honest assessment indicating that Ghorpade and Limaye's project is at a higher level of sophistication than some competing books. With regards to terseness, it is no Rudin, I can assure you.

Links to further, shorter yet positive, reviews can be found on http://www.math.iitb.ac.in/~srg/acicara/.

As alluded to by Wildberger, the authors take an approach to numerous topics that differ from most presentations—and justify their choice. As one example of many, they define continuity in terms of limits of sequences, as opposed to the more standard ϵ–δ approach. What I like is that the authors *also* later give that formulation, proving it is equivalent the limit-of-sequence approach, and do not treat it with disdain, but rather use it in some subsequent proofs, thus showing that both definitional approaches have value. In regards to this, quoting from the book, page 98:

> The ϵ–δ definitions of limits and continuity are often seen as a nemesis for a beginner in calculus. The approach taken here of utilizing the limits of sequences to introduce continuity and limits of functions of a real variable seems to be simple-minded and easier to understand. We have shown the equivalence of the sequential and the ϵ–δ definitions toward the end of our discussion of continuity and of limits.

This approach is not new, nor do the authors claim it is. The books by Beardon; Field; Hijab; and Little, Teo, and van Brunt (all mentioned below) also prefer to avoid ϵ–δ formulations. As a case in point, Field (2017, Sec. 2.4.1) gives the classic ϵ–δ definition, then a full page discussion of it and its issues, and then an equivalent definition based on (what is called, or he calls) sequential continuity.

My only critique of the book is that solutions are not provided for the (large and well organized) set of exercises in the book, nor its multivariate counterpart (see Section 7.1). This would not detract me from using this as the primary book, notably because I argue that use of more than one book is advantageous when teaching the material (such as, in this case, one with a lower level of presentation, and includes solutions to exercises, such as that from Jacob and Evans mentioned above).

URA Book Choice 6: Brian S. Thomson, Judith B. Bruckner, and Andrew M. Bruckner, *Elementary Real Analysis*, 2008, 2nd edition.

The pdf file is freely and legally available on the web.

It is about 980 pages, though this particular pdf page formatting is such that it would be, roughly, 700 to 750 pages if printed as a typical math book on A4 paper. It is thus still quite large, also because the authors cover some basic topology of, and differentiation on, $\mathbb{R}^n$, as well as having a large chapter on metric spaces, including the Baire category theorem. The book received quite some favorable reviews at Amazon, as did the next level book by the same authors, *Real Analysis*, 2nd edition.

Interestingly, Estep's *Practical Analysis in One Variable*, 2002, mentions, in the conclusions (and recommendations for next books) this book (well, the 2001 version), saying they "cover much of the same material as Rudin but in a more modern style."

URA Book Choice 7: Mariano Giaquinta and Giuseppe Modica, *Mathematical Analysis: Functions of One Variable*, 2003, 353 pages. This translation from Italian is nicely endowed with pictures of various books, some recent, some ancient, and many mathematicians, as well as sections such as 3.4, "Calculus: Some Historical Remarks". Less superficially, the book has some nice features, such as a discussion of Landau's notation (small o, big O) and asymptotic expansions; convexity; and inequalities. The last chapter, 6, is somewhat unique, beginning with differential equations, and then expanding to optimization, geometry, and a bit of graph theory.

URA Book Choice 8: Vladimir Zorich, Mathematical Analysis I, 2015, 2nd edition, 616 pages; translated from Russian. There is a part II, mentioned in the comments below. Taken together, and by the nature of the books—which is well addressed in the MAA review, these books should definitely be considered as the main text, or at least be brought to the attention of students as supplementary reading. From the MAA review https://www.maa.org/press/maa-reviews/mathematical-analysis-i-0,

> This is a thorough and easy-to-follow text for a beginning course in real analysis, at the sophomore or junior level.
> Weighing in at about 1300 pages, the present work is three times as long as Apostol and four times as long as Rudin. The extra length comes not from more topics or more depth, but because Zorich writes everything out in detail and because [he] includes a large number of worked examples. It does cover

some topics in greater-than-usual detail and does cover a few newer topics that are not in the classic works.

Bottom line: Will be popular with students because of the detailed explanations and the worked examples. The book does seem like overkill to me, and I wonder how many curricula could make good use of it. It would probably take about two years to work through in a course, and at the end the students would be experts at all aspects of calculus, but I feel that most math students would be better served by moving more quickly in to advanced analysis topics.

URA Book Choice 9: Jirí Lebl, *Introduction to Real Analysis, Volume I*, 2020, 280 pages; free legal pdf via Creative Commons. My impression, based on a brief skim, is that this is quite a dedicated effort, resulting a very attractive book, also with some useful and nontrivial graphics, as opposed to a set of free course notes somebody threw together and put on the web. The pages are reasonably dense, I guess 10 point in latex. The MAA review, https://www.maa.org/press/maa-reviews/basic-analysis-introduction-to-real-analysis starts with a seemingly negative statement, "This is a no-frills introduction to real analysis", but the review turns very positive, writing, among other things:

> This is a no-frills introduction to real analysis that is suitable for a basic one-semester undergraduate course. It is designed to serve both future mathematicians as well as students not intending to pursue mathematics in graduate school. For this reason it is perhaps a better match for classes with a mixture of abilities, motivations and career plans than commonly available alternative texts.

The author says in the preface that is book has similarities to (what is often used to teach his class) Bartle and Sherbert, a book I recommend below. A difference is stated by the author in the preface: "A major difference is that we define the Riemann integral using Darboux sums and not tagged partitions. The Darboux approach is far more appropriate for a course of this level." The MAA reviewer also notes this, writing:

> This book is similar in many respects to Bartle and Sherbert's Introduction to Real Analysis (and indeed the author provides a table of correspondences between sections of the two books).
>
> Lebl chooses to follow the Darboux approach to the Riemann integral. It is perhaps less elegant than Bartle and Sherbert's approach, but less sophisticated and more intuitive. Lebl also avoids proofs by contradiction as much as possible. He does this not so much for philosophical reasons as because he thinks that it is a common source of trouble for beginning students. It is important to note that Lebl is not dumbing-down real analysis: his approach is clean, clear and rigorous. He's just not tempted to get too fancy.
>
> One thing I particularly like about his approach is the way that Lebl organizes the book with a kind of capstone theorem. He uses Picard's theorem on existence and uniqueness of solutions of ordinary differential equations to pull together many of the things students have learned to provide a direct, more-or-less constructive proof. Then in the chapter on metric spaces he offers another proof using the contraction mapping theorem.

A minor critique is also given before summarizing:

> One thing I missed in the book are more counterexamples. There is a little too much emphasis on rigorous ratification of calculus. Why should students find it valuable to work hard to learn to prove things that no one could possibly doubt?

> This is an attractive book, one well worth considering for anyone about to teach introductory real analysis.

Regarding the lack of counterexamples, see below for a list of several books giving counterexamples (and solved problems) in basic real analysis. Such books can form a very good compliment to any real analysis textbook, and apparently very much so for Lebl.

URA Book Choice 10: D. J. H. Garling, *A Course in Mathematical Analysis: Volume I: Foundations and Elementary Real Analysis*, 2013, 300 pages. This is the first of a set of three very attractive books, somewhat similar to the series by Jacob and Evans. At least with respect to this first book, it is somewhat more advanced than Jacob and Evans. I would offhand say that Garling's first book is also a bit more advanced than many of the other books I mention here, e.g., the well-known (and I would say, the book that helps *define* this book niche) Bartle and Sherbert.

In this first book, Garling indicates that chapters 1 and 2, on the axioms of set theory, and number systems, can be initially skipped and returned to later. Chapters 3 to 8 cover the usual topics in basic univariate real analysis, up to Riemann integration. Chapter 9 is an introduction to Fourier series, and chapter 10 contains 10 subsections, all applications (e.g., infinite products, the gamma, beta, and Riemann's zeta functions, Stirling's formula, and the irrationality of e and π).

The MAA review, https://www.maa.org/press/maa-reviews/a-course-in-mathematical-analysis-volume-i-foundations-and-elementary-real-analysis, is short but positive, stating that "The competition for this book includes such classics as baby Rudin, Bartle and Sherbert, Royden (and Fitzpatrick), to mention just a few."

That statement indicates the reviewer's apparent admiration for Bartle and Sherbert, to which I concur, but mention of baby Rudin seems a bit inappropriate. More odd however, is mentioning Royden's book, which is blatantly graduate level—and so perhaps the reviewer means all three books by Garling taken together (but that is not what is stated). The reviewer goes on to say "Garling is a gifted expositor and the book under review really conveys the beauty of the subject." I can also agree with that.

URA Book Choice 11: Terence Tao, *Analysis I*, and *Analysis II*, 2015, 3rd edition, 347 and 218 pages.

The author is a living legend in math. That does not necessarily make him particularly desirable and competent in writing textbooks, and there might even be some negative correlation between math prowess and ability to teach. It is interesting to see that, as a star researcher no less, he took the time to write these books (there is another of relevance to us, on measure theory, and Tao has numerous other books) on material for which there is anything but a shortage of good books. Based on the reviews of his *Analysis* I and II, the books are apparently excellent; see, e.g., the MAA review https://www.maa.org/publications/maa-reviews/analysis-i-0. They were also recommended to me from a math professor with whom I emailed (not Tao), who is an author of one of the books I discuss herein.

The MAA review mentions something I also noticed in my reading of the first book, which I had bought and started reading a couple years ago:

> Tao spends a lot of time (and space) on the preliminary material from set theory: $\mathbb{R}$ doesn't appear until around p. 100 of part I, only after very thorough coverage of $\mathbb{N}$ *à la* Peano, $\mathbb{Z}$, and $\mathbb{Q}$, i.e. material that is often either relegated to a separate prerequisite course, or given short shrift in the analysis sequence itself.

> The topics subsequently covered in Analysis I, II are standard, to be sure, but are placed in a proper natural sequence, and are covered with exemplary thoroughness. Tao's treatment is reminiscent of Hardy's in his famous A Course of Pure Mathematics: everything is there, and it is done most elegantly, efficiently, and effectively. A gifted undergraduate should be urged to meditate on every line of such a book, and a class of strong students would thrive dramatically with Tao's books as texts.
>
> The aim of Tao's books is truly to prepare future mathematicians, in the genuine sense of the word. Realistically (and bluntly) this implies that a typical cross-section of today's undergraduate majors, i.e. an average group of such, would largely amount to swine facing pearls not really meant for them — at least much of the time — were one to foist Tao's books on them. So one should pick one's audience carefully if Analysis I, II is to be used, and treat these gifted kids like apprentices.

The two books, *Analysis* I and II, are (perhaps surprisingly, given the prodigy status of the author) at the appropriate level for a first course in real analysis. The first book begins with development of $\mathbb{N}$ and $\mathbb{R}$ that I found to be long and not interesting, and I (perhaps embarrassingly and as an indication of my incompetence and lack of math prowess) lost interest in reading the book. (It seems indeed a case of—copying from the reviewer above—*swine facing pearls...*) It is tempting to say that that chapter could presumably be skipped, but as the aforementioned MAA reviewer also says (and as further evidence of my naivety),

> [I]t would be an error not to stick very close to the text — it's very well crafted indeed and deviating from the score would mean an unacceptable dissonance.

My only counterargument is that Garling, in his *A Course in Mathematical Analysis: Volume I: Foundations and Elementary Real Analysis*, as discussed in the previous list entry, explicitly says that the initial chapters (in his book of course!) on set theory and number systems can be skipped (and returned to later, if and when interest arises). Adding a bit more ammunition, the MAA reviewer of Garling's book finishes his review with:

> As the author points out in the introduction, a newcomer may be advised, on a first reading, to skip Part one and take the required properties of the ordered real field as axioms; later on, as the student matures, he/she may go back to a detailed reading of the skipped part. This is good advice.

The first of the two analysis books from Tao, *Analysis I*, covers the standard set of topics in a first course (with less standard material including a section on the Axiom of Choice, and the Riemann-Stieltjes integral). Chapter 5 develops the real numbers, and the approach is via equivalent Cauchy sequences. (This same approach, albeit presented in a somewhat easier fashion, is used in Strichartz' book; see below.) Tao's *Analysis II* has eight chapters: Metric spaces, Continuous functions on metric spaces, Uniform convergence, Power series, Fourier series, Several variable differential calculus, Lebesgue measure, and Lebesgue integration. The inverse function theorem, and the implicit function theorem are covered, though the topic of line and surface integrals, and differential forms (e.g., Green's theorem and the General Stokes' theorem), are not covered, nor is there a dedicated chapter on multivariate Riemann integration (though comparisons are made between Riemann and Lebesgue). Both books, notably in light of the last two chapters of *Analysis II*, presumably offer all the needed preparation in order to work through another of Tao's books, *An Introduction to Measure Theory*, 2011. The pdf file available on the web is that book, but without publisher information or a date. It does not refer to his Analysis I and II series, but he says in the preface:

> Most of the material here is self-contained, assuming only an undergraduate knowledge in real analysis (and in particular, on the Heine-Borel theorem, which we will use as the foundation for our construction of Lebesgue measure).

Other introductory real analysis books that end with an introduction to Lebesgue (and include other topics typically associated with a second course, such as Fourier analysis) include Stoll's book, given below in this list; William J. Terrell, *A Passage to Modern Analysis*, 2019, discussed next; and (an oldie but a goodie, and one of my faves—and with all three five-star reviews at Amazon) Richard R. Goldberg, *Methods of Real Analysis*, 2nd edition, 1976. The pdf file of Goldberg we can find in the web is, no surprise, a scan, and thus not an optical joy to read. I have a—cherished—hard copy of the first edition, 1967, and it is very clear and well-printed; but even if the 2nd edition book is still in print (I did not check), without a viable pdf file, it simply cannot be used in a course given nowadays.

URA Book Choice 12: William J. Terrell, *A Passage to Modern Analysis*, 2019, about 600 pages. The preface states (note the first sentence):

> Thanks for turning to the Preface.
>
> ...
>
> The prerequisite for beginning the book is two semesters of the standard university undergraduate curriculum in elementary single variable calculus and an introductory course in proof technique often having titles such as Transition to Advanced Mathematics or Introduction to Mathematical Reasoning. This should suffice for Chapters 1–7. However, undergraduate introductions to multivariable calculus and linear algebra are prerequisites for the material from Chapter 8 onward, where the focus is on n-dimensional Euclidean space $\mathbb{R}^n$ and a few function spaces.
>
> ...
>
> This text provides a bridge from one-dimensional analysis to more general spaces, building on the core topics of differentiation and integration and a few well chosen application areas such as solving equations, inverting functions, measuring the volume of sets, and understanding basic properties of differential equations, including some basic Fourier analysis for application to partial differential equations. The text culminates with two chapters on the Lebesgue theory and a chapter on inner product spaces and Fourier expansion in Hilbert space.

The MAA review https://www.maa.org/press/maa-reviews/a-passage-to-modern-analysis is very positive. As a sample:

> The book is primarily aimed at advanced undergraduates, though the material in the latter part of the book is suitable for beginning graduate students.
>
> As mentioned before, the book covers a wide range of topics and does so in a standard way. However, there were a few pleasant surprises:
>
> - A vivid plausibility argument for the Schroeder-Bernstein Theorem (a detailed proof appears in an appendix).
> - The Contraction Mapping Theorem makes an earlier-than-usual appearance in the setting of the real line (it is applied to treat Newton's method) before it is introduced in the general setting of metric spaces. It is then used in the usual way to prove the Inverse Function Theorem and an existence and uniqueness theorem on initial value problems for systems of ordinary differential equations.

- The coverage of $\mathbb{R}^n$ is quite thorough for a book of its kind. For example, Fourier expansions, the spectral theorem for real symmetric matrices, and matrix norms are all treated in detail.
- The Morse lemma for functions on $\mathbb{R}^n$ is proved in detail as an application of the inverse function theorem.

The exercises are generally well-chosen and are a mix of extending theorems, providing alternative proofs, working out small details in proofs, verifying definitions in examples, constructing counterexamples, and computations. However, there are almost no hard exercises. The exercises generally range in difficulty from easy to moderate. The book contains a thorough and usable index. I found very few misprints, all of which were minor notational inconsistencies which should have no detrimental effect on the reader.

In closing, I think this book would be a good choice as a textbook for courses or self-directed study in single- or multi-variable analysis. It stands out from other books on these topics due to its detail, and its coverage of differential equations, applications of Fourier series, and Lebesgue integration.

URA Book Choice 13: Robert G. Bartle and Donald R. Sherbert, *Introduction to Real Analysis*, 2011, 4th edition. This is a popular, very good book, well organized and structured, and presumably (close to) typo-free in its 4th edition. (I had used the first edition as an undergraduate—and still have my copy—, though was too naive to know and appreciate how much better it was than, say, baby Rudin.)

The (3rd and) 4th edition has a chapter dedicated to the Gauge integral, which, according to Robert Bartle, in his article *Return to the Riemann Integral*, makes an appealing case for using the integral proposed by Kurzweil and Henstock around 1960, instead of the Lebesgue integral; see https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Bartle625-632.pdf. (There is also the short Appendix E, giving a continuous nowhere differentiable function, attributed to van der Waerden in 1930; and a space-filling curve due to Schoenberg in 1938.) Otherwise, the selection of topics is basically standard, and remains univariate. That is not a critique: The writing is nice, and we get occasionally entertained with tidbits such as learning that (page 73), for the harmonic series $\sum_{k=1}^{\infty} k^{-1}$, in order to reach a sum of at least 50, "A supercomputer that can perform more than a trillion additions a second would take more than 164 years to reach that modest goal."

The MAA review for the third edition is extremely positive, https://www.maa.org/press/maa-reviews/introduction-to-real-analysis-1, as is that for the fourth edition, https://www.maa.org/press/maa-reviews/introduction-to-real-analysis-0.

As a last comment (that ties into the subsequently discussed book by Magnus), the last chapter, 11, "A Glimpse Into Topology" discusses the rudiments of open and closed sets, compactness, the Heine-Borel theorem, compactness and continuity, and a few modest pages on metric spaces. Interestingly, Magnus' book (next in the list) goes further than banishing that material to the end of the book, and explicitly states that he does not discuss it at all, and explains why.

URA Book Choice 14: Robert Magnus, *Fundamental Mathematical Analysis*, 2020. To proverbially throw ones hat in the ring nowadays, amid so many good introductory real analysis books, the author must be quite confident he has written something very good. One has that impression from reading the preface and, more importantly, from the text. From the preface, we can see what is covered, but also what is *not* covered—such as a topic I had just assumed is mandatory:

> In this text there is no discussion of countability versus uncountability for sets. There are no open or closed sets (apart from intervals), and therefore no topology or metrics; and certainly no Heine–Borel theorem, though we go

> dangerously close to requiring it. This means that we stop short of a nice, necessary and sufficient condition for integrability. The integral is Riemann-Darboux; though I freely confess my view that the Lebesgue integral is the greatest advance in analysis of the twentieth century.

Note the title: Fundamental. This is what the author means. So, no Heine-Borel, but we do have Bolzano-Weierstrass on page 60, and the book has (also within the exercises) three proofs of it.

There are short sections of great interest, and that demonstrate some of the strengths of this book for the reader, such as section 4.3.2 "Thoughts About the Proof of the Intermediate Value Theorem", section 4.3.3 "The Importance of the Intermediate Value Theorem", and section 4.3.5, "Thoughts About the Proof of the Boundedness Theorem", which provides two more proofs of it. That (one page) section ends by motivating the material, saying:

> The sketched proof just given is not just an academic curiosity. The Bolzano–Weierstrass theorem is capable of great generalisation, into the area of multivariate calculus, and even beyond, into the realm of infinite-dimensional spaces. It means that versions of the boundedness theorem, and the extreme value theorem of the next section, emerge repeatedly in advanced work.

The MAA review https://www.maa.org/press/maa-reviews/fundamental-mathematical-analysis is very positive, including such excerpts as:

> The result is a rich but demanding text that is appropriate for advanced undergraduate students who have had a moderate amount of previous experience writing and reading proofs.
>
> ...
>
> In terms of style and format, this text is excellent. There is an incredible level of detail and granularity to the chapter/section/subsection divisions in the table of contents. Exercises occur at the end of each subsection, and they are by and large stated at the appropriate level of difficulty. Historical context or quotes are occasionally given to help guide a conversation or elucidate where an idea came from. One of my favorite elements of this text are the extra, exploratory sections known as "nuggets" (so-named to represent a "nugget of wisdom"). These are entire sections, scattered throughout the text, that do a momentary deep dive into an advanced topic. Nuggets include topics such as continued fractions, approximation by step functions, and Riemann's rearrangement theorem, to name three. These serve a clear role as supplemental material for class projects, or as a tempting morsel for an interested student.
>
> ...
>
> This text is well organized and peppered with insight and mathematical nuggets. Its high level of rigor might make it challenging to use in a class whose students are not ready to meet the challenge. Nevertheless, I found it an enjoyable read and am confident that a well-prepared student would gain a tremendous deal from this text.

Being a new book, and from Springer (which makes their books digitally available to subscribers), the pdf of the book is available on the usual web sites for such things: It is a perfect pdf file, so optically wonderful, with markable, and thus copyable (both "copyable" and "copiable" are given as valid spellings in online dictionaries), expandable, and searchable text.

Enjoyably for students in Zurich, the first actual chapter, chapter 2, begins with the following quote:[36]

[36] The quote, of course being a translation from German, is given on Wikipedia as beginning with "As

> As professor in the Polytechnic School in Zurich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt, more keenly than ever before, the lack of a really scientific foundation for arithmetic.
>
> R. Dedekind. Essays on the theory of numbers

Regretfully (in my opinion), and like several other books in this list, there are no solutions to exercises in the book, and there appears to be no instructor's manual containing the solutions. This of course does not prevent this book from being used as the main text, though see my thoughts on this issue in Section 2.3.

URA Book Choice 15: Manfred Stoll, *Introduction to Real Analysis*, Second Edition 2001; but note the Third Edition, 2021. At the time of this writing, I was not able to find the pdf of the 3rd edition.

Stoll is, like Bartle and Sherbert (and several others) a very good book. As mentioned, there is a third edition now available (also lending evidence that the book is doing well in the market), dated 2021, making its choice yet more attractive, provided a pdf of the book is available—recall the remarks in Section 2.1. A few years ago, I received the book (second edition) from the publisher for free—and read (and enjoyed) it. For the most part, it is (superficially) "ordinary" in terms of presentation and topics covered, but I mean this here in the good sense of unpretentious, and very well organized and well written. It also touches upon more advanced topics, such as Fourier analysis, the Riemann-Stieltjes and Lebesgue integrals, and normed linear spaces.

I emailed recently with a math professor (and author of one of the books I discuss herein) about Stoll's and other analysis books, and asking for his thoughts. He wrote me back, agreeing,

> "... as I happened to have a copy [of Stoll's book], I took a look at it. It is, as you said, clear, honest and unpretentious. As for boring, that is a matter of taste (I'm sure I would have liked it as a student), but I observed that he has some quite advanced problems that would be a challenge to most students (giving the Schröder-Bernstein Theorem as an exercise as he does on page 45, borders on cruelty)."

He indicated favor for Terence Tao's books, and also holds Körner's book (see below) in high regard. Körner (as well as a few other books in the next list) could serve as secondary, somewhat terser and deeper, reading.

URA Book Choice 16: William F. Trench (1931–2016), *Introduction to Real Analysis*, 2003, available as free legal pdf file (Open Textbook Initiative, and updated 2012; see http://ramanujan.math.trinity.edu/wtrench/misc/index.shtml). I own the 2003 hard-copy of Trench's book, and would also say it is "unpretentious" (like Stoll's book), and well-organized. I recall finding some typos—a true annoyance for students, because they are trying to learn material that, for them, is already challenging. Perhaps these have been corrected in the available 2012 version. A complete instructor's manual with solutions to the exercises is available; I have obtained it. Supplemental documents by Trench include *Functions Defined By Improper Integrals* (29 pages) and *The Method of Lagrange Multipliers* (31 pages), helping to further round out a solid, and legally free, package for instruction.

URA Book Choice 17: Michael Field, *Essential Real Analysis*, 2017, 450 pages. This is another newcomer to the game, with an impressively written preface motivating the book, its choice of style of presentation, its emphasis on applications (as opposed to simply preparing the student for "the next course in analysis") and coverage. The book is

professor in the Polytechnic School [autumn of 1858] in Zurich", thus indicating also when he is referring to.

clearly a bit more advanced than many of the books I list here. I juxtaposed and compared some common topics covered in this book and that by Jacob and Evans: Field moves much faster. This book might be better suited for a second course in real analysis that (i) moves relatively quickly and in more depth through the topics associated with a first course, and (ii) emphasizes multivariate calculus and metric spaces. As such, I include it in Section 7.1 (and perhaps could have included it in Section 7.4).

The review https://www.maa.org/press/maa-reviews/essential-real-analysis from the MAA is very strong, emphasizing some of the nice features of the book; in particular, coverage of topics not usually seen. Excerpts include:

> This is a well written text on Real Analysis that may be used for a course in Advanced Calculus. It can also serve as a reference for advanced topics in Real Analysis.
>
> The text contains very clear and detailed coverage of single and multivariable calculus. In addition, there are numerous examples of how the material is applied. The author includes examples and counterexamples that illustrate the material.
>
> The text covers a rigorous blend of classical and modern analysis. There may be more material than possible to cover in a semester, but there is much to choose from. This is a very well written text by an author who is well versed in the material.

The author, in his extensive preface, does not say how the book can be used. I would say it is appropriate for two courses, namely a first course in real analysis, using chapters 1 to 6 (possibly augmented with material from another, presumably easier, book); and then a second course, covering metric spaces (his chapter 7 on metric spaces is far more extensive than the brief presentations of most beginning books, also containing the Arzela–Ascoli theorem and the contraction mapping principle) and differential calculus on $\mathbb{R}^n$ (his last chapter, 9). Presumably, such a second course would be supplemented with material on multivariate integration, several suitable books for which are discussed below in Section 7.1.

Solutions are not given in the book, nor is there mention of a solutions manual. Notably being a recent book, from Springer, the pdf file is both available and perfect.

URA Book Choice 18: Stephen Abbott, *Understanding Analysis*, 2nd edition, corrected 2nd printing, 2016. This book is prized for being very well written, but it also contains a few topics not addressed in many comparable books, e.g., the generalized Riemann (gauge) integral, Weierstrass's continuous, nowhere-differentiable function (both of those topics also occurring in Bartle and Sherbert), connected sets, the Baire Category Theorem (those two topics not being in Bartle and Sherbert), etc.. Besides (mostly; see below) raving reviews at Amazon, the MAA review is more than enough to provoke severe envy in any author: From https://www.maa.org/press/maa-reviews/understanding-analysis-0, we read:

> This is a dangerous book. Understanding Analysis is so well-written and the development of the theory so well-motivated that exposing students to it could well lead them to expect such excellence in all their textbooks. It might not be a good idea to create such expectations. You might not want to adopt this text unless you're comfortable teaching from a book in which the exposition will nearly always be clearer than your lectures. Understanding Analysis is perfectly titled; if your students read it, that's what's going to happen.
>
> I should also mention that there are plenty of lovely exercises including many of a type I believe are particularly pedagogically potent, to wit: Construct an example of an object (function, sequence, series) that has property X (differentiability, convergence, absolute convergence), but does not have prop-

> erty Y (bounded derivative, reciprocals converge, absolute convergence when squared), or prove that no such object exists.
>
> This terrific book will become the text of choice for the single-variable introductory analysis course.

Returning to the Amazon reviews, some are quite negative. Omitting the ones complaining about the poor quality of the printed book, we have one that says "When I wanted to know a proof of some famous theorem, I opened this book. I didn't find a complete proof but exercises to complete the proof." Several complaints were along these lines: "I was disappointed this book had no solutions in the back." Interestingly (and in support of a book I praise), one person writes "It is an inexpensive book, but Bartle and Sherbert's masterful introduction is well worth 5 times its price."

To give an illustration of how Abbott delegates to the reader proofs and developments (and the understandable ensuing frustration for most readers), consider the very last paragraph of the main text, on page 303:

> As with Dedekind's approach, it can be momentarily disorienting to supplant our relatively simple notion of a real number as a decimal expansion with something as unruly as an equivalence class of Cauchy sequences. But what exactly do we mean by a decimal expansion? And how are we to understand the number 1/2 as both .5000... and .4999...? We leave it as an exercise.

That is no trivial exercise! The next book, by Strichartz, addresses this development early on (chapter 2), and does so in a very detailed, readable fashion. I know which book I would prefer as a student...

URA Book Choice 19: Robert Strichartz, *The Way of Analysis*, Revised Edition, 2000, 739 pages. This is a perfect example of "last but not least": This book choice *was anything but* stuck grudgingly at the end of my list. I hold this book (or, perhaps I should say the author) in very high regard. I use the venue for discussing his book also as an outlet to continue my thoughts, started in Section 2.3, on what constitutes a good mathematics book.

I was not able to locate any book review for this (seemingly reasonably popular) book. The customer reviews at Amazon are overall very positive, and some internet searching reveals that it is the primary book for some analysis courses, e.g., at John Hopkins University (`https://math.jhu.edu/~bernstein/math405/index.html`), and (unsurprisingly, given the affiliation of the author) Cornell University (`http://pi.math.cornell.edu/~rvale/math4130.html`). As seen from the page count, it is a large book, and it covers both univariate and multivariate real analysis (including detailed proofs of the Implicit Function theorem, and the multivariate integral change of variables), metric spaces (chapter 9), ordinary differential equations (chapter 11), Fourier series (chapter 12), as well as the Lebesgue integral (chapter 14). The book is obviously meant to be used for two, possibly three, courses.

A very nice feature of the book is its end-of-chapter summaries. This is not the usual post-chapter text summary typically seen in high school and freshman college textbooks, that everyone anyway skips, but is rather a clear and concise list of the definitions and theorems appearing in the chapter. I am surprised more books do not do this: Or, perhaps those authors still expect the students to do it on their own—a worthy and useful idea, certainly employed 50 and even perhaps 20 years ago, but it is just no longer viable in the internet age in which we have grown accustomed to service, and getting answers instantly. Definitions and theorems are precisely the objects you want to memorize, and you can test your understanding by assessing how much of the proof of each theorem you remember and could replicate. Such a summary is notably of great value in a book like this, in which the author is more chatty than usual. I will elaborate on this chattiness next.

I gave a quote in Section 2.3 from the preface of Strichartz. The last part of it alludes to the author being a bit more verbose than most books (certainly more than the sadistically terse books common in the 20th century) and giving (historical and mathematical) context to concepts. From the preface, we also read:

> My goal in writing this book is to communicate the mathematical ideas of the subject to the reader. I have tried to be generous with explanations. Perhaps there will be places where I belabor the obvious, nevertheless, I think there is enough truly challenging material here to inspire even the strongest students.

Given the aforementioned list of topics, and the length of the book, I think the author is correct about there being "enough truly challenging material here to inspire even the strongest students".

The book's presentation is a bit like the transcript of a (good) lecture. I initially wrote up some example sentences from chapters one and two to illustrate what I mean, but for brevity (noting the size of this document), I removed them, and just say: Go and read some of it yourself. As examples, the author explains about the power set of an infinite set being cognitively difficult for us humans to perceive (making me feel like I am not the only one who cannot easily grasp this concept); and making the reader aware of the existence of the constructionist group in mathematics (and how the Axiom of Choice, for countable and, worse, uncountable sets, "[leads] to a level of non-constructivity that is mind-boggling").

In chapter 2, the real numbers are constructed from Cauchy sequences, with each number being an equivalence class of Cauchy sequences of *rationals*. (This approach is of course in other books, e.g., Tao's *Analysis I*.) I found Strichartz' presentation to be outstanding. This is also because I am of the opinion that a textbook should be able to be "read and understood", without needing too much paper and pencil to deal with the ubiquitous "The reader can show", or "Proof: Omitted", or "Proof: See Exercise 22". Strichartz agrees: From his preface (that we already quoted in Section 2.3):

> This book was written to be read—not deciphered.

While reading his book, I have the impression as though Strichartz (i) is talking to me, trying to teach me, anticipating my questions; and (ii) takes me seriously, not treating me like a moron freshman. To give some explanation of what I mean by the latter statement, he, for example, does not go so far as to make a game out of (like some books do, namely some of the ones I mention at the end of Section 5.1) "I give you an ϵ, and you have to counter with a δ." I am not mocking books that go that far—on the contrary, it is in fact admirable that those authors have the patience (I certainly would not) to go to that level of explanation, and are trying to reach all students who want to (or for some reason have to) learn (or pass a course in) basic real analysis—including, and notably those who, let's just say, probably will not be going on to receive the Fields Medal.

Strichartz assumes the reader is smart enough to understand those simple ϵ–δ type of arguments, and instead concentrates on helping to spell out arguments that can be subtle and problematic, even for talented mathematics students. I give one example: He spends quite some effort (pages 42 to 44) developing the lemmas and proof required to show that $\mathbb{R}$ forms a field, namely the hardest part, that reciprocals exist, i.e., if x is defined by the equivalence class $x_1, x_2, \ldots$, then, for $x \neq 0$, x^{-1} is defined by the equivalence class $x_1^{-1}, x_2^{-1}, \ldots$. This is far from rocket science, and ultimately involves some bookkeeping to stay bounded away from zero, and a few invocations of the triangle inequality, but there are several small steps (notably, it has to hold for all sequences in the equivalence class), and the author spells them out brilliantly, and in such a way that I think he somehow knows exactly what I am thinking (namely about how to sabotage the procedure), and then shows that it indeed works.

At the risk of being even more verbose than Strichartz himself, I wish to emphasize: That above example for x^{-1} is a good case in point of how the reader *can just read the material, and understand it*—of course not like reading a Harry Potter novel—, but slowly, and intently, and even (as I do) adversarially, questioning the validity of everything. This writing style can be contrasted with a book (or oversized pamphlet) consisting of cryptic, bare bones, scaffolding of lists of definitions and results, with the author clearly priding himself on just giving enough definitions so that, with enormous effort, the reader formally could work out everything on her own. Such hilariously terse books, written by a seemingly angry, authoritarian, eccentric math professor with a two-digit social IQ and crumbs in his beard, behooving the reader to sit down and spend days and weeks filling in all the details, ideally struggling and suffering, are long in the dustbin of pedagogic history.

Repeating myself from Section 2.3, *some of that* "suffering" is important to help get the definitions, and techniques of correct proofs, to stick in the student's head, and train the student to do research, but let's not get stupid about the issue: Write a book that an average (not in the population, but self-selected to study mathematics) person can sit and read, and say "I actually understood every piece of this, and how it all fits together, and even enjoyed doing so".

For beginning students (but not just beginners), I think this semi-conversational style, with peripheral commentary and explanations, is highly valuable. I am not against using another book in conjunction with this one, notably a book that is (almost inevitably) more terse, but covers certain aspects and ideas in a different way, and/or that covers additional topics, e.g., as a simple and not overly important example, recall from Section 2.2 that Strichartz does not discuss (the convergence of the series associated with) Euler's constant.

One of the Amazon reviewers said something useful in characterizing the book:

> Most of the standard texts are great as references, but are often not very effective at conveying the material to a student approaching a subject for the first time. Refreshingly, this book does not suffer from this failing; instead, it is packed full of explanation, examples, context, and informational asides. In fact, it is so crammed full of text that often the actual point often becomes lost amid the explanation. This renders the text almost useless as a reference because it requires so much trudging to get to the point. I very much enjoyed some of the insights provided by this book, but ultimately I found it too much of an effort to sort out the central message from all the other fluff; I ultimately abandoned this book for a cleaner treatment.

My thinking is that many students beginning their journey with analysis will greatly appreciate how this book, notably compared to most others, is "packed full of explanation, examples, context, and informational asides"; and as they progress, understand the material, and up their maturity and sophistication, they will naturally want to access more focused books with "a cleaner treatment." (Regarding sorting out the central message from the fluff, note the already mentioned end of chapter summaries, these being a perfect complement to an otherwise chatty book (and an antidote to complaints about it being too verbose).

Another Amazon reviewer writes:

> Most books on mathematics simply dump concepts, equations and examples and let you figure out what to do. Not this one. The book is written in a passionate manner where the author takes pains to explain why we are going in a particular direction and the goals. The style is extremely lucid and informal, something unusual for a subject that is steeped in formal mathematics. Yet the author presents, explains and covers all the formal theorems, concepts

> etc. The book also has excellent exercises. A truly noteworthy achievement. I would highly recommend this to anyone (especially self-study) trying to learn this subject.

and finally (I cannot resist one more, who also mentions Rudin),

> I believe this is how a book aimed to teach should be written. Some of the reviewers believed that the book was too verbose. These folks must be geniuses; I really wonder about their intelligence. I have read Walter Rudin's book on someone's recommendation, it just sucked. Rudin is not a book from which you can learn analysis. It is one of the crappiest books ever written. It probably makes for a good reference but if the material was learnt it is more than likely u will not need a reference. If you are like me self studying analysis you will love this book.

Before closing this discussion, remember: Strichartz also covers multivariate analysis, and the Lebesgue integral—newbies might really appreciate his chatty tone for these subjects.

5.3 Comments

1. All of the first four books (in fact, to some extent, the first five), have elements of traditional first-year basic calculus (not rigorous proofs, but rather motivations, heuristics for proof, basic calculations, and applications), but also rigorous proofs and are genuine "real analysis" books, albeit at the most basic level. This seems to be one new trend in books in this field. Jacob and Evans literally splits the two aspects, with the first part of their book being an excellent calculus refresher (and of course setting the stage for the second half of the book), and the second part being a classic introductory real analysis book.

   Another (now seemingly old, and rather lengthy) book that also has elements of calculus along with rigorous real analysis proofs, suitable for a first course, is Emanuel Fischer, *Intermediate Real Analysis*, 1983, 770 pages. Despite the title, it is indeed for a first course in analysis, as also stated by the author in the preface, where we also read:

   > There are a great deal of books on introductory analysis in print today, many written by mathematicians of the first rank. The publication of another such book therefore warrants a defense. I have taught analysis for many years and have used a variety of texts during this time. These books were of excellent quality mathematically but did not satisfy the needs of the students I was teaching. They were written for mathematicians but not for those who were first aspiring to attain that status. The desire to fill this gap gave rise to the writing of this book.

   Thus, already in the 1970s, the need for more suitable books in analysis was clear. The content is, for the most part, standard for a first course, although does include Bernoulli numbers, the gamma and beta functions, "other elliptic functions and integrals", rudiments of complex numbers, and some other extra material. It is interspersed with elements usually associated with a first course in basic calculus, such as examples and exercises on basic techniques of integration (substitution, integration by parts), but, very admirably and not so often seen, a detailed discussion on the method of partial fractions.

   The pdf file is quite reasonable, given the date of the book, namely it is markable and searchable, but of course not as optically clear and pleasing as are those from newer books. There does not appear to be a second edition, nor an errata list, nor solutions to the numerous exercises throughout the text, making it difficult to justify using this as the primary book. However, as additional reading, I can recommend it.

2. The first eleven books in the list are part of a series, and thus have follow-up books:

   - Lax and (Maria) Terrell's follow up is on "advanced calculus", i.e., multivariate real analysis and an introduction to manifolds;
   - Jacob and Evans have subsequent volumes on both of the aforementioned topics (and plan to have seven books in total);
   - Sasane's is an introduction to functional analysis;
   - Canuto and Tabacco have their *Mathematical Analysis II*, covering topics in multivariate calculus.
   - Ghorpade and Limaye have a book on multivariate analysis ("advanced calculus");
   - Thomson, Bruckner, Bruckner has their follow up, more advanced, book "Real Analysis", covering measure theory and functional analysis;
   - Giaquinta and Modica have several follow up books, *Mathematical Analysis: An Introduction to Functions of Several Variables*, 2009; *Mathematical Analysis: Foundations and Advanced Techniques for Functions of Several Variables*, 2012, which is on measure and integration theory; and *Mathematical Analysis: Linear and Metric Structures and Continuity*, on functional analysis.
   - Zorich has his Mathematical Analysis II, 2016, 2nd edition, which begins with a chapter on metric spaces (and the Contraction Mapping Principle), then vector spaces and linear transforms and (higher order) derivatives (and the General Implicit Function Theorem). Then typical topics in multivariate analysis, e.g., multiple integration, surface integrals, field theory and the heat equation (topics of interest in physics, and also present in Lang's large Undergraduate Analysis book); followed by differential forms and integration on manifolds; then convergence, series and families of functions (including the Arzela–Ascoli theorem), then "Integrals depending on a parameter" (e.g., gamma); then Fourier series and transform, and finally asymptotic expansions (Laplace method). With the appendices, it is about 700 pages.
   - Lebl's *Introduction to Real Analysis, Volume II*, 2020, is about 187 pages "only", with four chapters: Several Variables and Partial Derivatives, One-dimensional Integrals in Several Variables, Multivariable Integral, and Functions as Limits (e.g., Arzela–Ascoli theorem).
   - Garling has his volume II, *A Course in Mathematical Analysis: Volume II: Metric and Topological Spaces, Functions of a Vector Variable*, and volume III, *A Course in Mathematical Analysis: Volume III: Complex Analysis, Measure and Integration.* Each book is about 300 pages.
   - Tao's two books form a sequence; and the second refers to the first. But topic-wise, noting the univariate and multivariate (and other topics) coverage of some other books under study, such as Conway (mentioned in Section 7.1), Field, Strichartz, and Terrell, and also the modest page length of Tao's two books, they could have been joined into one larger book for serving a first and second course in real analysis. (It is arguably more comfortable to have them separated, when using the physical book, though this is irrelevant with a pdf file, in which case, it is nicer to have them combined, and allowing for click-on entries to move around the document.)

William J. Terrell's book, *A Passage to Modern Analysis*, 2019 (about 600 pages) contains the content for both a first and second course in real analysis. The same can be said of Strichartz' rather large book, and also (the yet even larger) Thomson, Bruckner, and Bruckner, *Elementary Real Analysis.*

5.4 Alternative and Supplementary Books

Books such as Bartle and Sherbert, Lebl, Stoll, Trench, and many others from Section 5.2, are excellent, standing out for their pedagogic value, but not for some markedly different aspect, e.g., a particular methodology or writing style. To get a sense of what I mean, I give below a list of analysis books that have a certain "character" or personality about them making them substantially different than (but notice I did not say better than!) the majority of books.

Indeed, to enter the market now with a new book, it must have *something* to make it stand out in a large, tall crowd. If not, one can predict the book reviews. This is the case for example of Joseph L. Taylor (not to be confused with Michael E. Taylor, more than one of whose books appear in this document), with his 2012 book *Foundations of Analysis*, reviewed here, https://www.maa.org/press/maa-reviews/foundations-of-analysis. The book is appealing, notably because it covers both univariate and multivariate analysis (see Section 7 for more detail on the latter), and I like his section 10.5 on the (multivariate) change of variables formula; but, as pointed out by the reviewer:

> So, given that there are lots of books in this area of mathematics and that most of them cover pretty much the same material, it would seem that any new real analysis book should have some sort of distinctive feature to it, in order to set it apart from the pack. The book Real Analysis and Applications: Theory in Practice by Davidson and Donsig, for example, distinguishes itself by offering a number of chapters on very nontrivial applications; Rudin's classic Principles of Mathematical Analysis (aka "Baby Rudin") is legendary for the elegance and succinctness of its prose style (and difficulty of exercises, as I learned the hard way as an undergraduate); Stahl's Real Analysis: A Historical Approach does a splendid job of using history to motivate analysis, and Bressoud's A Radical Approach to Real Analysis also uses history (specifically, the problems encountered in dealing with Fourier series) as a springboard for teaching the subject.
>
> There are also books, such as Strichartz's The Way of Analysis or Abbott's Understanding Analysis, that have a deserved reputation for being vividly written and doing an excellent job of motivating real insight into the subject. (Abbott's book is limited to the single-variable theory, while Strichartz does multivariable and even some Lebesgue theory.)
>
> Unfortunately, the book under review does not have any readily apparent comparable distinguishing feature or "hook". It has what one might expect to find in a book of this nature: clear writing, logical organization, a decent supply of examples and a reasonably broad range of exercises. While there are no obvious defects in the book, there is also nothing particularly memorable about it, and while it would certainly be a satisfactory text for a course on this material, I think the same could probably be said with equal force for about ten or so other books.

As an aside, the pdf file of the book is rather poor (shocking for a recent book—but this is common for books coming from this publisher, namely somehow they apparently prevent perfect pdf files from entering into the web), making it far less enticing to adopt this book as the primary one for a course, knowing full well that students will immediately consult their well-worn bookmarked web page to download the pdf.

Turning now to another book, William R. Wade's *An Introduction to Analysis*, 4th edition, 2010, does not receive such explicit critique of being unremarkable, but the MAA review of it, https://www.maa.org/press/maa-reviews/an-introduction-to-analysis, is noticeably short, and has very little on the positive side to say. The reviews at Amazon are mixed, with few truly informative ones. One seemingly informative and competent review is not positive, and points out several flaws ("gaffes" no less), but I did not confirm these claims. The book is for both univariate and multivariate analysis, and while I did not take the time to read and evaluate it, one might wish to have a look, possibly also to augment those books listed in Section 7.1.

The following is a list of books that have—for better or for worse—a unique selling point and take a different angle on the usual pedagogical approach to real analysis. There is no particular ordering, except that I own and have read (at least parts of) the first eight, and thus can strongly comment on.

1. Donald Estep, *Practical Analysis in One Variable*, 2002, 621 pages. This book, like several others mentioned previously, has elements of a traditional (honors) calculus course, and also a first course in real analysis, and the author in fact says (parts of) it can be and have been used for both types of courses. The book is unique in its approach of being grounded in real applied problems, though still full of detailed proofs. The third and last part, 160 pages and entitled "You Want Analysis? We've Got Your Analysis Right Here" makes it clear that this is still an analysis book, written in a way to make the book attractive to both pure mathematicians and applied researchers, e.g., engineers.

   From the preface, we read:

   > Neither an art nor a science can be taught effectively in the abstract. Concepts and techniques that are perfectly well motivated in practical settings simply become a "bag of tricks" in the abstract. Moreover, technical difficulties often become overwhelming when there are no concrete examples to motivate the issues or provide a compelling reason to spend time on the complications. Too often, the mind lacks the firepower to leap past abstract technical mathematics to imagine how the underlying ideas might be used.
   >
   > Consequently, I present the basic ideas of real analysis in the context of a fundamental problem of applied mathematics, which is approximating solutions of physical models.
   >
   > ...
   >
   > [T]eaching introductory real analysis using a modern abstract approach, even from a beautiful book like Rudin's, is far from optimal. As noted, I have serious doubts as to the effectiveness of an abstract approach to teaching analysis. Moreover, this approach has some serious consequences. First of all, it perpetuates the faulty notion that there is some difference between "pure" analysis and the "dirty" topics important to numerical analysis and applied mathematics. This seeds the prejudices of pure and applied mathematicians that are so unfortunate for mathematics. Moreover, it makes the typical introductory real analysis course unattractive to the brightest students in science and engineering, who could benefit from taking such a course.

   The MAA review https://www.maa.org/press/maa-reviews/practical-analysis-in-one-variable is detailed and positive, saying:

   > The book contains most of the classical topics in real analysis, but they are presented in the context of approximating solutions of physical models, a fundamental problem in applied mathematics. This approach is due mostly to the author's research interests in applied mathematics and numerical analysis. As Donald Estep notes in the Preface: "This book attempts to place the basic ideas of real analysis and numerical analysis together in an applied setting that is both accessible and motivational to young students of all technical persuasions."
   >
   > ...
   >
   > Although the book is not a sequence of theorems followed by proofs (as in a standard text) it does contain quite rigorous proofs, usually placed as motivations, followed by the statements of the theorems.
   >
   > ...

> I confess that when I first started reading this book I was intrigued by the new approach of real analysis but did not quite see what it might be good for. In the end, however, I was convinced that it could be a very good text book, especially in courses taken mostly by engineering majors: I am sure these students would find the approach of the book attractive and motivating.

I think very highly of this book, and it would be high on my list as supplementary material, and I could be easily persuaded to use it as a main text, possibly equally so in conjunction with a second book from the list above. The author kindly responded to my email inquiry about periphery book material (e.g., errata, solutions) that used to be available, with bad news: "Those materials have long been inactive. Sorry." (I presume that will be the inevitable path for all of us, and our books...)

2. Konrad Königsberger, *Analysis I*, 6th edition, 2004. Not apparent from the title is that the book is in German. Even less apparent is that it is one of the books I own and have (partially) read (also to help learn German during my graduate student days). It receives mostly strong reviews at Amazon.de and is among the most popular books used by mathematics students in German-speaking countries. The next volume in the author's series, *Analysis II*, 5th edition, 2004, covers differential equations, differential forms, and the Lebesgue integral.

3. Steven Krantz, *Real Analysis and Foundations*, fourth edition, 2017. Unlike most books, that expand over time, this book started out much more advanced, and lost material over the editions. For example, the preface for the second edition begins by stating: "The book Real Analysis and Foundations, first published in 1991, is unique in several ways. It was the first book to attempt a bridge between the rather hard-edged classical books in the subject—like Walter Rudin's Principles of Mathematical Analysis—and the softer and less rigorous books of today. This book combines authority, rigor, and readability in a manner that makes the subject accessible to students while still teaching them the strict discourse of mathematics".

   In going from the 2nd to the 3rd edition, among other removals, we read from the preface: "We have removed the chapter on wavelet theory, as it is truly beyond the scope of a typical real variables course. We have removed the material on measure theory because it is just too difficult. We have removed the material on differential forms as it is really best suited to a more geometric course."

   In the fourth edition preface, we read: "In this new edition we endeavor to make the book accessible to a broader audience. We do not want this to be perceived as a "high level" text. Therefore we include more explanation, more elementary examples, and we stepladder the exercises. We update and clarify the figures. We make the sections more concise, and omit technical details which are not needed for a solid and basic understanding of the key ideas. ... In the same spirit, we have eliminated Chapter 13 on advanced topics and Chapter 14 on normed linear spaces. These are very attractive sets of ideas, but are probably best treated in a more advanced course"

   I own the 2nd edition, and recall enjoying reading parts of it. It was reviewed here, `https://www.maa.org/press/maa-reviews/real-analysis-and-foundations`.
   Krantz is a prolific and excellent author, and this book can certainly enter the running for use in a first course in real analysis—but with so many other good choices, and the bizarre history of this book, it might best serve as secondary reading, notably if a follow-up course uses Krantz' *Elementary Introduction to the Lebesgue Integral*, 2018, mentioned below. Before leaving Krantz, I also note another book from him (he has dozens), *Convex Analysis*, 2015, a topic of sure interest for students pursuing machine learning and optimization.

4. Serge Lang, *Undergraduate Analysis*, 2nd edition, 1997. Serge Lang writes (wrote) in a certain style that is unique and recognizable, arguably very elegant, but not necessarily

overly terse, though I cannot formulate in words how to characterize it.[37] I hold this book in high regard for its clarity and style, though it is a large book suitable for two full courses. In its favor (as possibly a supplement) is that it has some outstanding content, and there is also a separate entire book containing all the solutions to the exercises.

5. Beardon, 1997, *Limits: A New Approach to Real Analysis* uses the idea of limits and directed sets to prove nearly everything in basic analysis. I found this book to be very enjoyable and informative, but I would not use it as the core book.

6. Pugh, *Real Mathematical Analysis* is noticeably different in terms of structure and content than competing books, and, while formally for a first course, and not advanced, can be enjoyed best after having had a basic course in real analysis and some multivariate aspects.

7. Browder, *Mathematical Analysis: An Introduction*, 1995, starts off basic, but is more advanced and terse than the usual books. It includes a chapter on measures, and chapters on differential forms and integration on manifolds (and Brouwer's fixed point theorem). Thus, previous exposure to (univariate and multivariate) real analysis is highly recommended to appreciate this book. The presentation is first rate, albeit far too fast, and deep, thus (in my opinion) rendering it to be a fantastic book to study after the reader has seen most of the material already.

8. Omar Hijab, *Introduction to Calculus and Classical Analysis*, 4th edition, 2016. From the preface,

   > Now, every mathematician knows that analysis arose naturally in the nineteenth century out of the calculus of the previous two centuries. Believing that it was possible to write a book reflecting, explicitly, this organic growth, I set out to do so.
   >
   > I chose several of the jewels of classical eighteenth- and nineteenth-century analysis and inserted them near the end of the book, inserted the axioms for reals at the beginning, and filled in the middle with (and only with) the material necessary for clarity and logical completeness. In the process, every little piece of one-variable calculus assumed its proper place, and theory and application were interwoven throughout.

   Hijab has a similarity of sorts to Ghorpade and Limaye, and also Beardon's book mentioned above, and Little, Teo, van Brunt, mentioned below, namely, from the preface, Hijab writes:

   > Continuous limits are defined in terms of limits of sequences, limits of sequences are defined in terms of upper and lower limits, and upper and lower limits are defined in terms of sup and inf. Everybody thinks in terms of sequences, so why do we teach our undergraduates ϵ–δ's?"

   Further, his treatment of integration is highly non-standard at this level:

   > The second feature is the treatment of integration. We follow the standard treatment motivated by geometric measure theory...

[37] Some time after having written this, I was delighted to have found someone else remarking on Lang's style, namely Mark Hunacek from Iowa State University, writing a review for the recent book by Michael Taylor (*Linear Algebra*, 2020). At the end, Hunacek says

> ... this book reminded me of some of the books by another prolific author, Serge Lang, that I have read over the years: interesting, idiosyncratic, informative, and insightful, but probably more likely to be enjoyed by instructors than by their students.

I could not have said (and did not say) it better.

> The third feature is the treatment of the theorems involving interchange of limits and integrals. Ultimately, all these theorems depend on the monotone convergence theorem which, from our point of view, follows from the Greek mathematicians' Method of Exhaustion. Moreover, these limit theorems are stated only after a clear and nontrivial need has been elaborated.
>
> The fourth feature is the use of real-variable techniques in Chapter 5. We do this to bring out the elementary nature of that material, which is usually presented in a complex setting using transcendental techniques.

The online bookseller Amazon states for this book (of course, judicious and flattering) pieces of official reviews, such as by Steven G. Krantz (author of *Real Analysis and Foundations*, as discussed above; and *Elementary Introduction to the Lebesgue Integral*, 2018; discussed below), who writes "Chapter 5 is ... an astonishing tour de force...".

There is an MAA review, https://www.maa.org/publications/maa-reviews/introduction-to-calculus-and-classical-analysis-0, for the book's second edition, from which we read:

> The last chapter is full of interesting applications, among them Euler's gamma function, Stirling's approximation of $n!$, infinite products, Jacobi's theta function, and Riemann's zeta function.
>
> I find the unusual features in this book interesting, but I would be very reluctant to impose them on students in an undergraduate one-variable analysis course. I especially feel that doing measure-theoretic area, in order to develop integration, is inappropriate at this level. Also, there are too many personalized definitions. Finally, I object to books that provide answers to all of (or none of) the exercises; instructors should have a choice, with answers provided to some of the exercises.

My reading of it suggests that, while it is still clearly an undergraduate analysis book, it is significantly more sophisticated than the "entry level" books on the subject. It also has several sections explicitly dedicated to applications, making the book unique (though see Davidson and Donsig below). The author's impressive chapter 5 (beginning on page 193), has nine sections of applications, followed by the last chapter 6, "Generalizations", containing further advanced topics such as measurable functions and limit theorems (e.g., Summation Under the Integral Sign, Dominated Convergence Theorem, Continuity Under the Integral Sign, Differentiation Under the Integral Sign), among other topics.

Hijab kindly provides the solutions to the exercises at the back of the book (a feature I admire, unlike the reviewer above). His book is cited, for example, for a proof of exchange of derivative and integral, on page 103 of Hamza Alsamraee's 2019 book *Advanced Calculus Explored* (the book being a very nice but unconventional tour of univariate advanced calculus, and the author being a teenager!). (The exchange of derivative and integral is also discussed at length in Lang's *Undergraduate Analysis*, as discussed above, in section X.7, XIII.3, and XVII.8.)

I think Hijab is a unique and special book that can be used to supplement a course with its chapter 5, and also other parts of the book, notably on integration, as a preparation for a deeper study of Lebesgue measure and integration. The pdf file of the book is also perfect.

9. Kenneth R. Davidson and Allan P. Donsig, *Real Analysis and Applications: Theory in Practice*, 2010, which moves very fast through traditional theory, and then, similar to but even more so than Hijab's book, turns to applications—60% of the book in fact. An Amazon reviewer says something one might have expected, namely:

> One thing to note is that this book is probably not the best choice for a first course on analysis. There are many books that focus much more on the basics

where as this one gets them out of the way pretty quickly and moves on the bigger and better things.

10. Karl R. Stromberg (1931-1994), *An Introduction to Classical Real Analysis*, 1981, 577 pages; reprinted by American Mathematical Society, 2015. One "unique selling point" of this book is simply that it is a classic. It also skips Riemann integration and introduces Lebesgue first, which should easily qualify to be unique. From the preface:

    > One significant way in which this book differs from other texts at this level is that the integral which we first mention is the Lebesgue integral on the real line. There are at least three good reasons for doing this. First, the F. Riesz approach (after which mine is modelled) is no more difficult to understand than is the traditional theory of the Riemann integral as it currently appears in nearly every calculus book. Second, I feel that students profit from acquiring a thorough understanding of Lebesgue integration on Euclidean spaces before they enter into a study of abstract measure theory. Third, this is the integral that is most useful to current applied mathematicians and theoretical scientists whether or not they ever study abstract mathematics. Of course, it is clearly shown in Chapter 6 how the Riemann integral is a special case of the Lebesgue integral. Stieltjes integration is presented in a graded sequence of exercises.

    Here is (almost the entirety of) the review from Ian S. Murphy (1982), *Proceedings of the Edinburgh Mathematical Society*, 25(1), pp. 105-6.

    > For students taking a rigorous course the book discusses series in the setting of complex numbers and develops the exponential and trigonometric functions as sums of complex power series. Use of certain ideas (e.g. the number π) is scrupulously avoided until they have been formally presented in the text. The theory of metric spaces is treated in considerable detail in Chapter 3 with discussions of uniform convergence and the Stone-Weierstrass and Ascoli theorems. A novel idea is the introduction in Chapter 6 of the Lebesgue integral as a first integral on the real line.
    >
    > For the mathematically mature the book contains a host of interesting results, historical references and exercises. The author says in his preface that he spent at least three times as much effort in preparing the exercises as he did on the main text. As a result there are hundreds of eye-catching exercises, many with subdivisions and hints. Some are routine, some lead up to a main result (e.g. that the number e is transcendental), some bear twentieth century surnames, some offer alternative methods, some introduce and illustrate concepts not dealt with in the main text (e.g. Fourier transforms, Lagrange multipliers, Bernoulli numbers, Lambert series). The exercises of the last three chapters on Integration, Infinite series and products and Trigonometric series are particularly impressive.
    >
    > Students of Analysis will find the book inspiring but may also find it somewhat inhospitable in places: a case in point is the definition of continuity, another is the treatment of radius of convergence and the examples on it. Nevertheless the overall impression is of a fine achievement which will have appeal for analysts everywhere.

    Even more positive is the review from Alberto Torchinsky (1983), *American Mathematical Monthly*, 90(4), pp. 294-5. Again almost in its entirety, we read:

    > Skip the bills, exams, reviews and other distractions at hand, play one of Hindemith's wood-wind sonatas (the Beatles will also do), and, most importantly, read *An Introduction to Classical Real Analysis*, Karl Stromberg's expert work

of art and love, in the basic subject described by its title. Stromberg condenses over twenty years of teaching experience in an attractive treatise meant for those students who, having survived our merciful attempts to teach them Calculus without ϵ's and δ's, want to know how things can be really made to work. Such a book is especially welcome at this introductory level, for the main texts in print are the second editions of Apostol and Rudin, written in the midseventies. Both of these authors made a significant contribution to the study of Classical Analysis, or Advanced Calculus as the subject is sometimes referred to, and their highly successful texts do provide a solid preparation for undergraduate and beginning graduate students learning complex and abstract analysis. In fact, the clarity of their presentation affords in a sense an invitation to the students to "read through" the book. Stromberg's book, on the other hand, while just as clear as those of Apostol and Rudin, advances a somewhat different point of view: the author insists that to learn Mathematics (and this subject in particular) hard work is necessary, and he incorporates into the flow of the presentation the solving of sets of exercises, some supplied with copious hints. Most of the exercises represent assertions to be proved and lead the students to important and interesting results; this is an attractive asset.

Readers will have no difficulty in deciding that in the balance of rigor vs. intuition, the author has chosen an extremely rigorous approach which opens with a set of axioms introducing the real numbers and culminates with a discussion of the growth rate of partial sums of Fourier series. A safe assumption is made, namely that students know no mathematics, and a point of view is advanced, namely that no concept will be used until it is properly introduced; thus π is not mentioned until Chapter 5. The emphasis (and in some parts, as in the convergence tests for numerical series, it is heavily laid) is on the *classical* aspect of the theory. For instance, although the concepts of abstract topological and metric spaces are indeed introduced in Chapter 1, the book promptly reverts to real Euclidean space for examples, motivation, and development. Complex valued functions are also considered, but the theory of analytic functions is not discussed. As for functions of several real variables, although the Fubini-Tonelli theorem and formulas such as the change of variables involving Jacobians are given, the theorems of Stokes and Green are not covered. Students in engineering and other applied areas will be pleasantly surprised by a significant simplification, if that is the right word, which Stromberg proposes. The Lebesgue integral is introduced before other notions of integral. To support this concept Stromberg advances three reasons; I add two of my own: it saves time and it works well.

The different chapters that comprise the book can be read independently, the exercises are challenging and fun, and the content is supported by the clear, concrete, and correct mathematical thinking (in both level and presentation) that is such a crucial component of the book. Through the chapters students can learn about the number system, sequences, and series, limits and continuity, differentiation, the elementary transcendental functions, integration (and its main applications including L_p spaces, and some differential calculus in several real variables), infinite series and products, and trigonometric series. Stromberg indicates to the prospective teacher how to program a two-semester, or three-quarter, course by marking in the index with a single asterisk sections that may be omitted if time presents a problem and with a double asterisk interesting and useful applications of the theory.

I feel this book will become a source of inspiration for teachers and students of classical analysis. It has become one of my reference books and one of my

> first choices for the kind of course where students want to see the proofs of the theorems, and consequently the whole theory, come alive.

The last review we mention is our usual, namely from MAA, https://www.maa.org/press/maa-reviews/an-introduction-to-classical-real-analysis, from 2015. In addition to commenting on the impressive exercises, we read

> The narrative part of the book is well-done too, and contains many interesting things, but it is not such a standout as the exercises. The term "classical" in the title indicates that the book is slanted towards the concrete and has quite a lot on properties of particular series and integrals. In olden days it might have been titled Advanced Calculus, although it doesn't go very far into multi-variable calculus. In modern terms it is a text for a first rigorous course in mathematical analysis. This is a very competitive field, and as the present book was written in 1981, you would expect some better texts to have come out since then. Considering only the narrative part and not the exercises, I think Ross's *Elementary Analysis* would be a better choice for most courses. The present book is more advanced in some aspects, and in particular it develops the Lebesgue integral rather than the Riemann integral (through step functions rather than measure). Despite the slant towards concreteness, it does prove results in more generality when it's not much harder to do so. For example, it proves the Stone-Weierstrass theorem in its full generality rather than the Weierstrass approximation theorem.
>
> If your interest is primarily in the classical analysis aspects rather than rigorous analysis, there are some recent samplers that are valuable and interesting: Duren's *Invitation to Classical Analysis* and Chen's *Excursions in Classical Analysis*.

We have met Ross' book in Section 5.1, and we now turn to Duren, and then Chen.

11. Peter Duren, *Invitation to Classical Analysis*, 2012, 392 pages. This book is even more extreme in turning quickly to applications, and thus forms a nice trio with Hijab, and Davidson & Donsig. The first chapter is full review of elements from (a first course in) real analysis, while chapters 2 and 3 have some review, and some applications. Chapters 4 through 14 are all applications. The author states in the preface that "the reader is assumed to have acquired a good command of basic principles" from (but not necessarily more than) a first course in real analysis, and also emphasizes "its focus on functions of a single real variable". Measure theory and Lebesgue integration are not needed. The author motivates matters best in his preface, from which we excerpt:

> [I]t is designed for students who have learned the basic principles of analysis, as taught to undergraduates in advanced calculus courses, and are prepared to explore substantial topics in classical analysis. And there is much to explore: Fourier series, orthogonal polynomials, Stirling's formula, the gamma function, Bernoulli numbers, elliptic integrals, Bessel functions, Tauberian theorems, etc. Yet the modern undergraduate curriculum typically does not encompass such topics, except perhaps by way of physical applications. In effect the student struggles to master abstract concepts and general theorems of analysis, then is left wondering what to do with them.
>
> It was not always so. Around 1950 the typical advanced calculus course in American colleges contained a selection of concrete topics such as those just mentioned. However, the development could not be entirely rigorous because the underlying theory of calculus had been deferred to graduate courses. To remedy this unsatisfactory state of affairs, the theory of calculus was moved to the undergraduate level. Textbooks by Walter Rudin and Creighton Buck helped transform advanced calculus to a theoretical study of basic principles.

> Certainly much was gained in the process, but also much was lost. Various concrete topics, natural sequels to the abstract theory, were crowded out of the curriculum.
>
> The purpose of this book is to recover the lost topics and introduce others, making them accessible at the undergraduate level by building on the theoretical foundation provided in modern advanced calculus courses. My aim has been to develop the mathematics in a rigorous way while holding the prerequisites to a minimum. The exposition probes rather deeply into each topic and is at times intellectually demanding, but every effort has been made to include the background necessary for full comprehension. It is hoped that undergraduate students (and other readers) will find the material exciting and will be inspired to make further studies in the realm of classical analysis.

The MAA review is short and positive, https://www.maa.org/press/maa-reviews/invitation-to-classical-analysis, beginning with:

> This is a concise, very clearly-written undergraduate textbook in classical analysis, that includes a very broad selection of the most important theorems in the subject. The content and approach are severely classical, and in fact this book could have been written seventy-five years ago. (I am not complaining. No one did write it seventy-five years ago, and I am glad we have it at last.)

12. Hongwei Chen, *Excursions in Classical Analysis*, 2010, 320 pages. Chen jumps straight into applications: The book is more of a "problems and solutions" collection, and could have been placed in Section 5.5 below. From the preface, we learn a bit more of what to expect, and also about what is expected of the reader:

> The book aims to introduce students to advanced problem solving and undergraduate research in two ways. The first is to provide a colorful tour of classical analysis, showcasing a wide variety of problems and placing them in historical contexts. The second is to help students gain mastery in mathematical discovery and proof. Although one proof is enough to establish a proposition, students should be aware that there are (possibly widely) various ways to approach a problem. Accordingly, this book often presents a variety of solutions for a particular problem.
>
> This book is accessible to anyone who knows calculus well and who cares about problem solving. However, it is not expected that the book will be easy reading for many math students straight out of first-year calculus. In order to proceed comfortably, readers will need to have some results of classical analysis at their fingertips, and to have had exposure to special functions and the rudiments of complex analysis. Some degree of mathematical maturity is presumed, and upon occasion one is required to do some careful thinking.

This extends the aforementioned trio of books to four, perhaps best dubbed the "four horsemen of books giving applications from classic real analysis". The MAA review (from Henry Ricardo, whom we met in Section 3.2) https://www.maa.org/press/maa-reviews/excursions-in-classical-analysis remarks positively:

> Chen's book is a wonderful updated tour of classical analysis and would serve as an excellent source of undergraduate enrichment/research problems.

By "updated", Ricardo presumably means that Chen makes some use of computer algebra systems, whereas Duren and the other books mentioned above do not. Chen writes in the preface:

> With the continuing increases of computing power and accessibility, experimental mathematics has not only come of age but is quickly maturing. Ap-

> pealing to this trend, I try to provide in the book a variety of accessible problems in which computing plays a significant role

The review also mentions the related, and also highly praised *Excursions in Calculus: An Interplay of the Continuous and the Discrete* by Robert M. Young, 1996 (a book I did not know about until now, and wish I discovered much earlier).

13. Charles H. C. Little, Kee L. Teo, and Bruce van Brunt, *Real Analysis via Sequences and Series*, 2015. As its name suggests, this book has a bit of similarity to that from Beardon, though this one appears to be more suitable as a textbook. Chapter 9 is a highlight, discussing Newton's method, including error estimates, and then the wider scope of fixed-point problems. This book (or at least parts of it) would be wise reading before embarking on numerical analysis (see Sections 4.6 and 8). Also, as the authors emphasize in the preface,

> This approach not only has the merit of simplicity but also places the student in a position to appreciate and understand more sophisticated concepts such as completeness that play a central part in more advanced fields such as functional analysis.

14. James W. Cannon, *Two-Dimensional Spaces, Volume 1: Geometry of Lengths, Areas, and Volumes*, 2017, 119 pages. This book, while definitely meant to be instructional, and containing proofs and exercises, is not a traditional textbook. It is charming, well written, short, has many graphics, and would be a great companion to any student embarking on the study of more serious mathematics, e.g., a first course in real analysis. From the preface,

> An undergraduate student with a reasonable memory of calculus and linear algebra, but with no fear of proofs, should be able to understand almost all of the first volume. A student with the rudiments of topology—open and closed sets, continuous functions, compact sets and uniform continuity—should be able to understand almost all of the second volume with the exception of a little bit of algebraic topology used to prove results that are intuitively reasonable and can be assumed if necessary. The final volume should be well within the reach of someone who is comfortable with integration and change of variables. We will make an attempt in many places to review the tools needed.

With respect to volume I, the author writes:

> This volume is suitable for undergraduates who understand calculus and linear algebra and who want to understand a number of those beautiful results usually quoted to the undergraduate without proof. It explains an entire string of results that teased me as an undergraduate because they were stated without proof. I sorely wanted to understand why they were true. This book is written for the "me" who was a young college student.

The MAA review `https://www.maa.org/press/maa-reviews/geometry-of-lengths-areas-and-volumes` is clearly very positive, writing:

> The books overflow with mathematical charm. Many readers will be hooked by Cannon's aesthetics and proof exposition, where geometric intuition and topological arguments play leading roles.

It does however point out some issues and complaints. The pdf of the first of three books can be found on the internet, and is of perfect quality.

15. Ralph P. Boas Jr., revised and updated by (his youngest child) Harold P. Boas, *A Primer of Real Functions*, 4th edition, 1996. This is an unusual yet widely known and highly regarded text on real analysis because it is conversational and, dare I say, fun. It is suitable for a first course, and a bit beyond. The fourth edition adds a chapter on integration and Lebesgue measure, and still results in a short and readable book of 280 pages. It is a charm to read, and also includes solutions to exercises. Who would not want to read a book with the following text from the preface?

> I. To the beginner. In this little book I have presented some of the concepts and methods of "real variables" and used them to obtain some interesting results. I have not sought great generality or great completeness. My idea is to go reasonably far in a few directions with a minimum amount of special terminology. I hope that in this way I have been able to preserve some of the sense of wonder that was associated with the subject in its early days but has now largely been lost. I hope also that someone who has read this book will be able to go on to one of the many more forbidding systematic treatises, of which there is no lack.
>
> No previous knowledge of the subject is assumed, but the reader should have had at least a course in calculus. In general, each topic is developed slowly but rises to a moderately high peak; a reader who finds the slope too steep may skip to the beginning of the next section.
>
> II. To the expert. Experts are not supposed to read this book at all; ...

See also the positive MAA review, https://www.maa.org/press/maa-reviews/a-primer-of-real-functions.

16. Richard Beals, *Analysis: An Introduction*, 2004. I like this book, and will be making time to read all of it. As one of many examples, on page 8, when giving the Archimedean property (If r and s are positive rationals, then there is a positive integer N such that $Nr > s$), we subsequently read:

> (If we think of s as the amount of water in a bathtub and r as the capacity of a teaspoon, this says that we can bail the water from the bathtub with the teaspoon in at most N steps. Of course N may be large.)

While the book is certainly more terse than the usual crop of first-course books, it is impeccably and magnificently written, making me reflect on my own mathematics and writing skills—conveniently already packaged (again) better than I could have said it, from Harry Callahan (played by Clint Eastwood, in *Magnum Force*, the 1973 sequel to *Dirty Harry*): "*A man's got to know his limitations.*"

The purpose of the book, and for whom it is designed, is given in the preface. We read there (the very accurate statement)

> This text contains material for a two- or three-semester undergraduate course. The aim is to sketch the logical and mathematical underpinnings of the theory of series and one-variable calculus, develop that theory rigorously, and pursue some of its refinements and applications in the direction of measure theory, Fourier series, and differential equations.

followed by (the arguably less accurate statement)

> A good working knowledge of calculus is assumed. Some familiarity with vector spaces and linear transformations is desirable but, for most topics, is not indispensable.

I claim, to appreciate the second half of this book, the reader will require having had at least a first course in linear algebra and, crucially, at least a first course in real analysis. Formally, the author covers what is needed in the first half, but realistically, for most students, this would not be enough for a first exposure.

Indeed, the book starts modestly, with things like how to prove that two sets are equal (page 12), and on page 18, the author proves things like *If a and b are positive, then the sum $a + b$ and the product ab are positive*; and *If* $0 < a < b$*, then* $0 < 1/b < 1/a$.

Up until, and including, chapter 8, much material is standard, though includes sequential compactness and the Weierstrass Polynomial Approximation Theorem. Chapter 8, on "Calculus" is deceptively basic, being a short and fast 20 page coverage of standard topics such as differential calculus, Riemann sums, and Taylor's theorem. Then, in chapter 9, we find, among other things, the Fundamental Theorem of Algebra. Chapters 10 and 11 provide a relatively short and highly readable introduction to Lebesgue measure and integration, followed by 12 (function spaces), 13 and 14 (Fourier Series and Applications), and 15 (ordinary differential equations), finishing at about 240 pages, plus 15 for hints for some exercises.

Concentrating on the material beginning in chapter 10, as another example of what I like about this book, look at his (page and a half) section 10.A, in which he *actually motivates* why we would care about integrating the indictor function on $[0, 1]$, 1 if x is rational, 0 if irrational. He links this geometric question to the theorem of Banach and Tarski (the Banach-Tarski Paradox), with a sketch of the proof of it provided in an appendix. Chapter 10 starts on page 131, and by page 155, we arrive at (the end of chapter 11 and) Lebesgue's Dominated Convergence Theorem (along with the Monotone Convergence Theorem and Fatou's Lemma).

From `https://www.maa.org/press/maa-reviews/analysis-an-introduction`, the MAA review, we learn that it "is a serious textbook for serious students. Intended for advanced undergraduates, this book demands as much personal maturity from the reader as it does mathematical sophistication." Further,

> While, technically speaking, this book could be used for a first course in analysis, the title is perhaps something of a misnomer. The important introductory concepts are all discussed, precisely and completely, but often as a stepping-stone to more sophisticated results.
>
> Beals' writing style is characterized by a certain austere elegance. The author has an admirable command of the English language, and he appears unaffected by the excessive informality that has afflicted so many undergraduate textbooks.
>
> [It] is most appropriate for an undergraduate who has already grappled with the main ideas from real analysis, and who is looking for a succinct, well-written treatise that connects these concepts to some of their most powerful applications. Beals' book has the potential to serve this audience very well indeed.

More superficially, the available pdf is "perfect", and the layout is typical CUP—namely beautiful, making the text an optical joy to read. This book, among surely others I mention in this list of "books with a special character", could be used as supplementary reading. I also include it in the list of books in Section 7.3.

17. Michael E. Taylor, *Introduction to Analysis in One Variable*, 2020, 275 pages; and *Introduction to Analysis in Several Variables: Advanced Calculus*, 2020, 440 pages. The first volume is notably short, and moves quickly through standard topics to get to, after 190 pages, chapter 5, entitled "Further topics in Analysis", containing sections: Convolutions and bump functions, The Weierstrass approximation theorem, The Stone-Weierstrass theorem, Fourier series, Newton's method, and Inner product spaces. This

is followed by an appendix entitled "Complementary results", with sections: The fundamental theorem of algebra, More on the power series of $(1-x)^b$, π^2 is irrational, Archimedes' approximation to π, Computing π using arctangents, Power series for $\tan x$, Abel's power series theorem, and Continuous but nowhere-differentiable functions. This is arguably a bit like Hijab's book, with his impressive chapter 5, beginning on page 193, with 9 sections of applications, followed by the last chapter 6, "Generalizations".

There is a perfect pdf file of the first book, but for the second book, only a file that is "sort of readable" but clearly corrupted. It is enough to see the table of contents and see all the pages to asses its contents. The book seems rather advanced for this level, such as chapter 6, Differential geometry of surfaces, including the Gauss-Bonnet theorem.

18. Roger Godement, *Analysis I: Convergence, Elementary functions*, 2004, 430 pages, translated from French. This book easily deviates from the typical book on real analysis—the preface is already several pages long, delving into (admittedly interesting) discussions of politics. From the book's description at Amazon, "... the author emphasizes ideas over calculations and, avoiding the condensed style frequently found in textbooks, explains these ideas without parsimony of words. "

    As one Amazon reader writes (in the 4-star review), "The author will often introduce historical asides or personal views which generally are an entertaining complement to the material under discussion. At its best the author's approach results in inspiring and insightful reading, while at its worst it can seem disjointed and rambling. Fortunately, the positives far outweigh the negatives in most sections. 'Entertaining' is not generally used to describe mathematical exposition on topics at this level, but Godement's idiosyncractic style makes it seem appropriate."

    Nick Lord, in his review published in *The Mathematical Gazette*, 89 (514) (2005), 152-153, says it perhaps best:

    > Although the content is 'elementary', there are several reasons why I do not think this is an introductory book. There are no exercises (other than 'complete the details' type) and beginners will find it quite hard to sift the key results from the commentary in order to navigate a route through the book. It is much more likely to find a resonance with those thoroughly familiar with the material who will respect Godement's lifetime of reflection on the material and fully appreciate his more teasing remarks.

    The MAA review https://www.maa.org/press/maa-reviews/analyis-i-convergence-elementary-functions states:

    > To say Analysis I and Analysis II are idiosyncratic is not to do them justice. Roger Godement has given us a unique analysis text, bursting at the seams with beautiful and serious mathematics presented in a very unusual way and serving at the same time as a venue for historical commentary and extensive criticism of e.g. the American military-industrial complex.
    >
    > ...
    >
    > On to the mathematics, then, and in a word, it's superb. Analysis I and Analysis II, actually the first of four books, are encyclopedic in scope and are filled with marvelous and expansive expositions.
    >
    > ...
    >
    > Accordingly the long books read like a conversation with an expert who not only loves his subject but talks about it brilliantly.
    >
    > ...
    >
    > While these books should probably be kept far away from a randomly chosen beginning upper division student, lest he fail to stay "on sequence," they are truly magnificent sources for a deeper study of analysis.

> ...
>
> The reader, whoever he might be, whether an advanced undergraduate or a veteran, will finish these pages with a deeper understanding of the subject and an increased sense of mathematical culture.

19. Ethan D. Bloch, *The Real Numbers and Real Analysis*, 2011. This book covers the standard material (and appears very well written), but goes deeper into the real numbers, e.g., chapter 1 is on the construction of the real numbers, and chapter 2 is on properties of the real numbers.

20. John Stillwell, *The Real Numbers: An Introduction to Set Theory and Analysis*, 2013. This is a more advanced book emphasizing set theory, e.g., ordinals, axiom of choice, Borel sets, and measure theory. Online book reviews suggest it is excellent, and a reading of the preface indicates very strong and appealing authorship.

21. Thomas W. Körner, *A Companion to Analysis: A Second First and First Second Course in Analysis*, 2003. The clever title indicates well the positioning of the book, making is clear that it is somewhat more advanced than other the books listed above, but could serve well as a supplementary reading. Note also the book by the same author, *Vectors, Pure and Applied: A General Introduction to Linear Algebra*, mentioned in Section 4.3.

22. David M. Bressoud, *A Radical Approach to Real Analysis*, as one of several books from the author taking a different, namely historical approach to the development of analysis.

23. Saul Stahl, *Real Analysis: A Historical Approach*, 2nd edition, 2012. I read and enjoyed the first edition (1999), which is also favorably reviewed at MAA, https://www.maa.org/press/maa-reviews/real-analysis-a-historical-approach.

24. Mark Bridger, *Real Analysis: A Constructive Approach Through Interval Arithmetic*, 2007, 2019, Reprinted with corrections. As the title suggests, all proofs are constructive. From the preface,

    > In summary, then, this is neither a text in numerical analysis nor one intended solely to prepare students to be professional mathematicians. It is a thoroughly rigorous modern account of the theoretical underpinnings of calculus; and, being constructive in nature, every proof of every result is direct and ultimately computationally verifiable (at least in principle). In particular, existence is never established by showing that the assumption of non-existence leads to a contradiction. By looking through the index or table of contents, you'll see that nothing of importance for undergraduates has been left off or compromised by our approach. The payoff of the constructive approach, however, is that it makes sense–not just to math majors, but to students from all branches of the sciences.

    Perhaps Bridger was influenced by Harold Edwards: See, e.g., Edwards' 2005 *Essays in Constructive Mathematics*, where some history of this issue is discussed, and a better understanding of its underlying idea is given (as well as dispelling false ideas, e.g., constructivists do not accept the validity of proof by contradiction).

    Given that we have computers and they are since years now used ubiquitously by everyone including mathematicians—that is, we can compute things in a necessarily finite number of steps, and that, using (a subset of the) rational numbers—, it might make good sense to have future generations of mathematicians and scientists familiar with the constructivist approach.

25. John M. Erdman, *A Problems Based Course in Advanced Calculus*, 2018. As the title suggests, most results are posed as problems, to be worked out and proven by the student. Such a book could be of use for a student to test her skills, but perhaps during

or after having used a typical textbook, with detailed proofs, to learn the material in the more usual (and what I presume the author would say, passive) approach.

26. Robert R. Reitano, *Introduction to Quantitative Finance: A Math Tool Kit*, 2010, 709 pages. This seems out of place here, based on the title, but the contents is primarily material suited for a first course in real analysis (and not at all watered down), along with some very basic topology for metric spaces, and some elementary probability theory—all interspersed with applications in finance. This would be a serious choice, as either the main book, or supplementary, if you are in a finance department (as I am), and the director told you "we need to teach a course in analysis, but it absolutely, positively, must have relations to finance throughout." I discuss this book a bit more in Section 6.4.

5.5 Books of Problems and Solutions and Counterexamples

There is a niche of books consisting of solved problems and/or counterexamples. They cannot serve as stand alone texts, nor were they meant to, but most provide brief reviews of the material. This idea goes back to the famous books by George Pólya and Gabor Szegö, *Problems and Theorems in Analysis* (originally *Aufgaben und Lehrsätze aus der Analysis*, 1925).

One or more of the following would be an excellent complement to a mainstream book:

1. Teodora-Liliana Radulescu, Vicentiu Radulescu, and Titu Andreescu, *Problems in Real Analysis: Advanced Calculus on the Real Axis*, 2009. This book is for "mathletes", giving problems and solutions in real analysis. The online reviews are very strong for this book.

2. Andrei Bourchtein and Ludmila Bourchtein, *CounterExamples: From Elementary Calculus to the Beginnings of Analysis*, 2015. The book looks excellent.

3. Sergiy Klymchuk, *CounterExamples in Calculus*, 2010. In the preface, the author acknowledges Gelbaum and Olmsted, 1964, "a well-known resource for advanced calculus and analysis courses", with Klymchuk's book being similar but for calculus / beginning real analysis. The preface goes on to indicate one of the scores of examples:

   > Consider the following theorem from first-year calculus.
   >
   > If a function $f(x)$ is differentiable on (a, b) and its derivative is positive for all x in (a, b), then the function is increasing on (a, b).
   >
   > The following two statements look quite similar to this theorem, but both are incorrect:
   >
   > If a function $f(x)$ is differentiable on (a, b) and its derivative is positive at a point $x = c$ in (a, b), then there is a neighborhood of the point $x = c$ where the function is increasing;
   >
   > If a function $f(x)$ is differentiable on its domain and its derivative is positive for all x from its domain, then the function is increasing everywhere on its domain.

4. Masayoshi Hata, *Problems and Solutions in Real Analysis*, 2007. Also excellent: You will wish you knew everything shown in this book.

5. Asuman G. Aksoy and Mohamed A. Khamsi, *A Problem Book in Real Analysis*, 2010.

6. W. J. Kaczor and M. T. Nowak, *Problems in Mathematical Analysis I: Real Numbers, Sequences and Series*, 2000. There are also parts II and III available.

6 Interlude: Non-measure-theoretic Probability Theory, Mathematical Finance, Information Theory

> Mathematicians, pure and applied, think there is something weirdly different about statistics. They are right. It is not part of combinatorics or measure theory but an alien science with its own modes of thinking. Inference is essential to it, so it is, as Jaynes says, more a form of (non-deductive) logic.
>
> James Franklin, 2005.[38]

> Some books are correct. Some are clear. Some are useful. Some are entertaining. Few are even two of these. This book is all four.
>
> Many texts at this level are little more than bestiaries of methods, presenting dozens of tools with scant explication or insight, a cookbook, numbers-are-numbers approach.
>
> Presenting statistics this way invites students to believe that the resulting parameter estimates, standard errors, and tests of significance are meaningful—perhaps even untangling complex causal relationships. They teach students to think scientific inference is purely algorithmic. Plug in the numbers; out comes science.
>
> Freedman (2009, page xi)[39]

For students in mathematics, not long after they have enjoyed their sequence of undergraduate level real analysis courses, the next level is a course in which measure theory is introduced. Notably for applied researchers such as those in econometrics, statistics, and machine learning, the primary use of measure theory is its intimate relationship with probability theory. As such, most students in the aforementioned fields who get past a first course in analysis will have had, or want to take, a "first course in probability", which I define as calculus-based, but not with measure theory, and covering all the basics.

I call this section an interlude because I will be brief on all the topics mentioned in the title, despite applied probability and distribution theory being among my primary areas of expertise, having also written two textbooks in the field. It appears (as perhaps with other fields of inquiry) that there is a substantial number of "cookie cutter" (unremarkable) books in the important arena of "Introduction to Probability". Section 6.1 discusses some books in this first-course-in-probability niche that I think are worthy of mention.

I also give lists of books on stochastic calculus and mathematical finance—these being obviously relevant for students in quant finance (whether or not they wish to research in machine learning), but not necessarily for someone doing machine learning outside of finance. Notice that the prerequisites for stochastic calculus and mathematical finance are the same as discussed above, namely linear algebra and real analysis (and probability theory). Finally, I include a far-too-short section on information theory, just to bring this important topic to the attention of the student wishing to know all the core elements required to pursue machine learning, and wishing to gain a big picture overview and road map of how to proceed.

There are many paths one could take after such a first class in probability theory, including:

[38] From his book review article on three (not perfectly matched) books: *Probability Theory: The Logic of Science*, by E. T. Jaynes, 2003; *The Fundamentals of Risk Measurement*, by Chris Marrison, 2002; and *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*, by Trevor Hastie, Robert Tibshirani, and Jerome Friedman, first edition, 2001; having appeared in *The Mathematical Intelligencer* (2005), 27(2), pp. 83-5.

[39] From the Foreword, written by David Collier, Jasjeet Singh Sekhon, and Philip B. Stark, of David A. Freedman (2009), *Statistical Models: Theory and Practice*, revised edition, which came out just after Freedman (1938-2008) passed away. In case his name "rings a bell", he was indeed the co-author of the ubiquitous D. A. Freedman, R. Pisani, and R. A. Purves, *Statistics*, whose first edition appeared in 1978.
See the MAA review `https://www.maa.org/press/maa-reviews/statistical-models-theory-and-practice`, and notably the link therein to the stellar review of the first edition. Another review, by Paul Barrett (2009), appearing in *Structural Equation Modeling A Multidisciplinary Journal*, 16(2), pp. 391-5, is an onslaught of praise, stating "David Freedman has written a masterwork." and "I would recommend this book be bought by every teacher of research methods."

Prob Path 1: Classic material in stochastic processes (for which I grudgingly refrain from providing a list, as it is arguably a bit off topic, though note the mention earlier in Section 1 of the very attractive *Discrete Stochastic Processes and Applications*, by Jean-François Collet, 2018).

Prob Path 2: More advanced distribution theory (Section 6.2).

Prob Path 3: Basic stochastic calculus (Section 6.3).

Prob Path 4: Introductions to mathematical finance, developing the needed additional probability theory along the way (Section 6.4).

Prob Path 5: Information theory (Section 6.5).

Prob Path 6: A pursuit of measure-theoretic probability (Section 8.4).

Another (glaring and self-evident) path is statistical inference, which (despite it being one of my speciality areas) I do not formally elaborate upon in this document.

6.1 Introduction to Probability

Besides my book (which I naturally consider remarkable!), I happily also mention here a few "competitors" that I certainly have on my bookshelf, and can recommend as either a primary or secondary book for a course in this material.

1. Marc S. Paolella, *Fundamental Probability: A Computational Approach*, 2006. I was delighted to discover that my book was reviewed by the MAA. With nervous foreboding and childish excitement, I inspected it, https://www.maa.org/press/maa-reviews/fundamental-probability-a-computational-approach, and, to my relief, it is rather positive! Quoting,

    > [It] is designed for students who have studied calculus and linear algebra and assumes no previous background in probability. However, because of the amount of material and the depth of the treatment, it is most appropriate for beginning graduate students or strong upper-division undergraduates. The intended audience includes students of mathematics, statistics, biology, computer science and engineering, but the book is tuned somewhat for those with interests in the quantitative aspects of finance and econometrics. The author's goals are to focus on the practical aspects of the subject by emphasizing examples and computation, to push beyond the ordinary topics, and to include valuable techniques that are often omitted from introductory texts because they are computationally intensive.
    >
    > [The] treatment of combinatorics is a good deal more extensive than comparable texts, and combinatorial arguments are used broadly throughout.
    >
    > The author takes pains to help the student develop tools for evaluating and understanding new continuous probability distributions: identifying parameters, recognizing major applications, investigating behavior of the tails. By examples the reader is introduced to basic concepts used in finance including utility functions, stochastic dominance, value at risk, and expected shortfall. MATLAB programs used throughout the text help to integrate the computational and analytical. These MATLAB routines could easily be converted to Maple or a comparable language without much difficulty. Exercises are well-chosen, plentiful and graded by degree of difficulty. The publisher's website for the text has a complete set of solutions available online.
    >
    > One of the more unusual aspects of this text is the lengthy appendix (more than ninety pages) on "calculus review". The author wishes to "separate

mathematics from the probabilistic concepts and distributional results obtained in the text". He evidently expects students to review the appendix before beginning work on Chapter 1. I wonder how well this works in practice.

(To answer the reviewer's last question: Not great; but some students have remarked that the appendix, as an in-depth review of calculus and primer for real analysis, was a big help, for my class, but notably also for other classes requiring this material.) A full set of teaching slides, covering the primary parts of the book (as the reviewer picked up on), along with extra topics in the slides that make use of the material but that are not in the book, are available on *my own* website (as opposed to that of the publisher), https://www.marc-paolella.com/fundamental-probability, as well as the entire solutions manual, and computer programs, in both Matlab and R.

I was also proud (and relieved) to read the (long and positive) review of my book in *The American Statistician*, Vol. 62, No. 2, pages 179-80. Reviewer (Jane Harvill) begins by saying:

> [It] is the first of what is to be a three-volume set on probability and statistics. There is much to say about the first volume. I recently obtained the second volume,and am looking forward to reading through it. If Volume II is as well done as Volume I, then I will surely purchase Volume III when it becomes available. Volume I is a solid, 466-page coverage of combinatorics, probability spaces, counting, conditioning, and discrete and continuous random variables. Paolella also includes a detailed, self-contained appendix of calculus tools used throughout the book, appendices with tables which collect notational, distributional, and other information in a concise manner.

2. K. L. Chung and Farid AitSahlia, *Elementary Probability Theory: With Stochastic Processes and an Introduction to Mathematical Finance*, 4th edition, 2010.

3. Sheldon Ross, *A First Course in Probability*, tenth edition, 2020. I had used (what must have been) the third edition as an undergraduate, and the author's writing style had a clear influence on my own textbook writing in probability theory.

4. David Stirzaker, *Elementary Probability*, 2nd edition, 2003.

5. Geoffrey Grimmett and Dominic Welsh, *Probability: An Introduction*, 2nd edition, 2014. Review https://www.maa.org/press/maa-reviews/probability-an-introduction is, to be frank, not strong, with the reviewer having compared this book to that of Blitzstein and Hwang (see below), saying:

   > The main difference is in how much they hold the students hand. Blitzstein and Hwang try everything possible to help the student understand the material. Grimmett and Welsh present the material unaided. Blitzstein and Hwang have problems with applications to just about anything you can think of (Google's PageRank algorithm, legal, medical, ecology cryptography,genetics, computer science, etc), Grimmett and Welsh have only the typical probability problems (dice, cards, weather, etc). Blitzstein and Hwang have R code and an online companion website, Grimmett and Welsh do not. Blitzstein and Hwang have about 600 exercises, Grimmett and Welsh have about 400. Blitzstein and Hwang is close to 600 pages, Grimmett and Welsh is 270.
   >
   > What it comes down to, in my opinion, is that Blitzstein and Hwang is an excellent book for a wide variety of audiences trying to learn probability. Grimmett and Welsh are clearly focusing on math students — it is narrower, has fewer excursions, and is probably more difficult as a text. The material appears simple until you try to do the exercises, at which point you realize that there were many ideas contained in a few words.

6. Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Fourth Edition, 2020; and *One Thousand Exercises in Probability*, Third Edition, 2020. This book is considerably more advanced than either of the author's solo projects with elementary probability books (and also more advanced than my book). I mention it because it can serve as excellent supplementary reading, albeit far more terse, and far more extensive, than a gentle introductory book. As one sees, the authors have an entire book dedicated to problems and solutions, making this package yet more useful and impressive.

7. Joseph K. Blitzstein and Jessica Hwang, *Introduction to Probability*, 2014. The MAA review is very positive, https://www.maa.org/press/maa-reviews/introduction-to-probability, and this book is also praised in the MAA review of Grimmett and Welsh above.

8. Henk Tijms, *Probability: A Lively Introduction*, 2018. This book is pitched at the same level as mine, and I think it (is a very well written book and) can serve as a near perfect compliment to mine. The MAA reviewer mentions it is a clear competitor to (very popular, and for good reason) Ross' book, but Tijms is far cheaper in terms of price. To bolster my claim that Tijms book is a nice complement to mine and could be effectively used together, note that he includes a few topics I do not, one of which I am so angry I did not include, notably because I wrote my book while (and still am) in a finance department, and thus had a bit of a pitch to finance students; namely, Kelly betting. Good move Mr. Tijms![40] Mine, on the other hand, is much more extensive overall, and particularly with several topics, such as combinatorics; whereas Tijms' book was criticized in its (otherwise very favorable) MAA review, https://www.maa.org/press/maa-reviews/probability-a-lively-introduction. Quoting:

   > And speaking of things that did not find their way into the text, let me record one mild quibble here: I think the author's discussion of basic combinatorics, which basically comprises a five-page (six, counting some problems) appendix, could have been beefed up. The fact that this material is in an appendix rather than the main body of the text doesn't bother me, but I think five pages (and only four simple examples) is likely not a sufficient amount of time to spend on this topic, particularly since it is my experience (from teaching combinatorics, rather than probability) that many students may not be as familiar with this material as the author assumes they are.
   >
   > ...
   >
   > Bottom line: this book should be on the short list of any instructor who is shopping around for a text for an upper-level probability course. University librarians should also take a look at this text.

6.2 Distribution Theory

I give a short list, with just two entries, of books that build on basic probability theory, moving towards what is needed to do (beyond basic) statistical inference. The first book is from me, while the second is from Thomas Severini. The two books have, perhaps expectedly, quite some overlap (characteristic functions and inversion theorems, Laplace transforms and moment generating functions, sums of random variables, order statistics, various forms of convergence and their relations, multivariate normal distribution, central limit theorems, and saddlepoint approximations), and also numerous topics not shared. They are both at about the same level of mathematical sophistication.

Outside of having more than its fair share of typos, I would go so far as to include Severini's book on my personal list of "I wish I wrote it" books. If I were in a statistics department,

[40] Tijms also cites, on page 338, the standard and fantastic reference, namely Edward O. Thorp, *The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market*, https://www.sciencedirect.com/science/article/pii/B9780444532480500150. Regarding Thorp, see also http://www.edwardothorp.com/. For a textbook presentation that discusses the link between Kelly betting and information theory, see Section 6.5.

I would definitely use my book, but also his, for a second course in probability for master's students.

1. Marc S. Paolella, *Intermediate Probability: A Computational Approach*, 2007, 415 pages. This was also reviewed by MAA, https://www.maa.org/press/maa-reviews/intermediate-probability-a-computational-approach, positive but brief, concentrating mostly on the topics I cover, and concluding with:

   > The reader-friendly style of the text itself would make the book appropriate for self-study or classroom adoption, but it is strange that no exercises come with as much as a numerical answer, let alone with full solutions. This reviewer did not find any such information on the book's website either. Students need some way to check whether their answers are correct.

   I wish the reviewer had contacted me... I wrote the entire solutions manual! It, and computer codes are also available on my web page, https://www.marc-paolella.com/intermediate-probability, in both the original Matlab that I use and show in the book, along with R translations. As with my other probability book, teaching slides are also available on my webpage (along with the inevitable, humbling, errata list). Jane Harvill also reviewed my book, this time in JASA, Vol. 104, No. 487, pages 1285-6, writing (I swear I do not know her, nor paid her)

   > I thoroughly enjoyed *Intermediate Probability*. I was so thrilled with it (along with *Fundamental Probability*) that I shared it with some of my colleagues. They have called it a "gold mine" of problems and resources, and describing it as "amazing".

   (Thank you Jane!)

2. Thomas A. Severini, *Elements of Distribution Theory*, 2011, 515 pages. I consider this book to be outstanding. Having read parts of his book with a fine-toothed comb, I can venture my strong opinion; but also in doing so, I came across an annoyingly large number of typos (that I started to protocol).

   The MAA review https://www.maa.org/press/maa-reviews/elements-of-distribution-theory is very positive, saying (among much more):

   > This book is about the probability theory that is useful for statistics, done without measure theory. Its prerequisites are a decent background in analysis. The book could easily be used as a text or as a supplement to, for example, Casella and Berger's Statistical Inference. I can see many students using it to learn the details for topics that the standard texts don't have space to include. The material is well written, proofs are easy to follow, and motivation is clear.

   (The reviewer then goes on to discuss typos.)

Apparently, both Thomas and I have subsequent books dedicated to statistical inference, written about the same time no less, and are all slanted towards (or in Thomas' case, generally focused on) finance. They are:

1. Thomas A. Severini, *Introduction to Statistical Methods for Financial Models*, 2018, 370 pages.

2. Marc S. Paolella, *Fundamental Statistical Inference: A Computational Approach*, 2018, 564 pages.

3. Marc S. Paolella, *Linear Models and Time-Series Analysis: Regression, ANOVA, ARMA and GARCH*, 2019, 880 pages.

It would take us too far off track for me to protocol extensive lists of (what I deem to be) good books in statistical inference, linear models, and time series analysis, though in my books, especially the time series one, I mention and recommend numerous other books. I allow myself one exception, with that book being truly exceptional: Freedman (2009), from which I took the quote at the beginning of Section 6.

6.3 A First (or Second) Course in Stochastic Calculus

Stochastic calculus is inherently "advanced, abstract, difficult mathematics" for anyone who has not been exposed to, and learned, the material commonly seen in advanced undergraduate or graduate level mathematics courses. This will most often be the case for beginning master's students in business, economics, and finance. Realistic prerequisites, even for books that adorn their titles with words such as "Elementary", or "Introduction", include (i) a course in which mathematical proofs are emphasized, such as (and whose content is also relevant) a first course in linear algebra and also a first course in real analysis; and (ii) at least a first course in "serious, but non-measure-theoretic" probability theory, as given by some of the books in Section 6.1; and perhaps best to add a second course in (still non-measure-theoretic) probability, such as one or both of those in Section 6.2, and/or a course in "classic" stochastic processes (which invariably includes at least one chapter on Brownian motion).

Not being an area of mine for which I claim expertise, I provide only a modest list, concentrating on books I know, or "those that everyone knows and has on their shelf", e.g., Klebaner, Mikosch, and Shreve. I begin with four books that I admire and appreciate, and are the only ones in this list that I know well and have read all or parts of. Wiersema provides arguably the best introduction to the subject for a student having had only elementary probability theory, and, as I expected, is adorned with very many 5-star reviews at Amazon. Hassler's book is particularly well suited to students interested in econometrics. Lawler is a more general, and a bit terse, "first book on stochastic processes" that contains, along with "classic" topics in a first course in stochastic processes, also material on stochastic calculus, e.g., chapters entitled Brownian Motion, and Stochastic Integration. I have the first edition of Kijima's book, from 2003, and the second edition, among other changes, adds a chapter, "Change of Measures and the Pricing of Insurance Products".

The other books listed are in a different class of mathematical sophistication, and are better used for a second course in the material. Pardoux's book is (as the title suggests) more about Markov processes, but it also has a 50 page chapter, "Introduction to mathematical finance" well worth looking at.

1. Ubbo F. Wiersema, *Brownian Motion Calculus*, 2008.

2. Uwe Hassler, *Stochastic Processes and Calculus: An Elementary Introduction with Applications*, 2016.

3. Gregory F. Lawler, *Introduction to Stochastic Processes*, Second Edition, 2006.

4. Masaaki Kijima, *Stochastic Processes with Applications to Finance*, Second Edition, 2013. From the preface of the first edition, we read: "This book is suitable for the reader with a little knowledge of mathematics. It gives an elementary introduction to the areas of real analysis and probability. Ito's formula is derived as a result of the elementary Taylor expansion."

   The global financial crisis happened in between the first and second editions of the book. In the second edition, Kijima mentions it, and how quants and financial engineers were blamed, and adds, correctly so, that "while the theory is used to create such awful derivative securities, those claims are not true at all. Those who made the mistakes were people who used the theory of financial engineering without a thorough understanding of the risks and high ethical standards." He goes on to say:

> Hence, we need people who not only understand the theory of financial engineering, but who can also implement the theory in business with high ethical standards. The main goal of this book is to deliver the idea of mathematical theory in financial engineering by using only basic mathematical tools that are easy to understand, even for non-experts in mathematics.

5. Ovidiu Calin, *An Informal Introduction to Stochastic Calculus with Applications*, 2015.

6. René L. Schilling and Lothar Partzsch, *Brownian Motion: An Introduction to Stochastic Processes*, 2012. The title might be a bit misleading, given the intended audience and the prerequisites, namely: "The book is intended primarily for a graduate level course. It assumes basic measure theory, a course in probability (preferably one based on measure theory) and familiarity with discrete time martingales.", as was stated in the (very positive) MAA review `https://www.maa.org/press/maa-reviews/brownian-motion-an-introduction-to-stochastic-processes`. Quoting further from that review:

    > This book emphasizes the mathematical foundations of stochastic processes and Brownian motion. Those looking for applications in physics, economics or finance will have to look elsewhere. This is first and foremost a rigorous mathematics textbook.

7. Thomas Mikosch, *Elementary Stochastic Calculus with Finance in View*, 1998. The MAA review `https://www.maa.org/press/maa-reviews/elementary-stochastic-calculus-with-finance-in-view` is quite positive, writing:

    > When reading the title you may wonder how stochastic calculus can be elementary. Well, it really can't, it just depends on how you look at it. Elementary Stochastic Calculus is meant to provide an intuitive introduction to the subject. This is not a deep measure theoretic approach to explaining stochastic calculus. The author takes the other road — intuitive explanation with as few mathematical technicalities as possible. In only 200 pages he does indeed achieve this. Don't get me wrong, the book is technical to the extent of the actual theory of stochastic calculus. However, it does not even come close to some other texts, such as Introduction to Stochastic Calculus with Applications, by Klebaner, which is very technical.
    >
    > An advanced reader with a strong mathematical background in mathematical and functional analysis and measure theory will find this book inadequate. However, this is an introductory text and as such it should be read. It is a very good prelude to books such as Klebaner's Introduction to Stochastic Calculus with Applications by Klebaner or other more advanced texts on stochastic calculus.

8. Fima C. Klebaner, *Introduction to Stochastic Calculus with Applications*, Third Edition, 2012.

9. Steven Shreve, *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model*, 2005; and *Stochastic Calculus for Finance II: Continuous-Time Models*, (corr. 2nd printing), 2010.

10. Étienne Pardoux, *Markov Processes and Applications: Algorithms, Networks, Genome and Finance*, 2009, 293 pages. Translated from French. From the preface, we read that measure theoretic probability is desirable, but:

    > large parts of the book will be accessible to mathematicians who have only studied probability at undergraduate level, as well as by computer scientists, statisticians, economists, physicists, biologists and engineers.

The MAA review https://www.maa.org/press/maa-reviews/markov-processes-and-applications-algorithms-networks-genome-and-finance is short but positive, writing:

> After a first chapter devoted to introduce the fundamentals on Markov chains, the author explains how a Markov process can be applied to such diverse fields as mathematical finance, the genome and the analysis of queues and networks. These chapters are completely independent from each other, so that one can browse through the book in many ways, according to one's interests and needs.
>
> ...
>
> Well-written, this book is suitable as a textbook for teaching a postgraduate course on applied Markov processes. There is a wealth of interesting exercises (solutions to selected exercises are provided at the end); several exercises throughout the book are intended to be implemented on some suitable software, for instance MATLAB. The necessary prerequisites are: linear algebra, probability theory, random variables, calculus (measure theory + ordinary and partial differential equations).

6.4 Introductory Mathematical Finance

Someone wishing to do machine learning in finance, whether basic or highly sophisticated, will surely need to also know at least the basics of financial markets, financial products, and mathematical finance and its associated probabilistic tools and structures. I present two lists here, one of relatively basic books, and another with more sophisticated books, on mathematical finance. Both lists are blatantly incomplete, but enough to give the interested reader a head start into the literature.

Similar to what I said about "an introduction to stochastic calculus" being a bit oxymoronic when seen from the viewpoint of an undergraduate outside of a pure or applied mathematics department, mathematical finance is inherently mathematically sophisticated material that, at the higher end, presupposes familiarity (and comfort) with graduate level probability theory (e.g., measure theory, stochastic calculus, etc.). Still, there are several books, designed for undergraduates in finance, that begin very modestly (basic theory of interest, basic mean-variance Markowitz portfolio theory, rudimentary discrete probability, etc.), and progress slowly, touching upon more advanced topics without full-blown mathematical rigor, such as stochastic calculus, martingales, Brownian motion, Itô's formula, etc.. Here is my (rather incomplete) list of such introductory level books, knowing full well there are certainly more. They are listed chronologically.

1. X. Sheldon Lin, *Introductory Stochastic Analysis for Finance and Insurance*, 2006, 224 pages. The author does not precisely say what the formal prerequisites are, but we can glean them from a statement in the preface:

   > Most of my students have a Master's degree in science or engineering but had only taken one or two entry level probability courses prior to my course. My initial approach for teaching such a course was to focus on financial models and aspects of modelling techniques. I soon found that the lack of the sound knowledge in stochastic analysis often prevented students from fully understanding and mastering financial models, and the implementation of these models. I ended up spending much of the class time teaching stochastic analysis instead. I feel from that experience that a short course on stochastic analysis with an emphasis on techniques and methods used in financial modelling would greatly help students with a limited probability background to better study mathematical finance.

   The MAA review https://www.maa.org/press/maa-reviews/introductory-stochastic-analysis-for-finance-and-insurance is overall positive, but makes clear that this is an introduction: Measure theory is not required to read this book, and, as the reviewer states, "there is much more to stochastic analysis that is needed in advanced mathematical finance than what can be found in this book." She goes on to say:

> This book focuses more on the applied side of some more important theorems from stochastic analysis. Due to this approach the book is very suitable for practitioners who just need to understand the main connections between some financial concepts and stochastic analysis. One would have to have some background in finance to be able to understand the examples, as the author does not explain what bonds and options are, nor does he explain some of the computations related to them. This can serve as a short "stochastic finance" textbook.
>
> ...
>
> Filtrations, along with the notion of filtered space, are presented very briefly. The author gives couple of nice intuitive examples to support the theoretical concepts presented. Next, random walks, discrete Markov chains and martingales are explained. An example is presented with each topic. These examples are excellent for readers who don't have much experience in applying stochastic theory to financial problems.

The pdf file that can be found in the internet is optically acceptable, but not as crystal clear as what I call a "perfect" pdf file.

2. Robert R. Reitano, *Introduction to Quantitative Finance: A Math Tool Kit*, 2010, 709 pages. Inspecting the table of contents, we see that the book could be described, if forced to be quick, as: It covers (i) selected topics associated with a first course in real analysis, complete with theorems and proofs, along with an introduction to metric spaces; (ii) elements of a first and second (non-measure-theoretic) course in probability theory; and (iii) throughout, gives examples from finance.

   This large book has a correspondingly large introduction, 14 pages, making a strong case for a book like this. For example:

   > Anyone interested in meeting the field requirements in finance is left with the choice to either pursue one or more degrees in mathematics or expend a significant self-study effort on associated mathematics textbooks. Neither approach is efficient for business school and finance graduate students nor for professionals working in investment and quantitative finance and aiming to advance their mathematical skills. As the diligent reader quickly discovers, each such book presents more math than is needed for finance, and it is nearly impossible to identify what math is essential for finance applications. An additional complication is that math books rarely if ever provide applications in finance, which further complicates the identification of the relevant theory.
   >
   > The second gap is in the finance literature. Finance texts have effectively become bifurcated in terms of mathematical sophistication. One group of texts takes the recipe-book approach to math finance often presenting mathematical formulas with only simplified or heuristic derivations. ... The other group of finance textbooks are mathematically rigorous but inaccessible to students who are not in a mathematics degree program. Also, while rigorous, such books depend on sophisticated results developed elsewhere, and hence the discussions are incomplete and inadequate even for a motivated student without additional class- room instruction. Here, again, the unprepared student must take on faith referenced results without adequate understanding, which is essentially another form of recipe book.

Regarding prerequisites, we read: "The only mathematical background required of the reader is competent skill in algebraic manipulations and some knowledge of pre-calculus topics of graphing, exponentials and logarithms." That description (which is basically that of a senior in high-school) is too modest: I would say the minimum is the same prerequisites that many first-course real analysis books do, namely that students have had a two or three semester sequence of calculus, and ideally also a class that emphasizes proofs, such as linear algebra (which anyway is necessary in finance). Better would be to have had a course using one of the "easier" books listed in Section 5.2, notably one of the first three books listed.

Of value for students and instructors:

> For the book's practice exercises, a Solutions Manual with detailed explanations of solutions is available for purchase by students. For the assignment exercises, solutions are available to instructors as part of an Instructor's Manual. This Manual also contains chapter-by-chapter suggestions on teaching the materials. All instructor materials are also available online.

This is (outside of that in-need-of-a-reality-check statement about the prerequisites) an impressive book that I will be considering (first reading, and then) using for a course in our quantitative finance master's program or, possibly, as the anchor book for a PhD course. The MAA review `https://www.maa.org/press/maa-reviews/introduction-to-quantitative-finance-a-math-tool-kit` is equally impressed, writing:

> Robert R. Reitano has written an ambitious and beautiful book with the aim of creating a new breed of quantitative finance professionals ("quants") with a deep understanding of the underlying mathematics.
>
> I found this book a thrill to review — it is very rich in material and I will be studying it for some time to come. Those seeking a career as a quant and not studying the book as part of a standard academic program that covers the programming skills necessary to implement the models and ideas presented in this book are well advised to gain such know-how.

3. Sheldon M. Ross, *An Elementary Introduction to Mathematical Finance*, Third Edition, 2011, 322 pages.

4. J. Robert Buchanan, *An Undergraduate Introduction to Financial Mathematics*, Third Edition, 2012, 458 pages. I have the hard-copy and have read some of it. It is definitely suitable for undergraduates. The prerequisite mathematics is, as stated in the preface, "elementary multivariate calculus". Also, the book "assumes no background on the part of the reader in probability or statistics", though a previous basic exposure would be useful, given that the reader is already confronted with learning new concepts from finance. I would also add linear algebra to the prerequisites, for chapter 4 (the arbitrage theorem), which also makes use of linear programming.

5. Steven Roman, *Introduction to the Mathematics of Finance: Arbitrage and Option Pricing*, 2nd edition, 2012, 287 pages. (The author is the same person who wrote *Advanced Linear Algebra*, mentioned in the comments part of Section 4.1.) My quick inspection of this book indicates that it very appealing for the intended audience of students with a certain mathematical maturity (to be discussed next), but little or no exposure to mathematical finance. As examples of the content, chapter 4 is "Stochastic Processes, Filtrations and Martingales", and chapter 8 is "Optimal Stopping and American Options". From the preface,

> It is my hope that this book will be read by people with rather diverse backgrounds, some mathematical and some financial. Students of mathematics

> may be well prepared in the ways of mathematical thinking but not so well prepared when it comes to matters related to finance (portfolios, stock options, forward contracts and so on). For these readers, I have included the necessary background in financial matters.
>
> On the other hand, students of finance and economics may be well versed in financial topics but not as mathematically minded as students of mathematics. Nevertheless, since the subject of this book is the *mathematics* of finance, I have not watered down the mathematics in any way (appropriate to the level of the book, of course). That is, I have endeavored to be mathematically rigorous *at the appropriate level*. However, for the benefit of those with less mathematical background, I have made the book as mathematically self-contained as possible. Probability theory is ever present in the area of mathematical finance and in this respect the book is completely self-contained.

The MAA review https://www.maa.org/press/maa-reviews/introduction-to-the-mathematics-of-finance-arbitrage-and-option-pricing is extremely positive. We read:

> The biggest strength of the book, especially in view of the intended audience, is its attention to pedagogy. Authors who attempt to cover a lot of ground in relatively few pages often end up producing a mere compendium of results. This book avoids that danger. The author has crafted something that is more a tutorial than a reference. Many results are paraphrased either before or after they are formally stated and the examples peppered throughout the text are extremely helpful. So, although a definition-theorem-proof framework still prevails, there is enough connective tissue to aid the student in following the discussion.
>
> ...
>
> The second quote above hints at another of the book's strengths: Much thought has been put into how topics are ordered. Let the structure of Chapter 1 serve as an example. The chapter begins by describing some basic properties of options, information that will be welcomed by finance novices. Next follows a discussion of payoffs for calls and puts, complete with graphs showing payoff curves. Next, time premium is introduced, illustrated with graphs of option premium curves. One more graph then allows a concise yet intuitive introduction to delta: it is simply the slope of the tangent line to the option premium curve. The text flows well and is at just the right level for a first chapter. Later chapters are more mathematically challenging, of course, but the order of topics is still carefully crafted.
>
> ...
>
> The book is largely self-contained, with the necessary probability explained as needed. (Nevertheless, prior exposure to probability is recommended.)

6. Stephen Blyth, *An Introduction to Quantitative Finance*, 2014, 175 pages. This is among the shortest books in this category, and also extremely modest in terms of the requisite probability theory required. From the preface,

> No prior exposure to finance or financial terminology is assumed.
>
> ...
>
> The sole prerequisite for mastering the material in this book is a solid introductory undergraduate course in probability. Familiarity is required with: discrete and continuous random variables and distributions; expectation, in particular expectation of a function of a random variable; conditional expectation; the binomial and normal distributions; and an elementary version of

> the central limit theorem. The single-variable calculus typically associated with such a course—integration by parts, chain rule for differentiation and elementary Taylor series—is used at several points.
>
> ...
>
> The mathematical prerequisite is modest and no more extensive than that for an introductory undergraduate probability course, although adeptness at logical quantitative reasoning is important.

There is a book review in JRSS-A (2014), 177(4), page 989. The reviewer's opening and last statements are:

> In contrast with the common textbooks on financial derivatives, this book is clearly addressed to students of mathematical rather than economical subjects. Statistical basics, especially the central limit theorem and the normal distribution as well as the expectation of a function of a random variable, are required.
>
> ...
>
> In summary, the book has a clear structure and is understandable with undergraduate statistical prerequisites. However, solutions for the challenging exercises are missing as they sometimes contain proofs of key concepts (e.g. the replication proof for valuation of a currency forward).

7. Arlie O. Petters and Xiaoying Dong, *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition*, 2016, 483 pages. From the preface,

   > The required mathematics consists of introductory courses on multivariable calculus, probability, and linear algebra. Along the way, we introduce additional mathematical tools as needed—e.g., some measure theory is presented from scratch. *No background in finance is assumed.*

8. Thomas Mazzoni, *A First Course in Quantitative Finance*, 2018, 598 pages. After the introductory chapter 1, chapters 2 and 3 give primers on probability theory and vector spaces, respectively (and there is an appendix on complex analysis). As the author states, measure theory, and Hilbert spaces, are not required, though the author does define a σ-algebra, and *mentions* the term Lebesgue measure. It seems he actively holds himself back from further discussion of measure theoretic probability theory, and we read:

   > There is much more to say about probability spaces and measures than may yet appear. Measure theory is a very rich and subtle branch of mathematics. Nonetheless, most roads inevitably lead to highly technical concepts, barely accessible to non-mathematicians. To progress in understanding the fundamental principles of financial markets they are a "nice to have" but not a key requirement at this point.

   An excellent selection of topics is covered, and solutions to problems are included in the book. (There is also a further solutions manual available to instructors, giving answers to the "quick calculations" thoughout the book.) The book also features stereoscopic images, and while I don't doubt that they work, I personally was not able to get my eyes to see them correctly.

   Some (editorial and purchaser) reviews can be found at Amazon, and on the publisher's page. I found this further (and very positive) 2019 review given here, https://letyourmoneygrow.com/2019/06/10/reading-a-first-course-in-quantitative-finance-by-thomas-mazzoni/, from which we read:

> For your first course you'd better take the Shreve's book but if you are going to make a deeper 2nd course, you are highly recommended to read the Mazzoni's book in between.

(The author is referring to Shreve's book *Stochastic Calculus for Finance II: Continuous-Time Models*, 2010, mentioned in the list in Section 6.3.) To the extent that this is accurate, Mazzoni's book might not be the best starting place, but it still seems far more accessible than the advanced books I list below.

9. Donald G. Saari, *Mathematics of Finance: An Intuitive Introduction*, 2019, 144 pages. The preface (provides a nice motivation to the book and) indicates what is required of the student: "As for required background, students who have finished the calculus sequence (several variables) and an introductory course in probability and statistics have been successful." To give an example of the intuition of this short, concise book, we read on page 79,

   > The elimination of minuscule terms is a standard mathematical approach, so it is worth developing intuition. Consider a driver's worries about a newly purchased Lamborghini when driving at unallowable speeds. If that well-polished surface is hit by a rock and a fly, the first is of concern, while that minute splat of the fly can be ignored.

10. Pablo Koch-Medina and Cosimo Munari, *Market-Consistent Prices: An Introduction to Arbitrage Theory*, 2020, 446 pages. Regarding the prerequisites, from the preface, we read:

    > We assume of the reader only a working knowledge of linear algebra and calculus at the advanced undergraduate level. ... In spite of its elementary level, we believe this book can also appeal to mathematicians seeking to understand how the main financial ideas in arbitrage theory translate into rigorous mathematical questions.

    The original intent of the book was to be a second edition of Koch-Medina and Merino, *Mathematical Finance and Probability: A Discrete Introduction*, 2003, but grew into something larger and different. While there are no book reviews yet, we can look at some from the 2003 book. For example, from *Zentralblatt Math* (and as stated on Amazon), we read, among several other very strong statements praising the book:

    > The book offers a self-contained elementary but rigorous and very clear introduction to the pricing of derivative instruments in discrete time. ... For the interested reader who has not been exposed to modern probability theory before, the book provides an excellent starting point for studying the theory of derivative pricing. In particular, for a rigorous course on derivative pricing in an economics department or at a business school this introduction seems to be well-suited.

    We see this "excellent starting point" also in the new book, such as in chapter 9, where numerous concepts associated with measurability of random variables are very nicely introduced.

11. Giuseppe Campolieti and Roman N. Makarov, *Financial Mathematics: A Comprehensive Treatment in Discrete Time*, Second Edition, 2021, 589 pages. The first edition (and for which a pdf file on the web is available) was called *Financial Mathematics: A Comprehensive Treatment*, 2014, 798 pages; and additionally has several chapters on continuous time modeling, and computational techniques. According to the preface of the second edition, it is limited to just the discrete-time part, and has added about 200 pages compared to the discrete-time part of the first edition, including a chapter on

elementary probability theory, and answers to all the exercises. (It does not say if the intent is a further new book just on continuous time methods, rendering the partition thus similar in style to the two books by Steven Shreve.) The second edition thus seems suited for this first list (and the first edition more for the second list).

We now turn to a (very abbreviated) list of advanced books.

1. Ernst Eberlein and Jan Kallsen, *Mathematical Finance*, 2019, 789 pages.

2. Hans Föllmer and Alexander Schied, *Stochastic Finance: An Introduction in Discrete Time*, Fourth revised and extended edition, 2016, 610 pages.

3. Monique Jeanblanc, Marc Yor, and Marc Chesney, *Mathematical Methods for Financial Markets*, 2009, 732 pages. This is a very high-level book, by very highly regarded academics in mathematical finance. The MAA review https://www.maa.org/press/maa-reviews/mathematical-methods-for-financial-markets is short and does not discuss any of the contents of the book, but rather criticizes it for being so mathematical:

   > The book is labeled a "textbook" and the introduction suggests the book will be of use to practitioners in the financial field. I do not agree with either statement. The book is more a monograph than a textbook. There are 650 notationally dense pages with very little exposition, 50 pages of references, and seven(!) illustrations, none based on data.

   The reviewer (arguably rather comically) states

   > I cannot see how this book could be used for instruction, except via the dreadful phrase "you might want to have a look at this book."

   While the book is of course intimidating to novices and arguably anyone without the high requisite mathematics, I am not so sure the reviewer is correct. Marc Chesney is my colleague, and has been using this book (or drafts of it) since many years in the PhD program (and pieces of it at the master's level), and I have only heard good things about the book (and his teaching). So, indeed, you might wish to have a look at the book. A possible critique (and excuse for a second edition) also raised by the MAA reviewer is that it only addresses the global financial crisis (and the ensuing questioning of the validity of certain models and assumptions) by saying that the "book does not deal with these new and essential questions".

4. Marek Musiela and Marek Rutkowski, *Martingale Methods in Financial Modelling*, Second Edition, 2005, corrected third printing, 2009, 715 pages. The MAA review https://www.maa.org/press/maa-reviews/martingale-methods-in-financial-modelling is very positive, and also making a statement about the prerequisites:

   > For a good experience in reading this book one should have a good knowledge of probability and stochastic calculus. Authors do provide an appendix that covers some stochastic calculus.
   >
   > For a graduate student in mathematics this text should not represent too much of a challenge (assuming they have the required background); for practitioners in finance its accessibility will depend only on their mathematics background, as it is written at a sophisticated level.

   A "good knowledge of probability" means (also as the reviewer alludes to) graduate level, i.e., at the level of books described in Sections 8.3 and 8.4.

6.5 Information Theory

As part of my expectations management, I already announced twice that this section is unduly short, though the two books I mention are very powerful, and (at least parts of them, particularly MacKay) should be mandatory reading for anyone claiming to be scientifically literate in information science, probability, and statistical inference, and, of course, machine learning.

1. David J. C. MacKay (1967-2016; see his Wikipedia page), *Information Theory, Inference, and Learning Algorithms*, 2003, 628 pages. The electronic version of the book is freely and legally available, along with other material, here: http://www.inference.org.uk/mackay/itila/.

   There are a relatively large number of professional reviews for this book, and of the several I looked at, they are all very strong; as are the reviews by customers at Amazon. Noteworthy is the author's belief that information theory is not just key for machine learning, but that they are in fact two sides of the same coin. Letting the author say it, we read from the preface:

   > Why unify information theory and machine learning? Because they are two sides of the same coin. In the 1960s, a single field, cybernetics, was populated by information theorists, computer scientists, and neuroscientists, all studying common problems. Information theory and machine learning still belong together. Brains are the ultimate compression and communication systems. And the state-of-the-art algorithms for both data compression and error-correcting codes use the same tools as machine-learning.

   We also learn from the preface what the (hopefully by now, obvious) prerequisites are:

   > This book is aimed at senior undergraduates and graduate students in Engineering, Science, Mathematics, and Computing. It expects familiarity with calculus, probability theory, and linear algebra as taught in a first- or second-year undergraduate course on mathematics for scientists and engineers.

   I was delighted to see that MacKay easily exceeds my ability for caustic diatribes against the use of Fisherian p-values.[41] Quoting from page 64 of his book,

   > While we are noticing the absurdly misleading answers that 'sampling theory' statistics produces, such as the p-value of 7% in the exercise we just solved, let's stick the boot in. If we make a tiny change to the data set, increasing the number of heads in 250 tosses from 140 to 141, we find that the p-value goes below the mystical value of 0.05 (the p-value is 0.0497). The sampling theory statistician would happily squeak 'the probability of getting a result as extreme as 141 heads is smaller than 0.05—we thus reject the null hypothesis at a significance level of 5%'. The correct answer is shown for several values of α in figure 3.12. The values worth highlighting from this table are, first, the likelihood ratio when $\mathcal{H}_1$ uses the standard uniform prior, which is 1:0.61 in favour of the null hypothesis $\mathcal{H}_0$. Second, the most favourable choice of α, from the point of view of $\mathcal{H}_1$, can only yield a likelihood ratio of about 2.3:1 in favour of $\mathcal{H}_1$.
   >
   > Be warned! A p-value of 0.05 is often interpreted as implying that the odds are stacked about twenty-to-one *against* the null hypothesis. But the truth in this case is that the evidence either slightly favours the null hypothesis, or disfavours it by at most 2.3 to one, depending on the choice of prior.

[41] I should mention that my gripe is not against Fisher himself and his many undisputed contributions, nor how he suggested that p-values be used; see footnote 5 and, in particular, the discussion of this issue in my textbook mentioned in that footnote.

> The p-values and 'significance levels' of classical statistics should be treated with *extreme caution*. Shun them! Here ends the sermon.

For those that enjoyed that wonderfully sarcastic righteous indignation, we get reminded often throughout the book of the perils of (what is usually referred to as) frequentist statistical inference. On page 457, as one of many examples, we are told how things should be done:

> There are two schools of statistics. Sampling theorists concentrate on having methods guaranteed to work most of the time, given minimal assumptions. Bayesians try to make inferences that take into account all available information and answer the question of interest given the particular data set. As you have probably gathered, I strongly recommend the use of Bayesian methods.

As a bit of arguably unnecessary trivia, in addition to sharing great disdain for the (incorrect) use of p-values, I was pleased to see that MacKay and I also share a penchant for mocking politicians who seemingly compete for the status of lowest IQ. On his page 469, we read:

> Errors in the cues for memory recall can be corrected. An example asks you to recall 'An American politician who was very intelligent and whose politician father did not like broccoli'. Many people think of president Bush – even though one of the cues contains an error.

My mockery, in footnote 2 on page 229 of my book *Linear Models and Time-Series Analysis* (2019), about the next Republican president after George W. Bush, did not go so far as to mention any names. For fun I repeat it here in the footnote.[42]

MacKay does not explicitly mention Kelly betting (see, e.g., footnote 40), but does have Exercise 36.8 on page 455 (which I found by searching for "Cover and Thomas").

2. Thomas M. Cover and Joy A. Thomas, *Elements of Information Theory*, Second Edition, 2006, 792 pages. From the preface,

   > The mathematical level is a reasonably high one, probably the senior or first-year graduate level, with a background of at least one good semester course in probability and a solid background in mathematics. We have, however, been able to avoid the use of measure theory.

[42]Henry Louis Mencken was better known for his disdain of the 1920 Republican Presidential candidate, Warren Harding, and the folly of many voters, resulting in a now-well-cited quote. On July 26, 1920, he published a column in the Baltimore newspaper The Evening Sun, writing "The larger the mob, the harder the test. In small areas, before small electorates, a first-rate man occasionally fights his way through, carrying even the mob with him by the force of his personality. But when the field is nationwide, and the fight must be waged chiefly at second and third hand, and the force of personality cannot so readily make itself felt, then all the odds are on the man who is, intrinsically, the most devious and mediocre—the man who can most adeptly disperse the notion that his mind is a virtual vacuum." He went on to say, now famously, "On some great and glorious day the plain folks of the land will reach their heart's desire at last, and the White House will be adorned by a downright moron."

7 Course Proposal 4: "A Next Course in Real Analysis"

> That the human intellect struggles with [quantum theory and relativity] should not be surprising. It evolved so that a social primate could find food and mates and keep safe by interpreting a world halfway between the submicroscopic realm of the quantum and the cosmic vastness of relativity. It has become a commonplace that human brains are lumbered with these limitations—cognitively, socially and politically. How surprising and gratifying, then, that humanity occasionally manages to use mathematics, observation and experiment to transcend its own limits so spectacularly.
>
> The Economist, 28 August 2021.[43]

There are several directions one can go after a first course in real analysis. I consider three primary directions: multivariate analysis (often called "advanced calculus"); measure theory; and functional analysis. For the latter two, I split these into two subdivisions based on level of sophistication, with the more advanced of each being placed in Section 8.

If I were asked for my thoughts about the most valuable and recommended pure math subject, after the obligatory basic courses in linear algebra and real analysis, for master's students in (quant) finance with ambitions in machine learning, and—don't forget—who are on a tight time budget in terms of how much additional pure mathematics courses they can tolerate in their (already overfilled and often expensive) curriculum, then I would opt for metric and related spaces, discussed in Section 7.4, and include some elements of multivariate analysis (Section 7.1), notably omitting the topics that are of more relevance in, say, physics (e.g., Stokes' theorem, used in the Maxwell-Faraday Law, Ampere's Law, etc.). The topic of metric spaces covers, among other things, various fixed point theorems and other aspects of optimization. It is clearly highly relevant for machine learning; and also, noting that we are not in a math department, it is more of mathematics that "you can sink your teeth into", as opposed to the "ethereal" and abstract set-theoretic nature of measure theory. Coverage of metric spaces (and some elements of multivariate analysis) can also be viewed as the next step to prepare for a course in functional analysis, some (I think) good (and mostly non-measure theoretic) books for which are given in Section 8.2.

7.1 Advanced Calculus – Multivariate Analysis

Prerequisites include a first course in linear algebra, and a first course in real analysis, for which I have given detailed recommendation lists above. I begin with one of the most impressive mathematics books I have ever seen. Since its first edition appeared, other books have emerged that, while not necessarily superior, are highly worthy of consideration on their own. The remaining entries in the list are in no particular order.

Abbreviation "MRA" stands for Multivariate Real Analysis:

MRA Book Choice 1: *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach*, John Hamal Hubbard and Barbara Burke Hubbard, 5th edition, 2015. This is an optically and content-wise very impressive 800 page book, covering a large amount of material at a definitely intermediate level (as opposed to being a first year calculus book). I have the hardcover 3rd edition and can attest, it is well done, also including historical remarks, and other tidbits of interest. There is a large errata list (despite being in the 5th edition), and a solutions manual for all the (hundreds of) exercises is available. An MAA book review of the 3rd edition from 2007 is available, https://www.maa.org/press/maa-reviews/

[43]The Economist article is entitled "In praise of physics: To see a world in a grain of sand". The notion that our intellect evolved as stated, and is therefore why we do not naturally perceive and cognitively understand distances on extremely small or large scales (or can easily fathom time as being on the order of nanoseconds, or eons) is of course not new. We find it discussed, unsurprisingly, by (and choosing on purpose a Brit) Richard Dawkins, in his *The God Delusion*, (2006, pp. 367-8).

vector-calculus-linear-algebra-and-differential-forms-a-unified-approach, which refers to the more lengthy and powerfully positive review of the second edition from 2002 (by Warwick Tucker, The American Mathematical Monthly Vol. 110, No. 8 (Oct., 2003), pp. 754-759), in which we read "[This] book is a real gem. It has a breadth and depth that is rarely seen in undergraduate texts and it teaches real mathematics from a researcher's point of view instead of the standard off-the-shelf recipes that have little use outside the classroom."

The MAA reviewer states:

> [The] book is a guide for a pretty tough course. It can be used for a two semester sequence which integrates multivariable calculus and linear algebra quite seamlessly, and which along the way introduces mathematical proof, the all-powerful tool of mathematical thinking. However it also includes so many details (and proofs) in basic analysis, single and multivariable, that it can even be used for more advanced students in a one semester analysis course.
>
> ...
>
> It is clear to me that the authors have put their hearts and souls into this project. The book has many details sprinkled in, many anecdotes, many personal opinions about how one does mathematics; any student interested in mathematics would find it a valuable experience to even flip through its pages randomly.
>
> ...
>
> I was very impressed with the depth, clarity and ambition of this book. It respects its readers, it assumes that they are intelligent and naturally curious about beautiful mathematics. Then it provides them with all the tools necessary to learn multivariable calculus, linear algebra and basic analysis. I definitely recommend the book to anyone who is planning to teach or learn multivariable calculus.

MRA Book Choice 2: *Multivariable Mathematics: Linear Algebra, Multivariable Calculus, and Manifolds*, Theodore Shifrin, 2005, about 480 pages. The author explicitly discusses aspects of linear algebra (that appear in a first course, e.g., basic Gaussian elimination, the four fundamental subspaces, eigenstuff, change of basis, diagonalizability, and the spectral theorem), but more of the book is on the typical topics in multivariate calculus, "interweaving the material as effectively as possible and include complete proofs". I like the idea, the topics, and the author's style, and, formally speaking, he is correct stating in the preface that "the prerequisites for this book are a solid background in single-variable calculus and, if not some experience writing proofs, a strong interest in grappling with them." However, before using this book, I would recommend students having had what I state above, namely a first course in linear algebra, and one in beginning real analysis.

Complete solutions are not available, but there is about 14 pages of solutions to selected exercises near the end of the book. There is no MAA review for this book, while I would say that the Amazon reviews are "not very informative".

The pdf on the web is a scan (with occasional handwritten markings), is not searchable or markable, but is very clear and readable. One could only have wished for a recent update, with thus a searchable and clickable pdf file, perhaps some more material, and more solutions.

MRA Book Choice 3: *Advanced Calculus: A Geometric View*, James J. Callahan, 2010. This is an impressive book, with impressive graphics. There is a solutions manual available on the web, as well as a 5 page errata sheet.

MRA Book Choice 4: *An Illustrative Guide to Multivariable and Vector Calculus*, Stanley J. Miklavcic, 2020, about 300 pages. From the preface,

> [This is] a book that students can carry easily to and from class, can take out and leaf through on the library lawn, or in a booth of a pub, or while lying on the banks of a river waiting for the fish to bite.

It contains some nice graphics, generated by (and with code shown from) Matlab—a feature I like. The book is optically very appealing, the pages not dense at all, with judicious use of color, also to indicate bits of text meant as extra comments. There are other interesting teaching aids, such as flowchart for how to deal with limits on page 61.

MRA Book Choice 5: *Multivariable Analysis*, Satish Shirali and Harkrishan Lal Vasudeva, 2011. It is written in standard, no frills LaTeX, there are few graphics, and presented in theorem-proof-example style: I think it is looks great! Also, it contains detailed solutions to the (mostly algebraic) exercises. While this book is certainly not as visually stimulating and unique as Callahan, or Miklavcic, or Hubbard and Hubbard, it definitely deserves consideration. From a mathematical standpoint, it is a bit higher level than (at least parts of) Shifrin; and Hubbard and Hubbard, requiring the student to have had (and absorbed the material in) a first course in linear algebra, and a first course in real analysis.

Conveniently, there is an MAA review of it, https://www.maa.org/press/maa-reviews/multivariable-analysis, which (is very positive and also) refers to Shifrin's book, and that by Hubbard and Hubbard. We read:

> This is a textbook for a rigorous multivariable analysis course, intended for an undergraduate audience of juniors and seniors that has already taken introductory multivariable calculus and perhaps also some has background in one-variable elementary analysis.
>
> Worked-out numerical examples are of course given in the text, but are usually in aid of illustrating some theoretical point (the necessity of a given condition, for example) rather than as mere busy-work.
>
> The book contains detailed discussions (with proofs) of the usual topics in multivariable analysis: derivative of a function from one Euclidean space to another as a linear transformation, implicit and inverse function theorems, constrained optimization and extrema from a more sophisticated standpoint than usually covered in third-semester calculus, multiple integrals, (including Fubini's theorem) and differential forms and the generalized Stokes' Theorem.
>
> The book also includes some material not easily found in the existing literature. The discussion of the inverse and implicit function theorems, for example, struck me as more detailed and elaborate than one usually finds in other texts (multiple versions of the implicit function theorem are given, for example), and in the next chapter Shirali and Vasudeva provide not just a discussion of necessary conditions for extrema but sufficient conditions as well. This is another topic not generally covered in textbooks, although Shifrin's book, mentioned below, does have an exercise on this subject.
>
> Principal competitors for this text would include Shifrin's *Multivariable Mathematics: Linear Algebra, Multivariable Calculus and Manifolds*, and Hubbard and Hubbard's *Vector Calculus, Linear Algebra and Differentiable Forms: A Unified Approach*. Both of these books are pitched at a lower level than Shirali and Vasudeva's text and do not assume prior background in multivariable calculus (although both have enough material — in the case of Hubbard and Hubbard, an Appendix of more than one hundred pages of proofs — to make the book suitable for post-calculus analysis courses for students who do have this background), and both have some features not found in Shirali and Vasudeva: considerably more emphasis on linear algebra, use of manifold terminology, and, in the case of Hubbard and Hubbard, a discussion of the Lebesgue integral.

> The ready availability of "back of the book" solutions may not please instructors who will not be able to assign these problems for homework, but at the same time may make the book very attractive to people interested in self-study, as well as to instructors using different books who are looking for a source of additional problems for their classes.

Sounds like a keeper to me.

MRA Book Choice 6: *Several Real Variables*, Shmuel Kantorovitz, 2016, about 300 pages, 60 of which are detailed solutions ("of a large portion") to the exercises (which are analysis problems, and not numeric plug and chug nonsense). My reading of the first chapter indicates that this is a well written, very attractive book (also optically as a pdf file), requiring a first course in linear algebra, and definitely a first course in real analysis, but not more. Excerpts from the preface give an idea of what is, and what is not, covered:

> Standard geometric applications in $\mathbb{R}^3$ are discussed at a slower pace, using intentionally some tools of the basic theory of systems of linear equations, in order to stress again the role played by Linear Algebra.
>
> The Banach Fixed Point Theorem is also applied to obtain one of the versions of the Existence and Uniqueness Theorem for systems of ordinary differential equations. Consequences of the latter for linear systems are elaborated, with many details relegated to the Exercises section.
>
> The chapter on integration begins with partial integrals (or integrals depending on parameters) of real functions of several real variables. We prove Leibniz's formula for derivation under the integral sign, and a theorem on the change of integration order.
>
> ...
>
> The change of variables formula is stated for any dimension, but the proof is omitted.
>
> ...
>
> Green's theorem is proved in two dimensions. The three-dimensional case (the Divergence Theorem) is stated but the proof is omitted. The generalization of Green's theorem to closed curves in $\mathbb{R}^3$ (Stokes' formula) is also stated without proof.

MRA Book Choice 7: *Advanced Calculus*, Patrick M. Fitzpatrick, 2nd edition, 2009, about 580 pages. It is arguably two books in one, with the first nine chapters being standard topics in introductory real analysis, and the study of functions of several variables beginning in chapter 10, with it being "The Euclidean Space $\mathbb{R}^n$"

The pdf is a scan, and thus text cannot be copied, nor searched. It is however readable enough to be used. The MAA review https://www.maa.org/press/maa-reviews/advanced-calculus-0, of the 2009 second edition, is not very clear about an opinion. The reviewer does say "Chapters 2 to 10 provide a clear, concise and motivated rigorous presentation of one-variable calculus" (though I would have thought this is supposed to be chapters 1 through 9) and

> The examples in the book are chosen to illustrate or motivate the concepts and theorems being discussed. The proofs offer a fair balance between giving all the details and just sketching the main ideas. This is a good teaching aid that encourages the student to get involved from the very beginning by filling the details as needed before he/she attempts to solve the exercises at the end of each section.

Some of the reviews at Amazon are less kind. One writes, among other insults, "Read Abbott's Understanding Analysis or Terry Tao's Analysis to learn how to write a real textbook first before injecting stupidness into the poor students that were forced to use your book." Other complaints include "[T]here are many errors in the book. Some details of proofs are not right," (and goes on with examples); and "It is a book of typos and mistakes on many pages."

MRA Book Choice 8: *Multivariable Calculus with Applications*, Peter D. Lax and Maria Shea Terrell, 2017, 483 pages. I have already mentioned (the good) work from these authors earlier, *Calculus with Applications*, 2014, in the context of a first course in basic real analysis. This book is similar in that it is somewhere between an advanced (multivariate) calculus book and (multivariate) analysis book. It contains theorems, and many proofs, but it is not as sophisticated as some other presentations mentioned in this section, e.g., Callahan, Conway, and Shirali and Vasudeva. The MAA review https://www.maa.org/press/maa-reviews/multivariable-calculus-with-applications is overall positive, ending with:

> Minor quibbles aside, I find Lax and Terrell's to be a marvelous text! I will certainly keep my copy close at hand when teaching courses on multivariable calculus and mathematical physics. I hope the authors will write us more sequels, for instance on ordinary and partial differential equations, and continue with their exposition of the "essential relationship between calculus and modern science".

Noteworthy is that about 55 pages of the text is dedicated to solutions of exercises.

MRA Book Choice 9: *A Course in Analysis Vol. II: Differentiation and Integration of Functions of Several Variables, Vector Calculus*, Niels Jacob and Kristian P. Evans, 2016. I (read and) praised their first book, *A Course in Analysis Volume l: Introductory Calculus, Analysis of Functions of One Variable*, 2016, and note that this one appears just as good. Noteworthy is that it does not start so modestly as the first book—the books are meant to be continuations, and thus the "initial part with kid gloves" is over. Nevertheless, this book is not advanced, and appears to be at precisely the right level (and similar in that regard to other books mentioned in this list). From the MAA review https://www.maa.org/press/maa-reviews/a-course-in-analysis-volume-i-introductory-calculus-analysis-of-functions-of-one-real-variable, we read:

> Volume II of A Course in Analysis is almost entirely concerned with functions of several real variables, although it begins with an introduction to metric spaces. The authors do a nice job in their treatment of this material, balancing rigour and intuition. As in the first volume there are some real gems in volume II. For example, after a discussion of multi-variable integration by parts the authors give a quick derivation of a simple version of Poincaré's inequality. A Course in Analysis seems to be full of these little gems where the authors use the material or ask the readers to use the material to obtain results or examples that the reader will certainly see again in another context later in their studies of mathematics. Generally, the quality of exposition in both of the first two volumes is very high. I recommend these books.

MRA Book Choice 10: *A First Course in Analysis*, John Bligh Conway,[44] 2018, 340 pages. We list other books from Conway in this document, such as his 2014 *A Course in Point Set Topology*, and his 2012 *A Course in Abstract Analysis*, all of which receive strong reviews. In particular, for the book of interest here, the MAA review https://www.maa.org/press/maa-reviews/a-first-course-in-analysis-0 is positive, writing:

[44]Not to be confused with John Horton Conway (1937–2020), who held the John von Neumann Professorship at Princeton University until his retirement. He died from complications from COVID-19. See the Wikipedia page on him, https://en.wikipedia.org/wiki/John_Horton_Conway.

> Its strength is in the multivariate calculus part. There's nothing wrong with the single-variable treatment, but for that material there are lots of good books out there already.
>
> The mathematical approach for multivariate calculus is similar to Rudin's in his *Principles of Mathematical Analysis* (they're especially close on differential forms), but Conway is much easier to understand.
>
> This book provides a rigorous view of what the student has already studied in first-year calculus, and generally speaking does not introduce new topics. In particular it does not go in new directions such as the Lebesgue integral or function spaces. There are a few exceptions: there's a good bit on metric spaces, and there's some coverage of the Stieltjes integral. There's a whole chapter on differential forms (needed to make surface integrals rigorous) which will be new to most students.
>
> The book deals only with metric spaces and not with general topological spaces. It deals with sets of measure zero and not with Lebesgue measure. The Banach Fixed Point Theorem is stated and proved in full generality, which is not very complicated, and is immediately put to good use in the Inverse Function Theorem and Implicit Function Theorem. It includes the beginnings of manifolds (it covers simplexes and chains but doesn't go all the way to manifolds).
>
> There are a few gems that you won't find in most books. One is Darboux's theorem that derivatives have the intermediate value property (one of my favorite calculus theorems). Another is a simple but very clever proof that $\sqrt{x}$ can be uniformly approximated on $[0, 1]$ by polynomials; this result is used in proving the Stone–Weierstrass Theorem. The proof is based on a recursively-defined sequence of polynomials; this construction is familiar in operator theory but I've never seen it in a calculus context.
>
> The book has a good selection of exercises, most not very difficult; there are no answers or hints. A Very Good Feature is the inclusion of many short biographies of the mathematicians involved; they are unfortunately printed in footnotes in tiny type (about 8 points on 10 point leading) and so are a little hard to read.
>
> Bottom line: A well-done "theory of calculus" text, that is especially useful if you need the theory of multivariate calculus.

The review in The Mathematical Gazette, 2020 / 03 Vol. 104; Iss. 559, page 187, is also positive, noting:

> Not unlike 'baby Rudin', the style of the early chapters may be a bit more dense than ideal for a student covering the material as part of the first year or two of an undergraduate mathematics degree.
>
> ...
>
> That said, the second half of Conway's text is highly recommended. As it explores more technical theory, going towards higher dimensions and the famous theorems of Gauss and Stokes, the level of detail seems extremely appropriate.
>
> An unusual touch is the addition of quarter-page biographies of all the mathematicians whose work is referenced, which goes well beyond the basic details...

Based on these reviews, and also the reviews of other books by the same author, the recommendation is clear: Use a different book for a first course on (univariate) real analysis, as explored in depth in Section 5, and then give serious consideration to using this book for a follow-up course, emphasizing multivariate analysis.

Of value is also to know that the book is, in terms of typesetting and formatting, optically wonderful, and the pdf file in the web is perfect. What more can you ask for? (How about, a new version with corrected typos, and a solutions manual...)

MRA Book Choice 11: *The Way of Analysis*, Revised Edition, Robert Strichartz, 2000, 739 pages. I already discussed and praised this book in Section 5.2, based on the author's impressive coverage of the standard univariate topics. It also covers the major multivariate topics and—amazingly—also has a chapter dedicated to (introducing the) Lebesgue integral. I admit to not having read anything yet from those chapters, but, given Strichartz' impressive teaching style, observed from a careful reading of the earlier chapters, it stands to reason that he also deploys his pedagogic gifts to the subsequent chapters.

MRA Book Choice 12: *Mathematical Analysis II*, Claudio Canuto and Anita Tabacco, 2nd edition, 2010, 520 pages. I have their *Mathematical Analysis I* on my list of possible books for a first course in real analysis.

MRA Book Choice 13: *A Course in Multivariable Calculus and Analysis*, Sudhir R. Ghorpade and Balmohan V. Limaye, 2010. I discuss and praise at some length the authors' univariate prequel. For the multivariate book, an Amazon reviewer writes: "Following on the equally excellent *A Course in Calculus and Real Analysis*, the authors embark on a visually pleasing, thorough and unique journey that somehow exceeds their opening act." It also receives a short but favorable MAA review, https://www.maa.org/press/maa-reviews/a-course-in-multivariable-calculus-and-analysis (by a different reviewer than the first book), who says "I recommend this book (together with its one-variable version written by same authors) for undergraduate students in mathematics and professors teaching courses in multivariable calculus."

MRA Book Choice 14: *Calculus and Analysis in Euclidean Space*, Jerry Shurman, 2nd printing, 2019. I would say this book falls within the same equivalence class as (and is optically similar to, also being Springer) the aforementioned Ghorpade and Limaye, and Lax and (Maria) Terrell. I read some of it, found it fine, but I did not like the presentation of determinants (and stopped reading the book). The MAA review is very positive, https://www.maa.org/press/maa-reviews/calculus-and-analysis-in-euclidean-space, stating for example:

> The material in this text is, of course, found in other undergraduate texts, but it is not easy to find other books that exhibit such care and clarity in the presentation of the subject. The author's writing style is clear and easy to follow, but, more than that, it is exceptionally well-motivated and contains some useful pedagogical ideas. In addition, throughout the book, the author notes issues that are likely to cause trouble to beginning students, and takes the time and effort to single them out and discuss them thoroughly.

The reviewer also comments, and positively so, on the presentation of the material on the determinant, saying

> [The] notion of a determinant is introduced by first specifying axioms for a determinant function, deducing things from those axioms, and then proving that there exists a unique determinant function satisfying those properties. I first saw this approach to determinants many years ago in Hoffman and Kunze's Linear Algebra, was very impressed with it, and am pleased that a modern book takes steps to make this approach more readily available again.

I was not so pleased: either I am mathematically too incompetent to appreciate it, or Shurman did not do it as well as famous Hoffman and Kunze (or both). Either way, based on my partial reading of his book, and also the glowing MAA review, Shurman's book, despite my gripe about the presentation of determinant, should be considered, quite possibly (and as applies to all book suggestions here) used in conjunction with another book, precisely for the reason, as seen here, that a few topics might well be better presented elsewhere.

MRA Book Choice 15: Michael Field, *Essential Real Analysis*, 2017, 450 pages. I mentioned this book already as a possible main text for a first course in real analysis. See the discussion there for why this book fits (also) in this section.

MRA Book Choice 16: *Analysis in Vector Spaces: A Course in Advanced Calculus*, Mustafa A. Akcoglu, Paul F. A. Bartha, and Dzung Minh Ha, 2009. Quite different from the fun and joy of Miklavcic, this book is dry, with very few graphics (and when, they are very ordinary and simple), and rather terse. Still, I started reading it, and found chapter 2 to be very good, while parts of chapter 3 (bilinear and multilinear functions) seemed, to me at least, poorly expressed. I ordered (and paid for) the solutions manual, and plan to look further in the book, as it might offer some good additional readings. The MAA review https://www.maa.org/press/maa-reviews/analysis-in-vector-spaces is short but positive, where we read:

> The format is "classic analysis textbook": definitions; lemmas and proofs; and theorems (sometimes built from lemmas) and proofs. If that appeals to you, you won't be disappointed.
>
> The book covers all its topics thoroughly and with examples that beautifully illustrate the ideas.

Another review hints at the book being terse and rather high level:

> The authors do not shy away from doing the hard work involved in proving say, the change of variable theorem for integration, the inverse function theorem, and Stokes's theorem—work which is not generally seen in standard calculus books—and thus they are quite correct when they state that students need a firm grip on single-variable calculus and some linear algebra, and a good comfort level with the comprehension and construction of rigorous proofs. Includes many examples and an excellent selection of exercises.
>
> (CHOICE, November 2010)

I very much like the book just by the third author, Dzung Minh Ha, entitled *Functional Analysis, A Gentle Introduction*, which will be mentioned below in the obvious category.

7.2 Supplementary Texts for Multivariate Analysis

There are certainly other attractive books that might be of use as supplementary reading, or for a lower-level, or higher-level course. Either way, the books in the following list are, in my opinion, not perfectly suitable as the main text, because (i) the level is a bit too low, e.g., Flanigan and Kazdan, and Lang; or (ii) the subject matter goes beyond the core material, e.g., all the books with "differential forms" in the title; or (iii) the book is "special" in its approach, contents, and style, e.g., Körner, and Bressoud; or (iv) the book has "aged" (and while its contents and presentation might still be very good, it is best enjoyed as supplementary reading, notably also because the quality of the available pdf file might be less than desirable), e.g., Munkres, and Hoffman.

1. *Calculus Two: Linear and Nonlinear Functions*, 2nd edition, 1990, originally by Francis J. Flanigan and Jerry L. Kazdan in 1971, while for the 2nd edition, as we read from the preface, "the revisions were actually written [by] Bert E. Fristedt and Lawrence F. Gray, with assistance from David Frank".[45] I have a hard copy of this book and so know it—regretfully, a pdf of it does not seem to be available on the web. I hold it in high regard, and is about in the same equivalence class, in terms of mathematical sophistication, of Lang's book, given next.

[45] Fristedt and Gray are the authors of *A Modern Approach to Probability Theory*, 1996, a beautiful, long, dense, graduate level text that I love to hate, because I cannot appreciate it, and also that has received scathing reviews at Amazon for asking the reader to do most of the proofs.

2. *Calculus of Several Variables*, 3rd edition, by Serge Lang, 1987. (I used this book as an undergraduate, in 1988.) It is a sizeable, excellent book with plenty of material, albeit not as mathematically sophisticated as other "advanced calculus" texts. I would suggest it as supplementary reading, or as a prerequisite to be self studied before taking a formal class on this material.

3. *Manifolds and Differential Forms*, Reyer Sjamaar, 2017, 165 pages. This is not a published book, but rather a set of course notes from professor Sjamaar at Cornell University Department of Mathematics, which can be downloaded for free. It is full of impressive and high quality graphics (e.g., Klein bottle, Möbius strip) and is clearly destined to become a textbook. It is intended "for an audience of undergraduates who have taken a typical calculus sequence at a North American university, including basic linear algebra and multivariable calculus up to the integral theorems of Green, Gauss and Stokes."

4. *A Visual Introduction to Differential Forms and Calculus on Manifolds*, Jon Pierre Fortney, 2018. This book is suited as preparation for further study of manifolds and differential geometry. This is an attractive book (visually and content-wise), with relatively densely written pages (making it also annoying to read on a tablet), and about 465 pages. It has an introductory chapter reviewing essentials from linear algebra.

5. *A Geometric Approach to Differential Forms*, David Bachman, Second Edition, 2012.

6. *A Companion to Analysis: A Second First and First Second Course in Analysis*, T. W. Körner, 2003.

7. *Second Year Calculus: From Celestial Mechanics to Special Relativity*, David M. Bressoud, 1991.

8. *Analysis On Manifolds*, James R. Munkres. This is arguably a famous book. CRC has a version dated 2018, while Amazon gives the year as 1997; but the book mentions "First published 1991 by Westview Press". I have an old copy (in my office, 9000 kilometers away from where I am right now, so I cannot check) and I recall that it is optically "readable but not great", with small print. I also recall being enchanted with it, having read parts of it years ago.

9. *Analysis in Euclidean Space*, Kenneth Hoffman, 1975, reprint 2007, 2015. Much earlier in time (1975), when there was barely a choice of analysis books, and they were all terse, ominous, and foreboding, the appearance of Hoffman's book was quite a change. The review by Dennis Berkey, in the American Mathematical Monthly (1977), 84(7), pp. 581-582, is blatantly positive, writing about this fascinating feature of this strange new book, that appears to be written in such a way as to actually *teach* the student the material:

   > The text is written with the development of the student, not merely the rigorous presentation of analysis, in mind. The degree of responsibility placed on the reader increases throughout the text, and basic techniques and fundamental concepts are repeatedly emphasized.
   >
   > Finally, the range of topics is broad, as the student is involved in significant applications of linear algebra, elementary properties of matrix-valued functions (including $e^{\mathbf{A}t}$), and the elementary theory of complex analytic functions.
   >
   > *Analysis in Euclidean Space* is a text for those who wish to communicate to their students the essence, and not just the facts, of analysis.

   The MAA review `https://www.maa.org/press/maa-reviews/analysis-in-euclidean-space` is rather detailed about the approach and contents of the book, and the reviewer is clearly ecstatic about this book having been re-released. At the end, we read:

> Hoffman's *Analysis On Euclidean Space* is a forgotten classic and its reissue in this beautiful cheap edition is a cause for celebration for all lovers of mathematics from Harvard to Hunter. I would love to assign this as the text the first time I teach real analysis. Instructors who do this will be doing their students a great service — not only educationally, but financially. It's a book whose depth is such that everyone can learn something from it — doesn't matter if you're an honors freshman struggling with epsilon-delta arguments for the first time or you're doing your PhD thesis in operator theory at Yale. Buy your copy now — you'll be glad you did. I know I am. And we can all thank Dover for making it available again.

There are a handful of reviews at Amazon, a couple of which having considerable detail, and one of which opens with "This is a stunning and masterful exposition of real analysis." However, not everyone agrees: One person, apparently a professor of mathematics (gave the book a one-star rating, and) begins with (and note the very first sentence has two mistakes):

> I think I would rate this text as about the worse I used in my 35 career as a professor of mathematics. Disclaimer: I did not choose myself the text - it was used in the first semester so when I inherited the class I had to keep it. So, it might be that the first half of the text is better than the second half. As an instructor, my complaints are that this text is full of mistakes, from actual math mistakes to a plethora of typos that confuse the students.

The person goes on with more detail, including: "my students positively hate this text and complain often that they can't learn from it. I can't blame them."

I state the obvious when confronted with such conflicting reviews: Best you read it yourself. My personal reading list is already too large, and I will simply pass, given the presence of so many other new and excellent books that overlap enough in terms of content with Hoffman.

7.3 Undergraduate Introduction to the Lebesgue Integral and Basic Measure Theory (or to the Kurzweil-Henstock Integral)

There is apparently now a new and well-defined niche of books, having emerged in about the last 20 years. Every book I give here is in its first edition and written recently. The books are all relatively short, explicitly suited for (advanced) undergraduates (in mathematics), and dedicated to introducing the Lebesgue integral—or, for a couple of them, the gauge or Kurzweil and Henstock integral, the latter being (apparently) easier, and more general. The books are presented in chronological order. I follow this list a subsequent one, containing just two entries, of books emphasizing the historical development of the Lebesgue integral.

Abbreviation "GIMT" stands for Gentle Introduction to Measure Theory.

GIMT Book Choice 1: Lee Peng Yee and Rudolf Výborný, *The Integral: An Easy Approach after Kurzweil and Henstock*, 2000.

GIMT Book Choice 2: Michael Carter and Bruce van Brunt, *The Lebesgue-Stieltjes Integral: A Practical Introduction*, 2000, 228 pages. From the preface,

> We have assumed that the reader has a reasonable knowledge of calculus techniques and some acquaintance with basic real analysis. The early chapters deal with the additional specialized concepts from analysis that we need. The later chapters discuss results from functional analysis. It is intended that these chapters be essentially self-contained; no attempt is made to be comprehensive, and numerous references are given for specific results.

The review by I. Tweddle in the *Proceedings of the Edinburgh Mathematical Society*, 44(3), 2001, pages 661-662, indicates that the book is a bit lacking in certain important areas of coverage:

> The setting-up of the integral is done in some detail, but thereafter very little is proved, so that the reader ends up with a mass of facts but not a lot of real substance.
>
> A few examples refer to probability distributions, but otherwise I see no genuine applications in the text—it is arguable, of course, that those who want to apply the theory already know their applications and their need is to understand the tools they have to use.
>
> I am sure that the authors have produced an interesting and useful account of their topic. The overview presented in the book and the illustrative examples should be of value both to those whose interests are in applications and those who are concerned with the theory. However, I feel that the book can only be a prelude to further study if a proper understanding of the applications is to be secured.

GIMT Book Choice 3: John Franks, *A (Terse) Introduction to Lebesgue Integration*, 2000. From the preface,

> This text is intended to provide a student's first encounter with the concepts of measure theory and functional analysis. Its structure and content were greatly influenced by my belief that good pedagogy dictates introducing difficult concepts in their simplest and most concrete forms. For example, the study of abstract metric spaces should come after the study of the metric and topological properties of $\mathbb{R}^n$. Multidimensional calculus should not be introduced in Banach spaces even if the proofs are identical to the proofs for $\mathbb{R}^n$. And a course in linear algebra should precede the study of abstract algebra. Hence, despite the use of the word "terse" in the title, this text might also have been called "A (Gentle) Introduction to Lebesgue Integration". It is terse in the sense that it treats only a subset of those concepts typically found in a substantive graduate level analysis course.
>
> ...
>
> The text presupposes a background which a student should possess after a standard undergraduate course in real analysis. It is terse in the sense that the density of definition-theorem-proof content is quite high. There is little hand holding and not a great number of examples. Proofs are complete but sometimes tersely written. On the other hand, some effort is made to motivate the definitions and concepts.

The MAA review of this book (and jointly with two others, one by Bear, which we mention below, and the other by Richardson, 2009, *Measure and Integration: A Concise Introduction to Real Analysis*) is at https://www.maa.org/press/maa-reviews/a-terse-introduction-to-lebesgue-integration. (The author of the review is Kenneth Ross, author of *Elementary Analysis: The Theory of Calculus*, 2nd edition, 2013, which we mention above.) From this review, we glean that Richardson is the most advanced presentation of the three (and implicitly that it is not appropriate for the intended audience). For example, we read, regarding Richardson's book, "then Chapter 2 jumps right into abstract measure theory. The reader or student needs to be quite comfortable with abstraction to survive this concise treatment." The reviewer avoids overt rankings and praise, but it is clear he likes both Bear and Frank, with the critique of Frank limited to "I'm shocked that it doesn't even mention Fubini's theorem".

GIMT Book Choice 4: H. S. Bear, *A Primer of Lebesgue Integration*, 2nd edition, 2001. The MAA review of this book is (positive and) discussed above in the context of Franks' book. The reviews at Amazon are mixed and overall good, with one person having written a rather long evaluation, 3 out of 5 stars, noting:

> This book's primary flaw is that it makes a mistake common among modern mathematicians i.e., that abstraction of itself affords clarity.
>
> ...
>
> Likewise H. S. Bear's book, which is called a primer but fails in many respects—initially with the introduction of nets and general ordering relations—to be one. Primers should focus not only on the introduction of basic principles but should not place too a great strain on intuition for the reason that they, like primers of old, are intended to teach one how to read—in this case to read a topic on mathematics. Most students—even very smart ones—are not prodigies and need to develop confidence. Prodigies can start out with Shakespeare and the Aeneid in the Latin but most of us need Dick and Jane. Thus, among many things, in primers logical sequences should be carefully explained and illustrated with concrete examples. The obvious should be explained!

GIMT Book Choice 5: Richard Beals, *Analysis: An Introduction*, 2004. I discuss this book in the comments in Section 5.3. It is a stand-alone beginning real analysis book whose focus and intended goal is the basics of measure theory and Lebesgue integration.

GIMT Book Choice 6: Sergei Ovchinnikov, *Measure, Integral, Derivative: A Course on Lebesgue's Theory*, 2013. From the preface,

> [My] goal in writing this book was to present Lebesgue's theory in the most elementary way possible by sacrificing the generality of the theory. For this, the theory is built constructively for measures and integrals over bounded sets only. However, the reader will find all main theorems of the theory here, of course not in their ultimate generality.

GIMT Book Choice 7: Gail S. Nelson, *A User-Friendly Introduction to Lebesgue Measure and Integration*, 2015. From the preface,

> The prerequisite for this course is a standard undergraduate first course in real analysis. Students need to be familiar with basic limit definitions, and how these definitions are used in sequences and in defining continuity and differentiation. The properties of a supremum (or least upper bound) and infimum (or greatest lower bound) are used repeatedly. The definition of compactness via open coverings is used in this text, but primarily for $\mathbb{R}^n$. I also assume students have seen sequences and series of functions and understand pointwise and uniform convergence.

The MAA review https://www.maa.org/press/maa-reviews/a-user-friendly-introduction-to-lebesgue-measure-and-integration is strongly positive. The part about the author "providing many illuminating examples and helps the reader develop intuition for many of the proofs" can be contrasted with the above quote from the reviewer from Amazon on Bear's book.

> The entire book is very carefully and clearly written. For instance, the author provides many illuminating examples and helps the reader develop intuition for many of the proofs. I believe that an average student with the appropriate background can read and digest the book without too much assistance.
>
> In summary, Nelson has written a very readable account of Lebesgue measure and integration that should be accessible to students who have successfully completed a, more or less, standard first course in real analysis at the undergraduate level.

> As a final note, A Primer of Real Functions by Boas is perfect as background or a refresher for reading this book, and Measure and Integral by Wheeden and Zygmund would make excellent followup reading. Both of these texts are referenced in Nelson's book.

(The Boas' book referred to here is indeed a treat, and mentioned above in Section 5.3.)

Perhaps noteworthy, I (bought and read the book, and) wrote to the author (Oct. 3rd, 2018), politely inquiring about what I thought to be a mistake: "on page 6, in the line where you define m_1^* and m_2^*, I believe that m_2^* should be an inf, and not a sup." I never heard back (but am trying again now).

GIMT Book Choice 8: William Johnston, *The Lebesgue Integral for Undergraduates*, 2015. (Note that this is a different Johnston than Nathaniel Johnston, author of the two books *Introduction to Linear and Matrix Algebra* and *Advanced Linear and Matrix Algebra*.) I have the book and appreciate it, but did not find reading it to be "a walk in the park", nor do not think the prerequisites as stated by the author (basic calculus) are adequate, certainly not for chapter 5. The MAA review https://www.maa.org/press/maa-reviews/the-lebesgue-integral-for-undergraduates is quite positive, though agrees that, at least as the book progresses, it would be beneficial for the reader to have had a first course in real analysis. An errata list can be found here: https://www.maa.org/sites/default/files/pdf/pubs/Lebesgue_Errata.pdf.

GIMT Book Choice 9: René L. Schilling, *Measures, Integrals and Martingales*, second edition, 2017. From the preface,

> The purpose of this book is to give a straightforward and yet elementary introduction to measure and integration theory that is within the grasp of second- or third-year undergraduates. Indeed, apart from interest in the subject, the only prerequisites for Chapters 1–15 are a course on rigorous ϵ–δ-analysis on the real line and basic notions of linear algebra and calculus in $\mathbb{R}^n$. The first 15 chapters form a concise introduction to Lebesgue's approach to measure and integration, which I have often taught in 10-week or 30-hour lecture courses at several universities in the UK and Germany.
>
> Chapters 16–28 are more advanced and contain a selection of results from measure theory, probability theory and analysis.

A solutions manual is available.

GIMT Book Choice 10: Satish Shirali, *A Concise Introduction to Measure Theory*, 2018. The end of the book contains solutions to exercises (about 90 pages). Note also the author's more advanced book, *Measure and Integration*, as well as his book *Multivariable Analysis* with Vasudeva, 2011.

GIMT Book Choice 11: Steven G. Krantz, *Elementary Introduction to the Lebesgue Integral*, 2018. From the preface,

> Most texts on the Lebesgue theory are pitched to graduate students, and require considerable sophistication of the student. It is important and useful to have a text on the Lebesgue theory that is accessible to bright undergraduates. This is such a text. Typically a student would take a course from this book after having taken undergraduate real analysis. So this would be fodder for the senior year of college. We have endeavored to keep this book brief and pithy. It has plenty of examples, copious exercises, and many figures. The point is to make this rather recondite subject accessible.

The MAA review https://www.maa.org/press/maa-reviews/elementary-introduction-to-the-lebesgue-integral is highly positive. Omitting the technical parts (and comparison to Burkhill's 2004 book), we read:

> Steve Krantz writes good mathematics books: he has written many and they do the job well. In the past I've used his books in a number of courses I have taught at the junior and senior undergraduate level at my university, and they have always done justice to the material, as well as standing out for their readability. Krantz (actually, I've known him for over forty years, so I'll just call him what I always call him, Steve) is an excellent teacher and knows his audience, and, by transitivity, his books can aid us in at least trying to reach our own audiences in the same way. I am very happy that he's written a book on the Lebesgue integral.
>
> ...
>
> There are exercises galore, and good ones: accessible, and in concert with the ambient material. Selected exercises' solutions are appended. The book also comes with a useful glossary, and a table of notations. Finally, the book's final chapter concerns applications to harmonic analysis: very pretty stuff.

It is no surprise that Krantz includes some work on harmonic analysis: He does so also near the end in his beginning real analysis book *Real Analysis and Foundations*, fourth edition, 2017; and he has an entire book on the topic, *Harmonic and Complex Analysis in Several Variables*, 2017. He has 65 books (according to his Amazon page), and a new one, *Differential Equations: A Modern Approach with Wavelets*, 2020.

GIMT Book Choice 12: Alessandro Fonda, *The Kurzweil-Henstock Integral for Undergraduates: A Promenade Along the Marvelous Theory of Integration*, 2018. This relatively short book, about 220 pages, is not an introduction to the Lebesgue integral, but rather, as the title indicates, the Kurzweil-Henstock (or gauge) integral, which is also covered in the third and fourth editions of Bartle and Sherbert's book. This book (is optically wonderful and) starts very modestly, with the first two chapters (up to page 123) being very readable. The last chapter, 3, is on Differential Forms, followed by some appendices, including a proof of the Banach–Tarski Paradox.

GIMT Book Choice 13: Steve Cheng, *A Short Course on the Lebesgue Integral and Measure Theory* 2004, 53 pages, available for free at http://webeducation.com/wp-content/uploads/2021/05/Steve-Cheng-A-Short-Course-on-the-Lebesgue-Integral-and-Measure-Theory-Perseus-Books-Sd-1990.pdf, and note that in the link, "webeducation" is (crucially or it will not work) "webéducation". From the preface,

> This article develops the basics of the Lebesgue integral and measure theory. In terms of content, it adds nothing new to any of the existing textbooks on the subject. But our approach here will be to avoid unduly abstractness and absolute generality, instead focusing on producing proofs of useful results as quickly as possible.
>
> Much of the material here comes from lecture notes from a short real analysis course I had taken, and the rest are well-known results whose proofs I had worked out myself with hints from various sources. I typed this up mainly for my own benefit, but I hope it will be interesting for anyone curious about the Lebesgue integral (or higher mathematics in general).
>
> I will be providing proofs of every theorem. If you are bored reading them, you are invited to do your own proofs.

The size of the document, and the statements in the preface indicating a modest presentation with complete proofs, are the reasons why I include this manuscript here (not to mention what appears to be very clean, carefully written presentation).

The following two books (have received very strong reviews and) detail the historical development of the Lebesgue integral, and so can serve as supplementary reading.

1. David M. Bressoud, *A Radical Approach to Lebesgue's Theory of Integration*, 2008, 344 pages. From the book's online description,

    > This lively introduction to measure theory and Lebesgue integration is motivated by the historical questions that led to its development. The author stresses the original purpose of the definitions and theorems, highlighting the difficulties mathematicians encountered as these ideas were refined. The story begins with Riemann's definition of the integral, and then follows the efforts of those who wrestled with the difficulties inherent in it, until Lebesgue finally broke with Riemann's definition. With his new way of understanding integration, Lebesgue opened the door to fresh and productive approaches to the previously intractable problems of analysis.

    From the MAA review https://www.maa.org/press/maa-reviews/a-radical-approach-to-lebesgues-theory-of-integration,

    > I think that there must be hundreds of books that treat the Lebesgue integral, but I expect that very few of them are as readable as David Bressoud's *A Radical Approach to Lebesgue's Theory of Integration*. Several times, as I wrote this review, I found myself picking up the book, opening to a random page and reading — and continuing to read because I was captured by the story and I wanted to see what came next.
    >
    > Why does the book have this appeal? For one thing, it is written with clarity and directness, one scholar to another. For another, it is historically savvy, sensitive to the context of the developments it describes. The author says, "...this is a textbook informed by history, attempting to communicate the motivations, uncertainties, and difficulties surrounding the key concepts." We get a real sense of the challenges faced by Lebesgue, Riemann, Weierstrass, Cantor and a host of others. Pedagogically this makes a lot of sense. The author, paraphrasing Luzin, puts it this way: "...the ideas, methods, definitions, and theorems of this study are neither natural nor intuitive." He goes on to say, "Here, more than anywhere else in the advanced undergraduate or beginning graduate curriculum, the historical context is critical to developing an understanding of the mathematics."
    >
    > What I like best about Bressoud's treatment is the way he conveys the sense that real analysis, and the Lebesgue theory in particular, developed in time, in fits and starts, with contributions from real people who sometimes made mistakes. We also see real analysis as more cohesive and less driven by bizarre counterexamples (as it must sometimes seem to students). Neither does Bressoud avoid the anomalies — after all, they were a driving force in the search for a better integral. The treatment of the interplay between analysis and set theory on the real line is important and is handled very well here. I particularly liked the discussion of non-measurable sets, which faces head-on issues related to the axiom of choice, offering illumination instead of mystification.
    >
    > Together with the author's earlier *A Radical Approach to Real Analysis*, this book offers a strong and approachable introduction to analysis. The current book is aimed at advanced undergraduates or beginning graduate students. It does not strive for the same breadth as other treatments of graduate-level real analysis; the development is limited to the real line and several of the usual topics from functional analysis are not discussed. Nonetheless, this would have been my first choice as a student.

2. Thomas Hawkins, *Lebesgue's Theory of Integration: Its Origins and Development*, 2002, 227 pages. The MAA review https://www.maa.org/press/maa-reviews/lebesgues-theory-of-integration-its-origins-and-development is very positive, writing:

> Hawkins's account of 19th century analysis is a tale of failure and success, of gaffes and insights, of mathematicians great and small who paved the way for the incomparable Henri Lebesgue and his wonderful integral. It is a mathematical adventure story.
>
> At this point, two observations are in order. First, the story Hawkins tells is vastly more complicated than is suggested by this brief synopsis. He addresses other thorny issues from the time, such as when/whether the integral of the limit is the limit of the integrals. And he samples the work of mathematicians like Darboux, Thomae, Hankel, Harnack, and Baire. Some of these individuals will be familiar to modern readers and some will not, but each played a role in the tale.
>
> Second, as should be clear from the account above, Hawkins cuts no mathematical corners. His book is technically challenging and is not to be attempted without a solid grounding in real analysis. Be forewarned.
>
> And you need not take my word for its excellence. In 2001, Thomas Hawkins received the first Whiteman Prize from the American Mathematical Society for "notable exposition in the history of mathematics."

Before closing this section, I refer to Terence Tao's *An Introduction to Measure Theory*, 2011, 206 pages, a book that was mentioned in Section 5.2 when discussing Tao's introduction to real analysis books *Analysis I* and *Analysis II*. It could also (or perhaps should) have been placed in Section 8.3. The MAA review https://www.maa.org/press/maa-reviews/an-introduction-to-measure-theory is very positive, as are the reviews on Amazon. I did not take the time to carefully read and inspect the books in this section, so I cannot say if and how much more sophisticated Tao's book is compared to the ones in the above list. From the MAA review (and in agreement with what Tao states in the preface), "Most of the material of the text is self contained and addressed to the students with only an undergraduate knowledge of real analysis." Solutions do not seem to be available to the (sizeable number of) exercises in the book, which the author implores the reader to work on.

7.4 Metric Spaces

There are now several books dedicated to aspects of metric spaces. As mentioned above, study of this subject should be of great value to students and practitioners aspiring to learn, use, and develop machine learning algorithms. As one example of many, note the relation of the Expectation-Maximization (EM) algorithm (as is commonly introduced in intermediate classes in statistical inference, and also appears in many books on machine learning) and fixed point theorems; see, e.g., Ahsene Lanani, 2018, *Relationship Between the Fixed Point Theorem and the EM Algorithm*, Journal of Informatics and Mathematical Sciences, 10(4), pp. 697–702.

Abbreviation "MS" stands for Metric Space. Books are presented in chronological order.

MS Book Choice 1: E. T. Copson, *Metric Spaces*, 1968. This is a very attractive and well written book at the appropriate level for the audience. The available pdf file is, perhaps surprisingly, of rather good quality, and text can be marked and searched. From the description of the book on Amazon (with the preface of the book regretfully not being in the available pdf file),

> Metric space topology, as the generalization to abstract spaces of the theory of sets of points on a line or in a plane, unifies many branches of classical analysis and is necessary introduction to functional analysis. Professor Copson's book, which is based on lectures given to third-year undergraduates at the University of St. Andrews, provides a more leisurely treatment of metric spaces than is found in books on functional analysis, which are usually written at graduate student level. His presentation is aimed at the applications of the theory

> to classical algebra and analysis; in particular, the chapter on contraction mappings shows how it provides proof of many of the existence theorems in classical analysis.

Reviewer Claude A. Rogers writes, in *The Mathematical Gazette* Vol. 53, Issue 385, 1969, page 342,

> This book gives a very clear and detailed introduction into some aspects of the theory of metric spaces with detailed discussions of examples drawn almost exclusively from elementary classical analysis. The treatment culminates in the contraction mapping theorem and its application to prove a variety of existence theorems. The material could all be taught to second year undergraduates and I believe most of it should be.

The book is referred to for a proof (of the fact that, for a given metric space, if either of Heine-Borel or Bolzano-Weierstrass is true, then the other is true) on page 52 of Boas (1996), another very attractive and readable book.

MS Book Choice 2: Victor Bryant, *Metric Spaces: Iteration and Application*, 1985, 1994.

MS Book Choice 3: Mohamed Khamsi and William Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, 2001.

MS Book Choice 4: Satish Shirali and Harkrishan Vasudeva, *Metric Spaces*, 2006.

MS Book Choice 5: Mícheál Ó Searcóid, *Metric Spaces*, 2007.

MS Book Choice 6: Christopher Heil, *Metrics, Norms, Inner Products, and Operator Theory*, 2018. From the preface,

> This text is aimed at students who have some basic knowledge of undergraduate real analysis, and who have previous experience reading and writing mathematical proofs. The intended reader of this text is a motivated student who is ready to take an upper-level, proof-based undergraduate mathematics course. No knowledge of measure theory or advanced real analysis is required.

MS Book Choice 7: Dhananjay Gopal, Aniruddha Deshmukh, Abhay Ranadive, and Shubham Yadav, *An Introduction to Metric Spaces*, 2020. Instead of discussing its contents, I reflect on a statement the authors make in their preface, in "A Note to the Reader", namely:

> It is a basic principle in the study of mathematics, and one too seldom emphasized, that a proof is not really understood until the stage is reached at which one can grasp it as a whole and see it as a single idea. In achieving this end, much more is necessary than merely following the individual steps in the reasoning. This is only the beginning. A proof should be chewed, swallowed, and digested, and this process of assimilation should not be abandoned until it yields a full comprehension of the overall pattern of thought.

Fascinatingly, we find that paragraph, word for word, in the "A Note to the Reader", in George F. Simmons, *Introduction to Topology and Modern Analysis*, 1963 (reprint 1983). We know from Oscar Wilde that "Imitation is the sincerest form of flattery that mediocrity can pay to greatness", but this is outright plagiarism.

In the preface, we do in fact read:

> Without claiming originality of the results, we do claim simplicity and lucidity of presentation coupled with comprehensiveness of the materials. We have been influenced by many books on the same subject and have listed in the bibliography those books which have been of particular assistance to us in preparing this book.

and indeed, Simmons is cited in the bibliography.

The book *Introduction to Metric and Topological Spaces*, by Wilson A. Sutherland, 2009, could also have been placed in this list, though I chose to add it instead to the subsequent Section 7.5.

7.5 (Point-Set) Topology

A typical student wishing to study "finance" is often motivated by the prospect of getting a well-paid job in the finance or banking industry. A percentage of these students have intrinsic interest and talent in mathematics, and naturally get enticed by "quant finance", as a way to help ensure (vastly less competition for) that good job, only to realize that the ticket to entry has a high cost in terms of required mathematical prowess. What might surprise many (and was unknown to myself until I went and looked) is the emerging prominence of topological concepts in machine learning. This is made clear for example in the recent article by Adams and Moy (2021), along with the 2019 teaser by Brüel-Gabrielsson et al., entitled "A Topology Layer for Machine Learning"; see https://ai.stanford.edu/blog/topologylayer/. An internet search with the obvious keywords yields plenty more sources of information, as does the search for (the emerging field of) Topological Data Analysis (TDA). As such, we are behooved (and delighted) to include a section on introductory books on topology.

There is at least a dozen books that start with the rudiments of point-set topology and are aimed at undergraduates—not necessarily only in mathematics, due to the application potential of the subject (see notably the book by Adams and Franzosa). The beginning material in most introductory topology books can also be viewed as a valuable augmentation of a first course in real analysis, and also serve as a vehicle for learning about metric spaces (which in some topology books appear right at the beginning, while others present it later, as just one example of the theory).

We present a list of books that concentrate on (or at least start with) point-set topology, and are suitable for (mathematics) undergraduates, and thus with modest prerequisites, e.g., basic linear algebra, a first course in real analysis, and a modicum of the elusive "mathematical maturity". We start with the classic books by Munkres (not suitable as a first book) and Simmons (very suitable), and then proceed alphabetically.

1. *Topology*, James R. Munkres, 2nd edition, 2000, 537 pages. The MAA review https://www.maa.org/press/maa-reviews/topology-0 makes clear that this book (first edition, 1975) is a true classic. Quoting:

    > I would say, by way of rough analogy, that this book is to undergraduate topology what Rudin's Principles of Mathematical Analysis (aka "Baby Rudin") is to undergraduate analysis.

2. George F. Simmons, *Introduction to Topology and Modern Analysis*, 1963 (reprint 1983). This book was already mentioned (regarding a memorable and interesting statement in its preface about understanding proofs) above, in the discussion of Gopal et al. (2020).

    As we read in the MAA review of a different book on topology (Singh, 2013, discussed below), the first half of Simmons' book is an introduction to point-set topology, while the second half is an introduction to functional analysis at an undergraduate level. As such, this book also appears and is discussed below in Section 8.2, on functional analysis. From the MAA review of yet another book, namely Ault's *Understanding Topology* (discussed below) we read:

    > Topology is one of those subjects that can be taught to undergraduates in a number of different ways. I learned it, as an undergraduate, from the first part of Simmons's book Introduction to Topology and Modern Analysis; the course followed the usual point-set route of metric spaces, topological spaces, compactness and connectedness. I thought it was a great course at the time,

> because it helped put real analysis into what I thought was a better context, but only afterwards came to realize that there was much more to the subject, even at the undergraduate level. Simmons's book is weak on the geometric aspects of topology, and not only are things like surfaces not mentioned, neither are quotient spaces; in fact, there is nary a Möbius strip or Klein bottle to be found in the book. Also, Simmons (like most other undergraduate topology texts at the time) doesn't address basic algebraic topology at all.

3. *Introduction to Topology: Pure and Applied*, Colin Adams and Robert Franzosa, 2008, 489 pages. The MAA review https://www.maa.org/press/maa-reviews/introduction-to-topology-pure-and-applied is very positive. Regarding for whom it is intended, and the extent to which it has applications, we read:

   > This rather unusual book aims to introduce topology as a mathematical discipline and to demonstrate the value of topological ideas in various areas of science, engineering and mathematics. It is intended to serve as a text for a one or two semester undergraduate course for students who have completed an introductory course in abstract mathematics or for mathematically mature students who have taken the basic calculus sequence.
   >
   > A reader of this text will learn the basics of point set topology in the first seven chapters, and then be introduced to a variety of topics including knots, manifolds, dynamical systems, fixed point theorems, and graph theory. The authors regard point set topology as the core of the book. For each of the other topics they offer a introductions designed to elicit student interest and facilitate further investigation. Applications are sprinkled liberally throughout; they are well-selected and germane to real problems.

   The presentation of the book is very nice, and pdf file is nearly perfect, with markable text.

4. *Understanding Topology: A Practical Introduction*, Shaun V. Ault, 2018, 416 pages. The MAA review is lengthy, very detailed, very informative, and overall positive, https://www.maa.org/press/maa-reviews/understanding-topology, also comparing the book to a handful of others. The author says the book "requires only a knowledge of calculus and a general familiarity with set theory and logic", while the reviewer disagrees, notably for later chapters in the book.

   The pdf file is definitely good enough for enjoyable reading, though it appears to be based on a (good) scan, and such that text is not markable.

5. *Introduction to Geometry and Topology*, Werner Ballmann, 2018, 165 pages. Translated from German. From the preface,

   > The text is conceived as the basis for a semester-long lecture course in the middle of a bachelor's program.
   >
   > I assume familiarity with linear algebra and real analysis of several variables. The first two chapters of the book are devoted to introductions to topological spaces and manifolds.

   The book is rather advanced, and the stated prerequisites (at least perhaps for audiences outside of mathematics departments in Germany) might be more accurate if they included a previous exposure to topology, as well as manifolds (the latter arguably covered in his requirement of "real analysis of several variables", though not all books with such titles cover manifolds and differential forms).

   As such, formally, this book does not belong in this list. Nevertheless, the first chapter is accessible (and provides a proof of the Jordan Curve Theorem, not in full generality, but just for a piecewise linear curve). We also mention it because (arguably very superficially) the available pdf file is not only perfect, but the book (in the modern format used by Birkhäuser's "Compact Textbooks in Mathematics" series is optically heavenly.

6. *A Course in Point Set Topology*, John Bligh Conway, 2014, 142 pages. The MAA review https://www.maa.org/press/maa-reviews/a-course-in-point-set-topology is very positive, though does note that the brevity of the book implies instructors will not have a variety of topics to choose from. We read, for example:

> Conway has that rare and valuable ability to write reasonably informally without sacrificing precision. In addition, his books also generally seem to have, for want of a better word, *personality*.
>
> The prerequisites are modest: technically, just a good calculus background should suffice, but some prior exposure to "theoretical calculus", as, for example, in a real analysis course (or perhaps even a really good honors calculus course), would be a definite plus.
>
> There are only three chapters. The first is on metric spaces and covers the basic topological concepts (continuity, convergence, compactness, connectedness, etc.) in that context. I fully agree with the author that this is the best way to introduce the subject of topology: it builds on the reader's Euclidean intuition of distance and also provides an easy way to motivate the definition of a general topological space, which in this book is the subject of chapter 2.
>
> [N]on-specialists in the area (like me) can certainly benefit from reading an expert's opinion of what is, or is not, important. By and large, based on my own (admittedly non-expert) opinion, I thought the author did an excellent job selecting topics, with one exception — although there is a section on quotient spaces in chapter 2, there is no reference at all to either Moebius strips, the Klein bottle, or the projective plane. How can you have an introductory text on topology that doesn't even have a picture of a Moebius strip in it? Fortunately this is hardly a deal-breaker; an instructor who is of a mind to do so can easily supplement the book on this one point. However, having already defined quotient spaces, it does seem like a missed opportunity to not give these neat examples.
>
> No solutions are provided in the text, which I view as another pedagogical plus.
>
> To summarize: this is a well-written book that I enjoyed reading. Assuming that your idea of what to teach in a first-semester course in topology is in line with the author's, this book would make an excellent text for such a course.

The book is optically nice, and the pdf file is perfect.

7. *Principles of Topology*, Fred H. Croom, 2016, 312 pages. (Original, 1989). The MAA review https://www.maa.org/press/maa-reviews/principles-of-topology is very positive (but still points out a couple of minor issues), and also indicates the level of the book:

> This is a very nice introduction to (mostly) point-set topology, ideally suited for use as a text in courses where the students have diverse backgrounds and a very sophisticated treatment is unwarranted. The exposition here is careful, conversational, clear, and well-motivated. The only real background required is a good working knowledge of calculus and some practice in writing and reading proofs.
>
> The order of presentation of the material also has some pedagogical advantages. I have long thought, for example, that the best way for students to be introduced to topology is by doing metric spaces first. It is, after all, a

> reasonably small jump from working with absolute values on the real line to the more abstract idea of a distance function, so metric spaces are easy to motivate. They, in turn, then help motivate the more abstract notion of a topological space, because by the time topological spaces are ready to be defined, the student has already seen open sets in metric spaces and is familiar with their properties. That is the approach taken here.
>
> In fact, over the years I have had the pleasure of reviewing a number of other topology texts for this column: the classic by Munkres, other Dover reprints by Gemignani and Gamelin and Greene, and more recent books by Conway, Singh, and Manetti. These are all very good books, but looking at them solely as candidates as a text for a junior-level undergraduate course, I would put the book now under review at the top of the list.
>
> In addition to the books mentioned above, I should also cite Simmons' Introduction to Topology and Modern Analysis, a book for which I have considerable sentimental attachment: I learned topology from this text as an undergraduate, about 45 years ago, and loved it. It is still an excellent book, written with great style and clarity, but I now think it has flaws that I didn't notice as an undergraduate (no discussion of quotient spaces and things like Mobius strips, for example). Croom's book is about as clearly written as Simmons, but remedies some of these flaws, so I would probably rank this text even higher than Simmons.
>
> As a possible text for an undergraduate introductory topology course at the easy end of the spectrum, this book merits a very serious look.

There are numerous positive reviews at Amazon, one seemingly informative one stating:

> I'm an adult, self-study student, with a background in calculus, physics. I've now gone through several books on topology, and I find that even many of the undergraduate texts tend to be a bit "dense," in that they introduce too much, too fast. Croom's textbook takes a very step-by-step, hand-holding approach to introducing topology, focusing on concrete examples, yet still having a reasonable amount of rigor. (Of nine chapters, he doesn't even formally get to topology until Chapter 4. The first three chapters are a general intro, open and closed sets, and metric spaces.) The last chapter offers a basic introduction to algebraic topology. This is an excellent book for self-study, and also good for undergraduates with a physics or engineering orientation who want to get the intuitive principles, and also some sense for the formal math.

The available pdf file is perfect.

8. *Essential Topology*, Martin D. Crossley, 2005 (Corr. print 2010). From the preface, we read an explanation of the title:

> In fact, if I may be so bold as to say so, the subjects covered by this book are those areas of topology which *all* mathematics undergraduates should ideally see. In that sense, the material is *essential* topology.
>
> ...
>
> Anyone who has some basic familiarity with functions, such as from a beginning course on calculus, should be able to follow the first four chapters. From Chapter 5 onwards, a little knowledge of algebra is required, in particular equivalence relations for Chapters 5 and 6, some familiarity with groups for Chapters 8 to 11, and with linear algebra and quotient groups for Chapters 9 and 10.

The MAA review https://www.maa.org/press/maa-reviews/essential-topology is clearly positive. We read:

> The first two parts of the text give a crisp, uncluttered, and well-flowing presentation of standard point-set topology topics. Covered in these two parts are continuity, topological spaces, subspaces, connectivity, compactness, the Hausdorff property, homeomorphisms, product spaces, and quotient spaces. Omitted are some favorite topics — metric spaces, space-filling curves, and the Jordan curve theorem, for example — but these omissions streamline the presentation.
>
> The book is very readable and would be accessible to an undergraduate studying independently. Solutions to selected exercises are provided, something I would not like as an instructor of a course, but which might be helpful to a student reading the text on her own. The first two parts of the text are certainly accessible to undergraduates with sufficient mathematical maturity (a junior or senior). The third part requires knowledge of group theory, and, as I mentioned, is written at a more abstract level.
>
> On the whole, Essential Topology is a nice addition to the introductory topology textbook literature. The text gives a clean presentation of basic point-set topology without neglecting the geometric intuition and without including too many topics.

The book is in standard latex math-book formatting, and the pdf file is perfect.

9. *General Topology*, Jacques Dixmier, 1984, 141 pages. (Translated from French, 1981). This book is meant for a first course (in terms of the material covered), though is relatively terse (or possibly austere, somber, and unadorned—see the quotes below from a reviewer). There is no MAA review, but there is one in The American Mathematical Monthly, 94:5, 1987, pp. 475-9, by Robert Brown. Brown juxtaposes and discusses three books, namely Dixmier, one more advanced book that we do not consider, and that from Klaus Jänich, as discussed below in this list. Quoting,

> ... there are 13 illustrations in all of Dixmier; Jänich has that many by page 15.
>
> But where, the experienced instructor may ask, are the motivating remarks, special cases, and other ruffles and flourishes with which so many authors like to decorate their arguments, especially the difficult ones? It seems there aren't any. The proofs are neat, formal, and boy they are condensed!
>
> [A typical student might think] Dixmier's text looks like a catalogue from an auction of government-surplus property.
>
> An instructor who is willing to put an unreasonable amount of effort into the course could use Dixmier as what amounts to a very high quality outline, but it would be an uphill battle ... There are other good texts intended for American students that would be a lot easier to use. However, a more mature mathematician who wanted to review point-set topology, especially as it relates to functional analysis, might find Dixmier right on target.

It is noteworthy that Brown wrote this, namely "There are other good texts intended for American students that would be a lot easier to use", in 1987. Over 30 years later, and noting the rapid expanse of new, and excellent books on so many other topics (such as linear algebra, and real analysis, as discussed herein), one can be essentially assured that there exist newer, yet even better books, also in (introductory, point-set) topology. This appears to be the case, as this list of books indicates, starting with, but not limited to, Croom's (1989, 2016) book.

The pdf file of Dixmier's book found in the internet is readable, but not optically enjoyable, and on occasion, a bit of print is missing. Given the less than flattering review of the book, notably its terseness and (in my words) lack of personality, as well as the fact that nowadays, numerous other books are available—notably with viable, optically pleasant pdf files—, it is hard to motivate using this book as the primary one for teaching or learning the material. Nevertheless, the above review indicates that it has value perhaps as a reference or alternative presentation, after having used another, more suitable book. Interestingly, there are several strong reviews at Amazon for Dixmier. As such, it is worth knowing about the book, as a possible source of supplementary reading.

10. *Topology*, Klaus Jänich, 1984, 192 pages; translated from German, 1980; and with 181 illustrations—information famously provided by Springer. I personally like Jänich' writing style, having enjoyed his book on linear algebra. From the (very short) preface, we read:

    > This volume covers approximately the amount of point-set topology that a student who does not intend to specialize in the field should nevertheless know. This is not a whole lot, and in condensed form would occupy perhaps only a small booklet. Our aim, however, was not economy of words, but a lively presentation of the ideas involved, an appeal to intuition in both the immediate and the higher meanings.

    From the back of the book, we read:

    > This is an intellectually stimulating, informal presentation of those parts of point set topology that are of importance to the nonspecialist. In his presentation, and through many illustrations, the author strongly appeals to the intuition of the reader, presenting many examples and situations where the understanding of elementary topological questions will lead to much deeper and more advanced problems in topology and geometry.

    Brown, in his review mentioned above in the context of Dixmier's book, mentions Jänich's "wonderful enthusiasm for mathematics", and also how Jänich traverses several advanced topics (e.g., functional analysis, algebraic topology), and remarks how the student reader might wonder to herself how she is going to learn

    > all this trendy mathematics in a book of well under 200 pages, much of it occupied by pictures, [and] which doesn't assume any prerequisite knowledge on the part of the reader. The answer of course is that she isn't going to "learn" all this fashionable, advanced mathematics, rather she's going to "learn about it", which isn't the same thing at all.

    The author is aware of this and does it intentionally to give a motivating broad stroke of the subject, and its importance, writing on page 76 about the advanced topics mentioned: "Those things are admittedly way above what can be done with the tools and on the level of this book, and a critical observer may find it outrageous to talk of them here." The reviewer clearly likes Jänich's book (and does not like Dixmier), stating (as a comparison to the terse treatment in Dixmier):

    > In Jänich, the quotient topology gets an entire chapter, with some fascinating examples including a totally convincing picture proof that the Klein bottle is the union of two Möbius strips.

    The reviewer also notes Dixmier's "rather inadequate" index (and complains about not finding various entries, and thus is not sure if they are even in the book), as compared to Jänich: "[W]hat it does have, incidentally, is the most elaborate index I've ever seen. Not only are the technical terms listed, but their definitions are included in the index, along with the page number reference."

11. *Topology*, Marco Manetti, 2015, 309 pages, translated from Italian. From the back cover we learn that

> This is an introductory textbook on general and algebraic topology, aimed at anyone with a basic knowledge of calculus and linear algebra. It provides full proofs and includes many examples and exercises.

The MAA review https://www.maa.org/press/maa-reviews/topology indicates that it is "intended for an audience of upper-level undergraduates", with the first half covering (a one-semester course in) point-set topology, while the second half covers algebraic topology. We further read:

> Other differences concern the overall level of difficulty of the texts. Manetti seems to spend more time trying to motivate things for the student than does Munkres, whose text does not, for example, have any chapters that are comparable to Manetti's chapters 1 and 9. Overall I would say that Manetti's text is somewhat more reader-friendly than is Munkres', and for that reason more likely to be successful as a text at a typical university.
> Besides the two introductory chapters, there are other things about this text that make it reader-friendly. The author writes clearly and provides lots of examples, and there are also a good number of exercises, solutions to a relatively small number of which are available in a 25-page concluding chapter. The exercises range from routine to quite challenging. (I was amused by the notation that the author uses in the exercise sets: those exercises that have solutions are denoted by a heart symbol, perhaps because the students love seeing solutions; those exercises that the author views as more difficult are marked by a coffee-cup symbol).
> [T]here are occasional passages that remind the reader that this book is a translation from the Italian.
> However, these are all relatively minor quibbles, and are easily outweighed by the good features of this book. Overall, this is a solid entry in the undergraduate-topology textbook market, and anybody teaching a course in the subject should take a look at it.

The book is optically very nice, though, as the reviewer also complains about, the author uses the European (Bourbaki) bracket style, i.e., the notation $]a, b[$ for an open interval. For example, on page 129, we read (and having done this in latex, it looks virtually identical to the passage in the book):

> *Example 7.2* A sub-basis for the Euclidean topology on $\mathbb{R}$ is given by the open sets $]-\infty, a[$, $]b, +\infty$ as $a, b \in \mathbb{R}$ vary, since open intervals $]a, b[$ form a basis, and we can write $]a, b[=]-\infty, b[\,\cap\,]a, +\infty[$.

The pdf file available for the book is perfect.

12. *Introduction to Topology*, Bert Mendelson (1926–1988), Third Edition, 1975, 206 pages; unaltered and reproduced by Dover, 1990; the first and second editions appeared in 1962 and 1968. From the preface, we read:

> I have attempted to resist the temptation to include more topics. There are many excellent introductory topology texts which are first-year graduate school texts and it was not my original intention, nor is it now, to write at that level.

On the last page, 200, the author states further recommended reading, and mentions Simmons' book, among others. There is no MAA review, but there is review of the first edition from 1962, in the American Mathematical Monthly, 1963, Vol. 70; Iss. 7, page 782. It is (short and) very negative. We read:

> This book has two major weaknesses as a text. First, the level of writing in the proofs is excessively low. The author is not adept in the use of common mathematical language, and the proofs contain an excessive number of ambiguous statements. Second, there are far too few examples. A majority of the exercises are merely extensions of the theory. The development of the major ideas is poorly motivated. The result is rather unfortunate: the book is little more than a theorem list.

Apparently not discouraged (or perhaps motivated) by such a scathing review, there were two further editions. We conveniently have online bookstores to help us judge: There are 110 reviews at Amazon (and 290 ratings), already a powerful statement of the book's popularity, and the reviews are overall blatantly positive.

Importantly and perhaps surprisingly, the pdf file is very good—in fact, it is optically excellent and thus highly readable. It is, however, not markable or searchable.

13. *Topology: Point-Set and Geometric*, Paul L. Shick, 2007, 271 pages. The author states in the preface "It is intended to be covered in one semester by typical math majors... [The] text is intended to be quite rigorous, modeling for the students how one writes precise proofs. It covers the essentials of point-set topology in a relatively terse presentation, with lots of examples and motivation along the way."

    The MAA review https://www.maa.org/press/maa-reviews/topology-point-set-and-geometric is not highly informative, and appears to restate parts of the book's preface:

    > This book, intended to be covered in one semester by math majors, provides a rigorous introduction to the basic notions of topology.
    >
    > Paul Schick explains in a terse style, and with abundance of examples, the fundamental constructions in topology: products and quotients of topological spaces. As should be, metric spaces are introduced only at a later stage, as a very special instance of topological spaces (unfortunately, some other textbooks do not adopt the same strategy).

    We also see that this is an example of a topology book that explicitly does *not* begin with metric spaces (and the reviewer favors not doing so, whereas the reviewer of Conway's book makes clear that he favors beginning with metric spaces). The reviewer of Shick also writes:

    > The exposition is always clear and supported by a number of examples and illustrations. Moreover, the exercises proposed at the end of each section make the textbook suitable for a first undergraduate course on topology.

    The pdf file is very readable, though the footnotes, with their smaller font, are not so clear. It is also not searchable or markable.

14. *Introduction to Topology*, Tej Bahadur Singh, 2019, 452 pages. This book could be placed between the two groupings we use, namely introductory, and advanced. Given the strong review the book received, and the fact that the book is optically wonderful, with a perfect pdf file available, one is behooved to draw attention to this book for both groups of students.

    This book is in fact an updated edition of the author's *Elements of Topology*, 2013. The MAA review of the 2019 version, https://www.maa.org/press/maa-reviews/introduction-to-topology-2 is very positive, as was the MAA review of the 2013 edition, https://www.maa.org/press/maa-reviews/elements-of-topology. In the review of the 2013 version, the reviewer is highly positive, but makes the quibble "One mild complaint: it did seem to me that the Index could stand some improvement." The new version indeed has an enhanced index (and, interestingly, the author notes that this was done because the reviewer had pointed it out). Besides fixing typos and modifying some proofs and text, there is now a (200 page) solutions manual available, albeit only to instructors.

Regarding the level of Singh, from the preface, we read:

> This book is intended as an introductory text for senior undergraduate and beginning graduate students of mathematics.
>
> There are some excellent textbooks and treatises on the subject, but they mostly require a level of maturity and sophistication on the part of the reader which is rather beyond what is achieved in mathematics undergraduate courses at many universities. Our objective is to make the basics of general topology and algebraic topology easily comprehensible to average students and thereby encourage them to study the subject.
>
> [T]he reader of the book who has an elementary knowledge of real analysis, group theory, and linear algebra, usually taught at the undergraduate level, would gain a deeper appreciation of the contents.

The reviewer ends with:

> This book would make a good choice for a graduate course in point-set (with an introduction to algebraic) topology, and would also function well as a text for a fairly sophisticated undergraduate course.

15. *Introduction to Metric and Topological Spaces*, Wilson A. Sutherland, 2009, 206 pages. From the short MAA review https://www.maa.org/press/maa-reviews/introduction-to-metric-and-topological-spaces, we read:

> This is a brief, clearly-written introduction to point-set topology.
>
> The book assumes some familiarity with the topological properties of the real line, in particular convergence and completeness. The level of abstraction moves up and down through the book, where we start with some real-number property and think of how to generalize it to metric spaces and sometimes further to general topological spaces. Most of the book deals with metric spaces.
>
> It also has an interesting chapter on quotient spaces, focused on Moebius strips and tori with various numbers of holes.

The numerous reviews at Amazon are very strong, with one writing:

> One of the best introductions to metric and topological spaces there is. Bear in mind that it is an introductory book but for that purpose it is truly excellent. It has a very strong grounding in making sure that clear explanations and meaningful carefully laid out proofs are given throughout the book.
>
> In particular, the notions behind continuity, uniform continuity and compactness are very well explained in the context of metric spaces (and maths in general) and tied together to then easily step off into topological spaces.

The pdf file is “good enough”, having been made from a scan of the book. It is also markable and searchable. Full solutions are available (only to lecturers), as stated on the OUP website for the book.

8 A Next^2 Course

> There are two basic concepts in quantum mechanics: states and observables. The states are vectors in Hilbert space, the observables self-adjoint operators on these vectors. Eugene Wigner (1960)[46]

> This strongly infers that mathematics and the reality it describes are both part of the objective universe and await discovery. Alex Harvey (2011)[47]

> Numbers exist only in our minds. There is no physical entity that is the number 1. ... As mathematics developed, new numbers were invented (some prefer to say discovered) to fulfill algebraic needs that arose.
> John B. Fraleigh and Raymond A. Beauregard (1995, p. 454)

8.1 Numeric Analysis (Graduate Level)

Recall Section 4.6, where several books for a first course in numerical computing (requiring only basic calculus and linear algebra) were given. Here, we list a few books in numerical analysis that are more suited for beginning graduate students in (applied) mathematics, and who have had at least real (univariate and multivariate) analysis, as well as a first course, ideally a second course, in linear algebra, as well as some exposure to complex analysis and normed linear spaces.

1. *Numerical Analysis*, Walter Gautschi, Second Edition, 2012, 588 pages. Despite the size of the book, the author omits topics from linear algebra because he believes they require book-length treatments: Which is precisely what we have seen in Section 4.5. Regarding the changes in the second edition, the author writes in the preface "the subtitle "An Introduction", as suggested by several reviewers, has been deleted.". The preface also states that, new to this edition, "a complete set of detailed solutions to all exercises and machine assignments" is available to instructors upon request from the publisher's website. The website provided does not in fact work, but the pdf of the document can be found on the web. (It is indeed highly detailed, with analytic solutions and also Matlab codes.)

   From the preface, the author indicates the prerequisites for his book:

   > The reader is expected to have a good background in calculus and advanced calculus. Some passages of the text require a modest degree of acquaintance with linear algebra, complex analysis, or differential equations. These passages, however, can easily be skipped, without loss of continuity, by a student who is not familiar with these subjects.

   The MAA review https://www.maa.org/press/maa-reviews/numerical-analysis-2 is very positive:

[46] From his arguably famous and certainly oft-cited: "The Unreasonable Effectiveness Of Mathematics In The Natural Sciences", Communications in Pure and Applied Mathematics, 13(1), pages 1-14 (1960). For readers who are (also) fascinated by the question as to whether mathematics is "real", having an independent existence in some sense, and is just being discovered (as in uncovered); or if the entire apparatus is in fact a figment of our human mental cognition, its roots being tied to our evolutionary past and means of survival, have a look at some articles written in reference to Wigner's work, often with very similar titles. The latter view reminds us of a quote given much earlier, by Jules Henri Poincaré, "Geometry is not true; it is advantageous." It was also supported by Brouwer; see the discussion in Harvey (2011). The former view is supported by Max Tegmark, as showcased in his popular book *Our Mathematical Universe: My Quest for the Ultimate Nature of Reality*, (2014); while his most recent, *Life 3.0: Being Human in the Age of Artificial Intelligence* (2017), might be of much more relevance to budding machine learning experts. See also Sorin Bangu (2016), "On 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences' ", Models and Inferences in Science, pp. 11-29, and the references therein.

[47] From Harvey, "The Reasonable Effectiveness of Mathematics in the Natural Sciences", Gen Relativ Gravit (2011) 43:3657–3664.

> From the very beginning Gautschi's book distinguishes itself from scores of other numerical analysis texts.
>
> Gautschi notes that he breaks the tradition of incorporating all the major topics of numerical analysis in one text; his justification is that major subdisciplines such as numerical linear algebra and numerical solution of partial differential equations have become so substantial that they can no longer be practically or adequately treated in a single book.
>
> Having thus set the metes and bounds of his book, Gautschi has produced a very attractive textbook. Just exactly who would use it and how is another matter. The first chapter is especially impressive. Students may find the business of machine arithmetic rather dull but it is truly fundamental to the whole business of numerical analysis (and frequently the source of otherwise mysteriously bad behavior of algorithms). The author does a very nice job of homing in on the real difficulties. This is as good a treatment as I have seen, but it is pitched at a fairly sophisticated level.
>
> The text has a very complete bibliography as well as an early section pointing the student to a selection of general numerical analysis textbooks, more specialized books and monographs, and relevant journals. Solutions to selected exercises are provided ("to give students an idea of what is expected") and a full solution manual is available to instructors.
>
> The first chapters by themselves would probably not constitute a good introduction to numerical analysis by themselves because of the absence of numerical linear algebra (especially for student who would take only an introductory course). They would nonetheless make for excellent supplemental or reference material.

Gautschi uses Matlab for illustration of codes and output. The pdf file is perfect.

2. *Numerical Analysis: Mathematics of Scientific Computing*, by David Kincaid and Ward Cheney, Third Edition, 2009, 788 pages. The short MAA review `https://www.maa.org/publications/maa-reviews/numerical-analysis-mathematics-of-scientific-computing` is positive but not enthusiastic. We read, for example:

> Open to a wide spectrum of readers, the book is largely self-contained. Each chapter has an introductory section that covers relevant preliminaries at the undergraduate level. The result is a thorough compendium on scientific computing suitable as a reference source or course textbook.
>
> The chapters feature theorems [with] clearly explained proofs. Roughly once per section basic algorithms are given in pseudo-code. This makes it is easy to implement them in any programming language. Section examples are basic and illustrative, as befits the introductory nature of the text.
>
> The chapter problem sets are at higher level and will prove much more difficult to independent readers. Combine this with the absence of any solutions to the problem sets and an unprepared solo reading of this text may prove unfruitful.

3. *Numerical Analysis: A Mathematical Introduction*, Michelle Schatzman, 2002, 512 pages; translated from French. It appears to be a very enjoyably written book, mathematically mature, requiring on behalf of the reader everything mentioned at the beginning of this section. From the preface,

> The prerequisites are linear algebra, calculus, and a tiny bit of Lebesgue theory, which is used only in Chapter 5 on polynomial least-squares approximation, Chapter 7 on Fourier analysis, and Chapter 18 which introduces partial differential equations. I do not use the theory of distributions, though I disguise some of its ideas in the spline Chapter 6.

The pdf file is fully adequate for reading, and text can be marked and copied. It was obviously, however, generated from a photocopy.

4. *Introduction to Numerical Analysis*, Josef Stoer and Roland Bulirsch, Third Edition, 2004, 752 pages. The authors do not discuss the mathematical level of the book, but is clear that at least first courses in linear algebra and analysis are required. The pdf file is perfect.

5. *An Introduction to Numerical Analysis*, Endre Süli and David F. Mayers, 2003, 444 pages. From the preface,

> Our treatment is intended to maintain a reasonably high standard of rigour, with many theorems and formal proofs. The main prerequisite is therefore some familiarity with elementary real analysis. Appendix A lists the standard theorems (labelled Theorem A.1, A.2,..., A7) which are used in the book, together with proofs of one or two of them which might be less familiar. Some knowledge of basic matrix algebra is assumed. We have also used some elementary ideas from the theory of normed linear spaces in a number of places; complete definitions and examples are given. Some prior knowledge of these areas would be helpful, although not essential.

   This book is in places a bit forgiving in the sense of stating definitions that the authors presumably expect the reader to already know. For example, on page 60, the usual easy proof is given for the Cauchy-Schwarz inequality (followed by Hölder and Minkowski; and the statement of Cauchy-Schwarz, but not the proof, is given again on page 254 in the more general setting), and then a discussion of open and closed balls.

   The contents of chapters 8 and 9, possibly among others, makes it clear that normed linear spaces are required. To understand chapter 4, on simultaneous nonlinear equations, the reader will have to have been exposed to multivariate analysis ("advanced calculus").

The formatting of the book is very nice, and the pdf file is perfect.

8.2 Functional Analysis (and Functional Data Analysis)

> The rapid development of information and computer technologies and of the computational sciences has created an environment in which it is critically important to teach applicable mathematics in an interdisciplinary setting.
>
> J. Tinsley Oden and Leszek F. Demkowicz (2018, p. xvii)

In addition to some common elements of machine learning requiring concepts and results from functional analysis (e.g., SVM kernel functions, and reproducing kernel Hilbert spaces), the growing field of functional data analysis (FDA) is obviously based on functional analysis. This section concentrates on functional analysis books that explicitly aim to be accessible to a somewhat wider audience, and (for the most part) do not require measure theory. At the end of the section, I mention just a few books on FDA.

Except for the classics by Simmons, and Kreyszig, the following list gives recent additions (relatively recent, or literally so), and in chronological order. Abbreviation "FA" stands for Functional Analysis:

FA Book Choice 1: George F. Simmons, *Introduction to Topology and Modern Analysis*, 1963 (reprint 1983). The following books are also implicitly discussed:

(a) Muscat, *Functional Analysis*, 2014;

(b) Chacón, Rafeiro, and Vallejo, *Functional Analysis: A Terse Introduction*, 2017.

Simmons' book receives impressive praise from several Amazon reviewers. Quoting one in its entirety,

> I used this textbook during my undergraduate degree to understand material on metric spaces and functional analysis. The book is self-contained and also contains a very clear discussion of set theory. The book is written extremely clearly and with a great deal of precision. Everything about it shows a great degree of care. For example, whenever I do not understand a term used in the book, turning to the index I find that it always contains the term I seek, and points me to the page I need to go to to understand it. This should be standard in textbooks, but unfortunately it usually isn't, which makes it worth pointing out in this case.
>
> The discussion and proofs of theorems are done very clearly. Simmons also gives a battery of examples for everything he discusses, and makes the effort to put them into context.
>
> The exercises in the book are also remarkably good.

Another person (who wrote an enormous review also detailing the contents) writes:

> The author's attitude can only be characterized as magnificent, and, if one is to judge his utterances in the preface by what is found after it, one will indeed find perfect evidence of his delight in mathematics and his high competence in elucidating very abstract concepts in topology and real analysis. Indeed, this has to be the best book ever written for mathematics at this level. It is a book that should be read by everyone that desires deep insights into modern real and functional analysis.

We read much about Simmons' book in the MAA book reviews about other books, being compared to Simmons; including (i) https://www.maa.org/press/maa-reviews/functional-analysis-a-terse-introduction, on the book *Functional Analysis: A Terse Introduction*, by Gerardo Chacón, Humberto Rafeiro, and Juan Camilo Vallejo, 2017, in which we read:

> Another distinction between this text and Simmons is that here there is somewhat more of an effort made to provide applications to analysis, thereby answering my one objection to the content of my old course. Specifically, this book provides substantial applications to analysis of both the contraction mapping principle and the Baire category theorem. (Simmons does give one application of the contraction mapping principle to a proof of the Picard existence theorem in differential equations, but this text gives additional applications.) While these could be viewed as applications of topology rather than functional analysis, the applications involve function spaces and are therefore certainly in the spirit of functional analysis.

and (ii), https://www.maa.org/press/maa-reviews/functional-analysis, on the book *Functional Analysis*, by Joseph Muscat, 2014, in which we read:

> This text pulls off the neat trick of simultaneously covering quite a lot of functional analysis ... while still being (for the most part, anyway) accessible to senior-level undergraduates. It begins, as does Simmons' text, with a look at point-set topology, but here the author does not offer a semester's worth of material. Instead, he only discusses that which is necessary for future developments.
>
> ...
>
> One feature that I thought was particularly attractive, especially for undergraduates, was the strong emphasis on specific examples. The dual spaces of

many spaces are worked out in detail, for example, and other results (such as the density of polynomials in a number of specific function spaces) are also established. This is valuable both for the student and the instructor, who now has a nice source for many of these well-known-but-sometimes-hard-to-find examples.

> Throughout, the author's writing style is clear, reader-friendly and accessible. (A good background in linear algebra and introductory real analysis should take the reader a long way.)
>
> ...
>
> Some parts of the book may prove less accessible to undergraduates than others, of course. One issue that must always be addressed in introductory accounts of functional analysis, for example, concerns the extent to which Lebesgue theory is going to be used. In Simmons, it is not mentioned at all; in Rynne and Johnson, it, according to a review of that book on this site, "keeps sneaking in and being shooed away." Muscat provides a fairly quick account of the subject in chapter 9 in connection with discussions of various function spaces. The discussion here is sufficiently rapid that someone without prior exposure to this material (such as, for example, most any undergraduate) will likely find it fairly difficult. Likewise, a fair amount of the material at the end of the book, specifically including the spectral theorems, would likely be above the heads of most undergraduates. However, because there is much more in this book than could ever be covered in a single semester, it would seem that this less accessible material could be easily omitted.

FA Book Choice 2: Erwin Kreyszig, *Introductory Functional Analysis with Applications*, 1978. The impressive reviews at Amazon speak for themselves.

FA Book Choice 3: Karen Saxe, *Beginning Functional Analysis*, 2001. From the MAA review https://www.maa.org/press/maa-reviews/beginning-functional-analysis,

> The obvious question about a book like this is, "Would you use this in your classroom?" The answer is a definite yes.
>
> Advanced undergraduates and beginning graduate students are sure to learn a lot from this book. The subjects are covered at a good pace with proofs that can be easily understood, but are not devoid of depth.
>
> The author's writing style is crisp and clear with an excellent understanding of her audience.
>
> I truly enjoyed the historical perspectives.

Some of the Amazon reviews are far less flattering, and, to the extent that they are informative, significantly temper the excitement for this book.

FA Book Choice 4: Dzung Minh Ha, *Functional Analysis, A Gentle Introduction*, 2006. Jumping to the chase, I consider the presentation to be excellent, and easily rank this book among my choice favorites. I need to temper my opinion with the disclaimer that I am far from being an expert in functional analysis, and so augment my selected readings of Ha's book with published reviews from professional mathematicians. I begin with quoting from the 2007 MAA book review https://www.maa.org/press/maa-reviews/functional-analysis-a-gentle-introduction-volume-1:

> "Overall, I found the book to be an acceptable introduction to functional analysis in spite of the issues raised above and I would consider this book if I were selecting texts for a course. I would, however, strongly advise that students ignore the author's "only a working knowledge of linear algebra" comment and complete a real analysis course before attempting to read this book."

The “issues raised” were about the numbering of lines and theorems in the book. Her only further complaint is that the author states the prerequisites to be too modest. I completely agree on this point. A course in beginning real analysis (Section 5), while perhaps formally not necessary, is highly advised, along with, as stated by the author, linear algebra (Section 3).

The second review I mention is from Carl L. DeVito, appearing in SIAM, pp. 396-8. It is a lengthy review, discussing in detail and commenting on, the contents of the book. The overall assessment is extremely positive. Quoting,

> He does this sort of thing throughout the book; i.e., he brings an elementary result that is probably well known to his readers into the discussion of a more sophisticated topic. ... The idea, and it is a good one, is to relate the more sophisticated, and often somewhat abstract, topic to something that is familiar to most readers.
>
> ...
>
> The author goes to great lengths to make his book accessible to students. The pace, for example, is extremely leisurely. Proofs are given in complete detail, and previous results, when used in a proof, are carefully referenced. There are many exercises, and those with an odd number are solved in Appendix D.
>
> ...
>
> The author has set himself a monumental task: writing a functional analysis book that can be read by any well-motivated student, even one with only a modest background in analysis. He carried out this task with great patience and care and he did a remarkable job. Not many books are written this way and many students will certainly benefit from what the author has done.
>
> ...
>
> This is a remarkable book, one that will serve as a valuable reference for anyone interested in functional analysis.

There is seemingly no freely available pdf file, but it can be purchased from the publisher for half price (which I did) if you own the hardcopy (which I do). Finally, I have emailed with professor Ha—he responded immediately and kindly, which is of use if one were to adopt his book.

FA Book Choice 5: Barbara MacCluer, *Elementary Functional Analysis*, 2008, 218 pages. There are (excerpts of) several extremely flattering reviews from various mathematical outlets, including (the extraordinarily positive one from) MAA, and positive reviews from purchasers. The book is in Springer’s GTM (graduate texts in mathematics) series, and the author says in the preface that she has used it for both graduate and undergraduates. “The prerequisites of the book include undergraduate courses in real analysis, linear algebra, and basic point set topology (say, in metric spaces). ... Beyond this, some familiarity with measure theory and the Lebesgue integral is desirable, but not essential.” Similar to the book by Saxe, MacCluer also apparently adds some historical commentary.

FA Book Choice 6: Bryan P. Rynne and Martin A. Youngson, *Linear Functional Analysis*, 2008, 324 pages. From the review `https://www.maa.org/publications/maa-reviews/linear-functional-analysis-0`, by Allen Stenger (who also wrote the MAA review on the book by Swartz, and also ends by stating his preference for Saxe’s book), we read:

> This is an undergraduate introduction to functional analysis, with minimal prerequisites, namely linear algebra and some real analysis. The book is kept at an undergraduate level by avoiding topology (nothing but metric spaces and

> convergence of sequences) and by using Riemann integration and continuous functions (Lebesgue integration keeps sneaking in and being shooed away).
>
> The book proceeds at a leisurely pace, and the reasoning is easy to follow. It is extensively cross-referenced, has a good index, a separate index of symbols (Very Good Feature), and complete solutions to all the exercises. It has numerous examples, and is especially good in giving both examples of objects that have a given property and objects that do not have the property.
>
> ...
>
> A competing book is Saxe's Beginning Functional Analysis , which has the same coverage and same audience. I browsed through Saxe, and I think it does a better job of putting the subject in context, both by moving the applications earlier in the narrative and by providing brief biographies of the big names in functional analysis and what they contributed.

FA Book Choice 7: Charles Swartz, *Elementary Functional Analysis*, 2009, 192 pages. Note that Swartz is also the author, with John DePree, of *Introduction to Real Analysis*, 1988. A djvu file is available, which can be easily converted to pdf, but the text, while still very readable, is not searchable or markable. From the preface,

> Only a very basic knowledge of linear algebra and introductory real analysis including a knowledge of metric spaces at the level of Apostol, Bartle, DePree/Swartz or Rudin is necessary to follow the developments in the text.
>
> There are a number of excellent texts on functional analysis. There are two things that distinguish this text from other texts in functional analysis. The first is that there are minimal prerequisites, especially there is no requirement of the knowledge of the Lebesgue integral or general topology. There are a number of functional analysis texts which require similar minimal backgrounds such as, [John Pryce, *Basic Methods of Linear Functional Analysis*, 1973, with Dover edition 2011, 2014] or [Martin Schechter, *Principles of Functional Analysis*, 2nd edition, 2002], but these texts do not contain many applications of the abstract results of functional analysis to concrete problems in function spaces...

(The author spells Schechter as "Schecter", in the preface and the bibliography, though admittedly, for German speakers, "schlechter" would have been a worse typo—but it is a real last name itself. Schechter's book gets strong reviews at Amazon, and the pdf file is available. The book indeed does not require knowledge of measure theory on behalf of the reader.) From the MAA review of Swartz' book, found at https://www.maa.org/press/maa-reviews/elementary-functional-analysis, by Allen Stenger, we read:

> This is a very concise but comprehensive introduction to functional analysis. The prerequisites are minimal: some real analysis and linear algebra, but no topology or Lebesgue integration.
>
> The book's strength is the wide variety of applications of functional analysis it gives, to many diverse areas of mathematics.
>
> Karen Saxe's Beginning Functional Analysis is a very different book that has the same prerequisites. Saxe's book gives a carefully selected coverage of the most important and interesting results of functional analysis, without any attempt to be comprehensive. Swartz's book is an impressive accomplishment, presenting quite a lot of functional analysis and its applications in only 180 pages and with almost no prerequisites. But all in all I think Saxe's book is the better choice for a course at this level, not least because it is very charming.

FA Book Choice 8: Yutaka Yamamoto, *From Vector Spaces to Function Spaces: Introduction to Functional Analysis with Applications*, 2012. From the MAA review https://www.

maa.org/press/maa-reviews/from-vector-spaces-to-function-spaces-introduction-to-functional-analysis-with-applications, we see much praise:

> This is an unusual textbook, a rigorous introduction to functional analysis without advanced prerequisites, together with a serious introduction to some of its important applications, all within the space of 268 pages.
>
> The author is unusually careful to address questions such as why we define things as we do, with an eye to what is needed in the applications.
>
> The somewhat quick pace may frighten away the less mathematically mature reader. It may be hoped that a future edition could contain an increased number of exercises. On the other hand, it may be worthwhile to produce a workbook to supplement the present volume.
>
> The book will be of interest to those who want a rigorous approach to functional analysis with a view to its applications. It will also serve as an example of good exposition for those writing in any area with a view to motivation and application.

FA Book Choice 9: Markus Haase, *Functional Analysis: An Elementary Introduction*, 2014, 372 pages. The MAA review https://www.maa.org/press/maa-reviews/functional-analysis-an-elementary-introduction, by Allen Stenger (whom we have seen above for the MAA review of two books, in both of which he states his preference for Saxe's book) states:

> This book sneaks up on functional analysis. It assumes little background in analysis or linear algebra and develops the needed material in the first half of the book. (There is also a series of appendices that provide even more mathematical background.) It then starts working on problems with a functional-theoretic flavor, such as integral equations, but they are tackled in an ad-hoc manner. The general theorems, such as Hahn-Banach, are not developed until the end of the book.
>
> I like this approach, especially for the extra attention given to Baire's category theorem and to the contraction mapping theorem. It is not strictly a historical approach, but it does have some echoes of the way in which functional analysis was developed in the 1920s to pull together many diverse threads of mathematics. The book has a very valuable historical overview of the development of functional analysis at the end. The book is in a graduate series, but appears to be aimed at upper-division undergraduates.
>
> ...
>
> A similar though more concise book is Saxe's Beginning Functional Analysis. It too assumes little knowledge of analysis and develops the needed background material from scratch. Both books have a lot of historical information. The big difference in the books is that Saxe proceeds more directly to the theorems of modern functional analysis and does the applications afterwards. Neither book is particularly abstract or theoretical, but Haase is more oriented to applications.

FA Book Choice 10: Balmohan V. Limaye, *Linear Functional Analysis for Scientists and Engineers*, 2016, 250 pages. We have discussed his two books on introductory real analysis (univariate, and multivariate) above. From the preface, "The book is accessible to anyone who is familiar with linear algebra and real analysis."

FA Book Choice 11: Amol Sasane, *A Friendly Approach to Functional Analysis*, 2017. The MAA book review https://www.maa.org/press/maa-reviews/a-friendly-approach-to-functional-analysis states:

> The "friendly" aspect promised in the title is not explained, but there are three things I think would strike most students as friendly: the slow pace, the enormous number of examples, and complete solutions to all the exercises. The prerequisites are minimal: some calculus and some linear algebra. The book develops all the topology needed from scratch, and spends most of its time developing properties of normed linear spaces and inner product spaces. It hedges on the Lebesgue integral: it does use it for the real line throughout, but in a way that does not require any detailed knowledge, and it advises the student to think of it as the Riemann integral. There's a brief appendix outlining properties of the Lebesgue integral on the real line.
>
> This may be that rare thing: a book with too many examples. I thought they often interfered with the exposition and made it hard to see where we were going. ... A more compact book, with the same prerequisites but maybe less friendly, is Saxe's *Beginning Functional Analysis*. It covers more topics, and in particular develops the theory of Lebesgue integration on general spaces.

FA Book Choice 12: J. Tinsley Oden and Leszek F. Demkowicz, *Applied Functional Analysis*, Third Edition, 2018, 609 pages. From the preface of the first edition (1995), we read:

> Prerequisites for the course for which this book is written are not extensive. The student with the usual background in calculus, ordinary differential equations, introductory matrix theory, and, perhaps, some background in applied advanced calculus typical of courses in engineering mathematics or introductory mathematical physics should find much of the book a logical and, we hope, exciting extension and abstraction of his knowledge of these subjects.

From the second edition (2009),

> The book attempts to teach the rigor of logic and systematical, mathematical thinking. What makes it different from other mathematical texts is the large number of illustrative examples and comments. Engineering and science students come with a very practical attitude, and have to be constantly motivated and guided into appreciating the value and importance of mathematical rigor and the precision of thought that it provides. Nevertheless, the class in which the book has been used focuses on teaching how to prove theorems and prepares the students for further study of more advanced mathematical topics.

FA Book Choice 13: James Robinson, *An Introduction to Functional Analysis*, 2020. So far, a pdf of this book is not available. There are two Amazon reviews, both 5-star, as well as highly praiseworthy excerpts from three professional reviews (with the reviewer's names and affiliations). From the description of the book provided there, we get the following idea of its level: "Familiarity with the basic theory of vector spaces and point-set topology is assumed, but knowledge of measure theory is not required, making this book ideal for upper undergraduate-level and beginning graduate-level courses."

I end this list by mentioning one more book on functional analysis, suitable for Ph.D. students of mathematics, by Vladimir I. Bogachev and Oleg G. Smolyanov, *Real and Functional Analysis*, 2020 (translated from an earlier version, in Russian). The MAA review https://www.maa.org/press/maa-reviews/real-and-functional-analysis is positive, writing that the authors "have succeeded in providing a modern and highly informative description of all the material traditionally covered in the context of a real and functional analysis course spread out over two semesters and intended for graduate students." The reviewer further comments on another aspect of the book, and is why I include mention of it here:

> The makeup of the vast bibliography (that includes more than 700 titles) was obviously the subject of painstaking literature review. The authors have taken the

time to include in the appendix a summary over several pages of the historical development of functional analysis and the various directions in which this discipline and its applications are used. There is then an orienting bibliography that will point the interested reader toward appropriate resources.

I now turn, very briefly, to books on functional data analysis (FDA). Certainly compared to more established fields, there are very few books available for FDA. Here are the major ones, with the first two entries being modest in terms of prerequisites:

1. James Ramsay and B. W. Silverman, *Functional Data Analysis*, Second Edition, 2005; and *Applied Functional Data Analysis: Methods and Case Studies*, 2002. The editorial reviews (excerpts of which can be seen on Amazon), as well as Amazon customer reviews, are very positive.

2. Piotr Kokoszka and Matthew Reimherr, *Introduction to Functional Data Analysis*, 2017.

3. Tailen Hsing and Randall Eubank, *Theoretical Foundations of Functional Data Analysis*, 2015. As the title suggests, this is not an introductory book. From the preface, a "solid mathematics background at a graduate level is needed to be able to appreciate the content of the text. In particular, the reader is assumed to be familiar with linear algebra and real analysis and to have taken a course in measure theoretic probability." Knowing that authors tend to downplay prerequisites, it might suggest itself to have already also had an exposure to functional analysis at the level of the books given in the previous list.

8.3 Graduate Level Analysis and Measure Theory

This is for a course that might be deemed a "traditional first year graduate analysis course". I envision it being taught appropriately for students outside of pure mathematics, i.e., at a slightly lower level and/or less ambitiously, and so I limit the list of books to account for this. A list of more advanced books is given subsequently.

Abbreviation "GRA" stands for Graduate Real Analysis, which implies coverage of Lebesgue integration, among other topics.

GRA Book Choice 1: (First part of) Jacob and Evans, 2018, *A Course in Analysis Volume III: Measure and Integration Theory, Complex Valued Functions*. The MAA review https://www.maa.org/press/maa-reviews/a-course-in-analysis-vol-iii-measure-and-integration-theory-complex-valued-functions-of-a-complex makes clear that this volume is right in line with (my thoughts and) what was said about the first two volumes: It is excellent. (The reviewer of the first two volumes is the same person who reviewed the third volume, namely Jason M. Graham, University of Scranton. Volume IV from Jacob and Evans, covering Fourier analysis, ordinary differential equations, and the calculus of variations, is a different person, and the evaluation of volume IV is equally strong.) We read:

> The treatment of measure theory is carried out via a careful, axiomatic development. Everything is proven in great detail and the approach is useful for students interested in both analysis as well as probability theory. The treatment of complex analysis in part II is equally detailed and also emphasizes the geometric nature of the theory of complex variable functions.
>
> Part I begins with careful definitions of the essential set-theoretic concepts for measures, i.e., σ-field, measurable function, etc. Many examples are provided and some of relevant topological notions are introduced.
>
> All of the important limit theorems for integrals are given careful treatment as are the Radon-Nikodym theorem and the Lebesgue differentiation theorem. The authors do a very nice job in leading the reader through the technical

details of measure theory while simultaneously helping the reader to develop an intuition for the field and some of its typical applications. One could reasonably use part I of the book as a stand-alone course in measure theory and integration.

Part II is where the book really shines. The treatment of complex analysis is beautiful. The authors stress the geometric feel of complex function theory and really help the reader to develop an intuition for complex analysis. Since part II largely does not rely directly on results from part I, the reader could start with part II. Moreover, one could use part II of the book as a text for a course in complex variable theory either before or after an advanced course in real analysis.

Volume 3 of A Course in Analysis is a great book for a first year (U.S.) graduate student. One of the nice features of the book is that the book contains full solutions for all of the problems which make it useful as a reference for self-study or qualifying exam prep.

GRA Book Choice 2: Sheldon Axler, 2020, *Measure, Integration, and Real Analysis* (MIRA). This is the same author who gained fame (notoriety?) from his book on linear algebra, *Linear Algebra Done Right*. This is a Creative Commons book, meaning, an electronic version is legally free (and the printed version is very modestly priced). The book is handsomely formatted, and the author also provides a document, "Supplement for Measure, Integration & Real Analysis", the intent of which should be obvious, namely "can serve as a review of the elementary undergraduate real analysis used in this book."

There are many exercises, and—like most author mathematicians—he writes in the preface, correctly so, that working these is a must. As conveyed to me from professor Axler, a complete solutions manual is in the works, and will be available (to instructors) perhaps early 2022. In light of the strong authorship, optical pleasantness, low price of a hardcopy and free availability of an electronic version, and with the author's own solutions eventually available (particularly welcome for novices and hobbyists such as myself for this material), this book is poised to be another market dominator from the author.

There are several reviews on Amazon, all (informative ones) 5-star, praising the book, e.g.,

> Stay away from this book unless you want to get addicted to beautifully written math textbooks. This book is right up the alley of "Linear Algebra Done Right" – beautifully written, amazing for self-study (I'm self-studying and sometimes it's hard to put the book down), and leaves you wanting for more.
>
> Don't get me wrong – it's by no means an easy text. But it keeps everything interesting and motivated, the proofs are clear and very well written, and there are enough (at least for me) end of chapter exercises to keep everyone happy and drive home the key ideas from the associated chapters.

and

> This has to be the best introduction to measure theory around, and free too. The text is clear and interesting at all times. It puts my lectures at Oxford in the 60s to shame, the approach was contorted and diabolical!

GRA Book Choice 3: Bruckner, Bruckner and Thomson's 2nd edition, 2008, *Real Analysis*. (Note this is a different book than their *Elementary Real Analysis*.) From the preface, we read:

> We have tried to write a book that is suitable for students with minimal backgrounds, one that does not presuppose that most students will eventually specialize in analysis.
>
> We provide a certain amount of historical perspective that may enable a reader to see why a theory was needed and sometimes, why the researchers of the time had difficulty obtaining the "right" theory. We try to motivate topics before we develop them and try to motivate the proofs of some of the important theorems that students often find difficult. We usually avoid proofs that may appear "magical" to students in favor of more revealing proofs that may be a bit longer. We describe the interplay of various subjects—measure, variation, integration, and differentiation. Finally, we indicate applications of abstract theorems such as the contraction mapping principle, the Baire category theorem, Ascoli's theorem, Hahn-Banach theorem, and the open mapping theorem, to concrete settings of various sorts.

GRA Book Choice 4: Christopher Heil, *Introduction to Real Analysis*, 2019. This appears to be among the most approachable graduate level analysis books; see, e.g., the excellent book review:
https://www.maa.org/press/maa-reviews/introduction-to-real-analysis-2
as well as those on Amazon. Further, the author has greatly augmented versions of his chapter 0 and 1 on the web, some solutions to problems, (and a short errata list). A full solutions manual is apparently available via the publisher.

GRA Book Choice 5: Tom L. Lindstrøm, *Spaces: An Introduction to Real Analysis*, 2017, about 360 pages. There is coverage of metric spaces, normed spaces, linear operators, a chapter entitled "Differential Calculus in Normed Spaces", two chapters on Lebesgue measure and integration, and a last one on Fourier Series. From the preface, we learn the intended audience:

> [The book] is written for students with a good background in (advanced) calculus and linear algebra but not more".

We read further about the contents:

> I don't redo Riemann integration but go directly to Lebesgue integrals, and I do differentiation in normed spaces rather than refining differentiation in euclidean spaces. Although the exposition is still aimed at students on the second level, these choices bring in material that are usually taught on the third level, and I have tried to compensate by putting a lot of emphasis on examples and motivation, and by writing out arguments in greater detail than what is usually done in books on the third level. I have also included an introductory chapter on the foundation of calculus for students who have not had much previous exposure to the theoretical side of the subject.

I read the first three chapters (by far the easiest ones) and can attest that the book is a joy to read. The lone three reviewers at Amazon seem to agree, with one person (a verified purchase) going so far as to state:

> If Tom Lindstrøm could write books on every mathematical subject, the world would be a better (simpler, at least) place. This book is incredibly well written, with elegant and properly explained proofs, and enlightening examples and comments. I think this book will become a classic.

GRA Book Choice 6: Satish Shirali and Harkrishan Lal Vasudeva, *Measure and Integration* 2019. This is in the Springer SUMS (Springer Undergraduate Mathematics Series), and indeed, from the preface, we read:

> A reader with a modest background in Mathematical Analysis in one variable, including rudiments of Metric Spaces, which are included in Chap. 1, will find the material covered in the book well within reach.

Note the other books by the author, also included in this document: *A Concise Introduction to Measure Theory*, 2018; and *Multivariable Analysis*, 2011.

GRA Book Choice 7: Elias Stein, Rami Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, 2005. This is part of a sequence of books that have been overall highly praised. Regarding the series of books, from the publisher, we read: "Real Analysis is accessible to students interested in such diverse disciplines as mathematics, physics, engineering, and finance, at both the undergraduate and graduate levels.". This book received a (short but) positive MAA review, https://www.maa.org/press/maa-reviews/real-analysis-measure-theory-integration-hilbert-spaces, while the reviews at Amazon are very convincingly in favor of this book, noting also its level is appropriate for (advanced) undergraduates.

GRA Book Choice 8: Herbert Amann and Joachim Escher, *Analyis I*, 2005, original German, 1998; *Analysis II*, 2008, and *Analysis III*, 2009. Note that Amann is retired from the math department at UZH.

The first book is a 426 page intermediate analysis book in the sense that it covers, more deeply than usual, the main topics in an introductory real analysis course, though not covering integration (neither Riemann, nor Lebesgue). Book II covers numerous aspects of integration, but not Lebesgue, while part III covers Lebesgue integration. Taken all together, in terms of quantity and depth, the three books constitute at least three years of advanced undergraduate / beginning graduate analysis coursework. There is a partial parallel to the first three books of Jacob and Evans, *A Course in Analysis*, though at least the first of these books is lower level than the first of Amann and Escher, and possibly all three.

From the MAA review of the first volume, https://www.maa.org/press/maa-reviews/analysis-i-0, we read:

> This is the first of a three volume introduction to analysis, which appeared recently in English translation, after 2 editions in German. It is a wonderful book that distinguishes itself by the clarity of presentation, by the fact that it is self-contained and has many exercises at various degrees of difficulty (among many other great qualities.)
>
> The book is intended to be used both as a self study and as a textbook for a course in analysis. The consequence of this goal is the thorough treatment (with complete and elegant proofs) of many topics which do not usually appear in an analysis textbook.
>
> ...
>
> In summary, we have here a very-very good, self-contained, well written book, including many applications and exercises, that can be used as a text for courses (real analysis, foundations, etc), as self-study, and/or as a basis for research in mathematics or even other disciplines. This book is a wonderful read.

There is no need to read between the lines in such a review. The MAA review for volume II, by the same person, says

> There are only three chapters in this book, but the depth of the treatment, the richness of examples and exercises, the rigor of the proofs, make it too long for a one-semester course. Instructors could choose parts of the book

> to be used in several courses, or (better yet) this book could be used for independent studies, or self-studies by students, or even by more experienced mathematicians in their study and research.
>
> Every student of analysis could benefit from reading (parts of) this book, and every analyst will find useful to own this book, as a reference when teaching an analysis course, or as a source of information.

GRA Book Choice 9: John DePree and Charles Swartz, *Introduction to Real Analysis*, 1988. I make two claims that I am not sure about: First, this book is not very popular, and second, it should be. It starts off very modestly, assuming only basic calculus on behalf of the reader, though progresses quickly, covering the standard set of first-course real analysis topics, and, by page 147, moving to more advanced topics, such as the gauge integral, Lebesgue integral, multiple integrals, and metric spaces, including the Baire Category Theorem. There is an entire chapter (albeit short) on compactness, including Arzela-Ascoli.

I am impressed with the writing style—it is certainly not chatty, but the authors take time to write text and explain things, and you (almost) have the feeling the authors are talking to you. The available pdf file is (perhaps surprisingly) high quality, with markable, searchable text.

In light of the other books now available, this one might best serve as supplementary reading. I added it in this primary list because I find it impressive enough to deserve mention.

Author Charles Swartz also has a book from 2001, *Introduction to Gauge Integrals*, for which "basic knowledge of introductory real analysis is required of the reader, who should be familiar with the fundamental properties of the real numbers, convergence, series, differentiation, continuity, etc." He is also author of *Elementary Functional Analysis*, 2009.

GRA Book Choice 10: Vicente Montesinos, Peter Zizler, Václav Zizler, *An Introduction to Modern Analysis*, 2015, 863 pages. Regarding for whom this is written, and what the prerequisites, from the preface we read:

> This text is directed at undergraduate students in mathematical sciences who wish to have solid foundations for modern analysis, a meeting point of classical analysis with other parts of mathematics, like functional analysis, operator theory, nonlinear analysis, etc. These foundations are necessary for applications of mathematics in sciences or engineering.
>
> It is assumed the reader has a good understanding of elementary linear algebra and arithmetics, as well as some training in simple logic.
>
> The text consists of a rigorous yet gentle self-contained introduction to real analysis with various visual supplements.

The usual first-course in real analysis topics are covered, and the book indeed starts off very modestly, in line with what is stated in the preface. However, there are chapters throughout that contain more advanced material. These include (already) chapter 3, on measure, including Lebesgue inner measure; but chapter 4 returns to basics, covering the basics of functions (though does cover measurable functions). Convergence is covered in chapter 5, this being at the level of a typical first-course book, while chapter 6 turns to metric spaces, covering the completion of a metric space, Polish spaces, compactness, the Baire Category Theorem, the Arzela–Ascoli theorem, and the Banach Contraction Principle. Chapter 7 returns to basics, doing the Riemann integral, but also Lebesgue, the latter in about 50 pages. Chapter 8 is on convex functions, a topic not often (enough) seen. Chapter 9 is on Fourier series, and chapter 10—what is this doing here?—is "Basics of Descriptive Statistics". Chapter 11 is a long chapter, over 100 pages, and is clearly where the authors intended to go, on functional analysis. Chapter 12 is an appendix, finishing on page 630. The final chapter 13 is exercises, 197 pages of them!

The book has many graphics, with 11 pages requires to list them. For example, Figure 4.18 on page 161, which could be in a high school text, very nicely showing the linearity of a function as one focuses on ever decreasing intervals; or page 178, (nicely) showing that $\sin(a+b) = \sin a \cos b + \cos a \sin b$, again being a graphic we find in high-school and undergraduate books on trigonometry.

And *à propos* convex functions, it is worth mentioning a book dedicated to the topic, from an author we have seen more than once, Steven G. Krantz, *Convex Analysis*, 2015, about 150 pages.

Finally, we come to its MAA review https://www.maa.org/press/maa-reviews/an-introduction-to-modern-analysis, which begins by reminding us of other books that have "modern" in the title (Whittaker and Watson's *A Course of Modern Analysis*, first edition 1902; and Dieudonné's *Foundations of Modern Analysis*, first edition 1960).

> [The book] is closer in spirit to Dieudonné's text, but there are substantial differences. There is no classical nor modern complex analysis, no calculus/analysis of several real variables, and no development of general measure theory. The core of the text is an exposition of single real-variable calculus, and it ends with a beautiful exposition of basic functional analysis. This last chapter ends with rather sophisticated material aimed at whetting the appetite of budding Banach space theorists. Both advanced undergraduates and graduate students will enjoy the proofs in this text, and would do well to try their hand at the vast collection of problems (the large majority of which come with hints) that are collected at the end of the book.
>
> We then arrive at the final Chapter 11 that sets this book apart from other standard texts at this level, the "Excursion to Functional Analysis". The writing here is distinguished by its clear proofs and strong emphasis on geometric intuition.
>
> When all has been said and done, the authors must be congratulated on writing a useful textbook that includes plenty of bonuses for both students and instructors.

GRA Book Choice 11: Anthony W. Knapp, *Basic Real Analysis*, Digital Second Edition, 2016, 811 pages. The length of the book is not definitely not congruent with the meaning of the author's last name in German, namely brief, curt, scant, scarce, short, skimpy, terse, etc..

This book, and its more senior counterpart, *Advanced Real Analysis*, as well as two books, again "Basic" and "Advanced", on Algebra, are also available from the author as free digital releases. See https://www.math.stonybrook.edu/~aknapp/.[48] To clarify adjectives, the "Basic" in the title is referring to beginning graduate level analysis (thus rendering Knapp's second volume to be indeed quite advanced). Nevertheless, *Basic Real Analysis* starts off in chapter 1 with a (quick, but mostly elementary, and very readable) development of topics associated with a first course in undergraduate real analysis, and its appendix A covers further elements of a first course, along with a few other topics, notably some aspects of linear algebra, and polynomials. The preface states:

> Beyond a standard calculus sequence in one and several variables, the most important prerequisite for using Basic Real Analysis is that the reader already know what a proof is, how to read a proof, and how to write a proof. This

[48] From his online CV, Knapp joined Stonybrook's math department in 1986, as a full professor, and thus him and I overlapped, with me having been an undergraduate there, though I never had a class from him.

> knowledge typically is obtained from honors calculus courses, or from a course in linear algebra, or from a first junior-senior course in real variables. In addition, it is assumed that the reader is comfortable with a modest amount of linear algebra, including row reduction of matrices, vector spaces and bases, and the associated geometry.

I think this is a reasonably accurate statement, notably in light of the author's chapter 1 and appendix A, though I would never consider using this book for students who have not had a formal exposure to at least one, ideally two, undergraduate courses in real analysis, covering the usual undergraduate-level material on metric spaces and multivariate analysis. That should not surprise, given the pace the author assumes, moving quickly past the basics. Also, chapter 1 uses "cuts" to construct the real numbers; and I presume (though he does not say) Dedekind cuts. This approach is done better, and with more detail, in other books, and I anyway prefer the method based on the use of Cauchy sequences of rationals (as done, e.g., in Tao's and Strichartz' books).

The size of the book helps confirm Knapp's desire for "completeness", here in its colloquial sense of covering a large selection of topics that might otherwise go overlooked in typical analysis courses:

> [I]t is often the case that core mathematics curricula, time-limited as they are, do not include all the topics that one might like. Thus the book includes important topics that may be skipped in required courses but that the professional mathematician will ultimately want to learn by self-study.

Appendix B is 70 pages, and serves as a self-contained short course on complex analysis. Chapter 2 is on metric spaces, and chapter 3 is on multivariate analysis, with the usual topics, and based on the Riemann integral. Chapter 4 is on (systems of) ordinary differential equations. As such, these first four chapters (about 265 pages) could be used as, say, supplementary reading in a two-course sequence of undergraduate real analysis, with the goal of getting the reader acquainted with Knapp's quite large and reference-like book, that goes well beyond a first and second course in real analysis.

Perhaps somewhat surprisingly, and no doubt welcoming to most students, the author provides partial or complete solutions to most of the problems, generating nearly 80 pages of such. (This was also in the first edition of the book, in 2005. This was, I would say, notably for an advanced book, quite forward looking.)

Neither of Knapp's books on analysis were reviewed by MAA, but both his algebra books have been, and both received blatantly positive reviews. For his books on analysis, we have the review by Neil Falkner, in *The American Mathematical Monthly*, 2009, 116(7), pp. 657-664. It is a very detailed review, as seen from the length, and not particularly positive. For example,

> He precedes many topics with helpful motivating discussions. My initial impression of these books was positive. But the more I examined them in detail, the more I found that the proofs were too often inefficient or unenlightening or both.

Another critique:

> Convexity is a key to a clear understanding of many of the fundamental inequalities of analysis. In Basic, Knapp aims to develop analysis from the ground up, so I find it surprising that he includes no discussion of convex functions or Jensen's inequality.

On a more positive note,

> Any textbook must include a good assortment of exercises. These books do. One enlightening group of exercises in Basic explains the fast Fourier transform in terms of the Poisson summation formula on a finite cyclic group. Fairly detailed solutions to essentially all of the exercises are given at the end of each book. Students may like this, though an instructor who wishes to assign homework to be turned in for credit may not.

Falkner clearly read the books, notably the advanced one, carefully, and comments on many specifics (that are above my pay grade). He ends with:

> It seems to me that Knapp's presentation of analysis falls short of the standard set by well-known earlier analysis textbooks and that Knapp missed some golden opportunities to improve on these textbooks. And while it's impossible to write a book that is free of mistakes, I have pointed out a couple of places where Knapp has committed serious errors, not just minor slips.
>
> Knapp has collected together a lot of important analysis in these two volumes. It is clear that he put a tremendous amount of work into writing them and I respect that. But I would not include them in my personal short list of analysis textbook classics.

Knapp most surely knows of this review, and reacted. He mentions in the preface of the (digital) second edition that:

> Along with the general comments and specific suggestions were corrections, well over a hundred in all, that needed to be addressed in any revision. Many of the corrections were of minor matters, yet readers should not have to cope with errors along with new material. Fortunately no results in the first edition needed to be deleted or seriously modified, and additional results and problems could be included without renumbering.
>
> ...
>
> The corrections sent by readers and by reviewers have been made. The most significant such correction was a revision to the proof of Zorn's Lemma, the earlier proof having had a gap.

On the author's web page, there is an errata list, a reasonable two pages, dated October 27, 2017, that applies to this second edition.

GRA Book Choice 12: Richard F. Bass, *Real analysis for graduate students*, Version 4.2, 2020, 462 pages. The book is freely available as a pdf file; see http://bass.math.uconn.edu/books.html. From the preface, we read:

> There are a large number of real analysis texts already in existence. Why write another? In my opinion, none of the existing texts are ideally suited to the beginning graduate student who needs to pass a "prelim" or "qual." They are either too hard, too advanced, too encyclopedic, omit too many important topics, or take a nonstandard approach to some of the basic theorems.
>
> Students who are starting their graduate mathematics education are often still developing their mathematical sophistication and find that the more details that are provided, the better (within reason). I have tried to make the writing as clear as possible and to provide the details. For the sake of clarity, I present the theorems and results as they will be tested, not in the absolutely most general abstract context. On the other hand, a look at the index will show that no topics that might appear on a preliminary or qualifying examination are omitted.
>
> All the proofs are "plain vanilla." I avoid any clever tricks, sneaky proofs, unusual approaches, and the like.

The prerequisites to this text are a solid background in undergraduate mathematics. An acquaintance with metric spaces is assumed, but no other topology. A summary of what you need to know is in Chapter 1. All the necessary background material can be learned from many sources; one good place is the book [W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., 1976].

Supplementary Books Now comes a list of graduate real analysis books that could serve as supplements. Wheeden and Zygmund (2015) is a well-recognized "classic" (first edition was 1977) that arguably belongs in the main list of books for beginning graduate level analysis (but I instead place here, due to the emergence of highly attractive new books); while the books by Yeh, and Vestrup, are enormous (in size, and in scope), highly detailed, and, for Yeh, a separate book of solutions, itself 499 pages, to all the exercises in his book.

1. Robert G. Bartle, *The Elements of Integration and Lebesgue Measure*, 1966, reprinted by Wiley, 1995; 179 pages. This is (or was) a very popular book, with strong positive reviews on Amazon (the most recent being from 2017). From the preface, "Its only prerequisites are a understanding of elementary real analysis and the ability to comprehend ϵ–δ arguments."

   The review from Götz Trenkler (Computational Statistics & Data Analysis, 1997, Vol. 23; No. 4, page 565), is short and positive, including "Only some familiarity with the Riemann integral is assumed. The book offers just enough of the theory of integration and Lebesgue measure to give a basic understanding and the most important results. ... The book makes very pleasant reading and can be recommended to everybody looking for the first knowledge in measure theory and Lebesgue integration." A bit longer review is provided by L. A. Baxter (The Statistician, 1996, Vol. 45, No. 4, pages 528-9), who says "This is an outstanding book. It is so well and so carefully written that all the material is easily understood on a first reading. Indeed, the clarity and lucidity of the style are such that the reader can master very difficult and abstract material with relatively little effort."

2. Frank Jones, *Lebesgue Integration on Euclidean Space*, 1993. I have read portions of this book and appreciate it immensely—notably the author's willingness to use graphics to assist understanding. My only gripe with the book (and apparently more than one reviewer at Amazon) is that the book's contents is heavily tied to the exercises, and there are no solutions available.

3. Charalambos Aliprantis and Owen Burkinshaw, *Principles of Real Analysis*, 3rd edition, 1998. Impressively, there is an entire separate book by the same authors, giving the solutions to all the exercises in the book, *Problems in Real Analysis: A Workbook with Solutions*, 2nd edition, 1999. There is no MAA review. The reviews at Amazon are mixed, from excellent to terrible. In the case of the latter, we read "I found Royden (the required text for our class) to be very sparse and was hoping for something to fill in the details. This book did little to help. It covers integration in a rather idiosyncratic way, devotes little time to differentiation, and says nothing about convexity."

   The available pdf file is a scan, so the quality is acceptable but not great, and the text is not markable or searchable.

4. Heinz Bauer, *Measure and Integration*, 2001, translated from German, 248 pages. I was made aware of this book from the book review of Royden that I mention at the beginning of Section 5. The prerequisites for this book are not explicitly stated by the author, but a perusal of the book indicates that they would be the same as required to read the books by Vestrup, and Yeh, mentioned above—namely, substantially more than a first course in real analysis.

   The pdf file is "good enough" (but not like that of a more recent book), with markable, searchable text.

5. Eric M. Vestrup, *The Theory of Measures and Integration*, 2003, about 590 pages (and relatively small print). I have a hard copy, but cannot say I read it. The book is blatantly for graduate students in mathematics, and appears magisterial. From the preface,

   > This book assumes that the reader has studied advanced calculus and elementary analysis. For example, the first eight chapters of Walter Rudin's Principles of Mathematical Analysis should provide a strong preparatory framework for the material in this book.
   >
   > ...
   >
   > In the writing of proofs and in my choice of notation, I have tried to be as explicit as possible. Elegance or a sense of aesthetics have without exception deferred to clarity, although one can certainly be both clear and elegant in certain situations. If anything, I have erred on the side of perhaps showing too many details, although I believe most readers will at some time be grateful for the details.

   There is currently no MAA review, but Amazon has two detailed reviews, both the full 5 stars. From one, a small excerpt reads:

   > This is a fantastic book on measure theory. The focus is on measure theory on its own right and not on probability. I was lucky to come across this book while canvassing the measure theory books at our library. I looked at the books by Billingsley, Halmos, Chung, Resnick, Rao, Rudin, Pollard, Dudley, Nielson, Stroock, Williams, Pitt, and many others. Hand-down, Vestrup is the best.
   >
   > I believe after scrutinizing so many books, I have a very good baseline to judge Vestrup's work. Here are a few specific reasons:
   > (1) If you don't like detail and revel in banging your head against the walls to figure out the skipped details in Billingsley, this is not the book for you. But if you are a first timer to measure theory, this is as good as it will get: All the major results of measure theory are presented in detailed and clear manner with few skipped details and few not-so-obvious "it is obvious" remarks.

6. Gerard Folland, *Real Analysis: Modern Techniques and Their Applications*, Second Edition, 2007. This is apparently a popular book, to which other books are compared. There is no MAA review, but there is one for another one of his books, much shorter (106 pages), *A Guide to Advanced Real Analysis*, 2009.

7. William C. Bauldry, *Introduction to Real Analysis: An Educational Approach*, 2009 (first edition). This book is aimed at "secondary teachers", and is suited for graduate mathematics students. The first chapter of 35 pages is Elementary Calculus, which seems rather unexpected in a graduate level book. Chapter 2 is Introduction to Real Analysis, which also is a bit unexpected, and it covers the usual topics in a first course in real analysis in about 70 pages. Chapter 3 introduces Lebesgue theory, chapter 4 is Special Topics, and the rest are appendices on definitions, theorems, history, and projects. The whole book is about 250 pages. It contains some nice historical information as interludes, as well as a nice discussion of Littlewood's Three Principles.

   It is a tough book to judge, but seems to have some strong points, terseness being one of them, but also starting rather modestly for a book at this level. It could possibly be used in conjunction with another, more detailed, book. From the description on Amazon, "...Riemann-Stieltjes integrals and the Lebesgue measure are also introduced to broaden the presented perspective. The book concludes with a collection of advanced topics that are connected to elementary calculus, such as modeling with logistic functions, numerical quadrature, Fourier series, and special functions."

   The MAA review is short and positive, stating:

> Here we have a text that is very nice for the audience for which it was intended. It is meant as a last course in real analysis for those intending to go into teaching, giving them that necessary layer of knowledge above calculus so that they can teach calculus well and with confidence.
>
> The writing is good, the author includes many references, the pictures are pretty...
>
> This is an admirable text that I hope will be widely used.

8. John Bligh Conway, *A Course in Abstract Analysis*, 2012. I provide a quote from the preface of this book in Section 2.3; and also from the preface, there is a short note intended for students that is very inspiring. Otherwise, I cannot claim to have read any of it, it being simply well out of my league. The MAA review of this graduate level analysis book, `https://www.maa.org/press/maa-reviews/a-course-in-abstract-analysis`, is blatantly positive.

9. J. Yeh, *Real Analysis: Theory of Measure and Integration*, 3rd edition, 2014, 815 pages. From the MAA review `https://www.maa.org/press/maa-reviews/real-analysis-theory-of-measure-and-integration`, we read:

   > It's all very complete, very dense. Every i is dotted and every t is crossed (as one of my old teachers used to say). At this point, after 500 pages of reading, the avid reader might wish to pause to catch her breath before plunging into the remaining 300 pages of this tome.
   >
   > A word on exercises: this book has 394 of them and they run the gamut from nearly obvious to difficult. They are accessible for anyone who has done the reading and has a burning desire to learn the material. It was a pleasant surprise to me that a fair number of them actually bridged the yawning chasm between multivariable calculus and real analysis.
   >
   > I can easily imagine this book being used as a review source for graduate students before taking qualifying exams. The material is very dense, though, and I can only imagine a hapless undergraduate feeling much as if they were staring into the maw of Moby Dick as they page through this tome. On the other hand, if one is dedicated to learning all the details of measure theory and integration I can assert with absolute confidence that they are all here for the interested reader.

   Impressively (and essential for some of us), the author has an entire book of solutions, 499 pages, *Problems and Proofs in Real Analysis*, 2014.

10. R. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*, 2nd edition, 2015. This book starts more modestly than Bauer, Vestrup, and Yeh, but is still a graduate level text presupposing the reader has had a solid dose of undergraduate analysis. From the preface of the first edition (which is included in the second edition), we read (the arguable understatement):

    > The book presupposes that the reader has a feeling for rigor and some knowledge of elementary facts from calculus. Some material that is no doubt familiar to many readers has been included; its inclusion seemed desirable in order to make the presentation clear and self-contained.
    >
    > The approach of the book is to develop the theory of measure and integration first in the simple setting of Euclidean space. In this case, there is a rich theory having a close relation to familiar facts from calculus and generalizing those facts.

    The pdf file of the 2nd edition is "perfect" and wonderfully readable. The preface of the 2nd edition boasts of adding many new exercises. The closest the book comes to providing solutions to the exercises is, as stated in the preface, "At the end of each chapter, we list a number of problems as exercises, sometimes with parenthetical hints at solutions."

11. Piermarco Cannarsa and Teresa D'Aprile, *Introduction to Measure Theory and Functional Analysis*, 2015 (from the Italian original, 2008, translated by the authors). From the preface, "This material is intended to render the exposition completely self-contained for whoever master's basic linear algebra and mathematical analysis." It is, however, not clear what constitutes "basic mathematical analysis" for these authors, though it seems that the level is approximately the same as Wheeden and Zygmund (2015).

    The MAA review https://www.maa.org/press/maa-reviews/introduction-to-measure-theory-and-functional-analysis is relatively short and overall positive:

    > This book, written by leading experts, is a well-crafted textbook covering a medley of relevant topics in measure theory and functional analysis in a rather get-to-the-point-quickly fashion, yet resulting in a very readable and enjoyable journey. The material covered is largely classical, heavily influenced by the good old textbooks in the field (i.e, Rudin, Royden, and Yosida), but the authors' own agenda and point-of-view are tangibly present and give the book a unique feel.

    Solutions are not provided for the exercises in the book.

12. Hugo D. Junghenn, *Principles of Analysis: Measure, Integration, Functional Analysis, and Applications*, 2018, 520 pages. The MAA review https://www.maa.org/press/maa-reviews/principles-of-analysis-measure-integration-functional-analysis-and-applications avoids being either positive or negative, and writes:

    > This book is intended to provide a detailed and rigorous treatment of the essential aspects of measure theory, integration and functional analysis for graduate students of mathematics. It has an unusually broad scope for such a text and probably has enough material for a three or four-semester course. ... The book has an encyclopedic feel to it as if the author wanted to include all the topics he thought every future analyst needed to know.

8.4 Measure Theoretic Probability

For this subject, among the most well-known books is Patrick Billingsley's *Probability and Measure*, 3rd edition, 1995, (first edition 1979), though there are now quite a few alternatives to be considered. Omission of a book in my list is not a signal that I deem it poor—there are presumably other good books, and I simply do not know about all of them. I only learned about some of these books while assembling this document, and I am glad I did: These include Roussas' (2014) book, for which the MAA review is stellar, and, from the pdf of the book available in the web, the book indeed looks extremely inviting. Next is Nair's (2019) book, which appears to be among the shortest and most approachable; and finally Shorack's (2017) book, which gets included because it looks excellent and useful, albeit, at least for me, rather advanced.

1. Santosh S. Venkatesh, *The Theory of Probability: Explorations and Applications*, 2013, 827 pages. This is another one of those books for which I say "I wish I wrote it". Besides a magnificent (advanced undergraduate) treatise on probability, the author introduces the basics of measure theory. Regarding this, he says in the preface:

    > I have attempted to satisfy the demands of intuition and rigour in the narrative by beginning with the elementary theory (though a reader should not confuse

> the word elementary to mean easy or lacking subtlety) and blending in the theory of measure half way through the book. While measure provides the foundation of the modern theory of probability, much of its import, especially in the basic theory, is to provide a guarantee that limiting arguments work seamlessly. The reader willing to take this on faith can plunge into the rich theory and applications in the later chapters in this book, returning to shore up the measure-theoretic details as time and inclination allow. I have found to my pleasant surprise over the years that novices have boldly plunged into passages where students with a little more experience are sadly hampered by the fear of a misstep and tread with caution.

Regarding prerequisites, the author clearly wrote this book to be a bit "outside of the factory norm", meaning, at least in my opinion, it does not fit nicely into the typical university teaching structure, and was not written to fit into a publisher's need to have a book filling a well-defined book niche. As such, he is a bit vague about what a student should bring to the table, but we do read:

> Of course there is always the question of what background may be reasonably assumed. My audiences have ranged from undergraduate upperclassmen to beginning graduate students to advanced graduate students and specialists; and they have come from an eclectic welter of disciplines ranging across engineering, computer science, mathematics, statistics, and pure and applied science. A common element in their backgrounds has been a solid foundation in undergraduate mathematics, say, as taught in a standard three or four-course calculus sequence that is a staple in engineering, science, or mathematics curricula. A reader with this as background and an interest in mathematical probability will, with sufficient good will and patience, be able to make her way through most of this book; more advanced tools and techniques are developed where they are needed and a short Appendix fills in lacunae that may have crept into a calculus sequence.

Finally, from the preface, the author makes a nice statement about judging books, as well as a reference to Chung's book:

> Inevitably, the price to be paid for an honest account of the foundational theory and its applications is in coverage. One cannot be all things to all people. I suppose I could plead personal taste in the shape of the narrative but as Kai Lai Chung has remarked in the preface of his classical book on probability, in mathematics, as in music, literature, or cuisine, there is good taste and bad taste; and any author who pleads personal taste must be willing to be judged thereby.

2. Kai Lai Chung, *A Course in Probability Theory*, Third Edition, 2001, 419 pages. The pdf available for this book is "just good enough" to read, and certainly cannot compare optically to the newer books.

3. David Pollard, *A User's Guide to Measure Theoretic Probability*, 2001.

4. Robert B. Ash (Author), Catherine A. Doléans-Dade, *Probability and Measure Theory*, Second Edition, 2000. From the preface, "We assume the reader has had a course in basic analysis and is familiar with metric spaces."

5. Krishna B. Athreya and Soumendra N. Lahiri, *Measure Theory and Probability Theory*, 2006, 618 pages. The book is intended for first-year Ph.D. students in mathematics and statistics. From the back cover:

> Prerequisites are kept to the minimal level of an understanding of basic real analysis concepts such as limits, continuity, differentiability, Riemann integration, and convergence of sequences and series. A review of this material is included in the appendix.

From the preface:

> The traditional approach to a first course in measure theory, such as in Royden (1988), is to teach the Lebesgue measure on the real line, then the differentation theorems of Lebesgue, Lp-spaces on R, and do general measure at the end of the course with one main application to the construction of product measures. This approach does have the pedagogic advantage of seeing one concrete case first before going to the general one. But this also has the disadvantage in making many students' perspective on measure theory somewhat narrow. It leads them to think only in terms of the Lebesgue measure on the real line and to believe that measure theory is intimately tied to the topology of the real line. As students of statistics, probability, physics, engineering, economics, and biology know very well, there are mass distributions that are typically nonuniform, and hence it is useful to gain a general perspective.

Positive excerpts from several book reviews can be seen on the Amazon page of the book, as well as numerous very positive customer reviews. Chapters 5 to 12 cover (obligatory material for a book on measure-theoretic probability, namely) probability measures, convolutions, probability spaces, independence, laws of large numbers, convergence in distribution, characteristic functions, central limit theorems, conditional probability and expectation. Less commonly, the remaining chapters cover martingales and stopping times, Markov chains and MCMC, Brownian motion, the Black-Scholes formula, limit theorems for dependent processes, the bootstrap, and a short chapter on the Bienyeme-Galton-Watson branching process. Appendix A is a fast-paced 21 pages of analysis and metric space review (and then five pages of exercises).

6. M. Thamban Nair, *Measure and Integration: A First Course*, 2019, 215 pages. From the preface:

   > [T]he reader will definitely find that the presentation of the concepts and results differ from the standard texts, in the sense that it is more student-friendly."
   > To use this book for a course on measure and integration, no pre-requisite is assumed, except the mathematical maturity to appreciate and grasp concepts in analysis, though it is recommended that it be taught after a course on real analysis.

   The MAA review https://www.maa.org/press/maa-reviews/measure-and-integration-a-first-course is positive, though emphasizing that this is for a first graduate-level course, and, similar to Athreya and Lahiri's book, "especially when the subject matter is meant to be broadly applicable (with a focus on general measurable spaces, not just Euclidean measurable spaces)."

7. Bert E. Fristedt and Lawrence F. Gray, *A Modern Approach to Probability Theory*, 1996, 756 pages. I already mentioned their book in foonote 45. A document containing solutions to some of the exercises in the book can be found here, https://www-users.cse.umn.edu/~grayx004/pdf/solutions.pdf.

8. George G. Roussas, *An Introduction to Measure-Theoretic Probability*, 2014. From the MAA review https://www.maa.org/press/maa-reviews/an-introduction-to-measure-theoretic-probability, we read:

> Well, it's clear, then, that this book is serious probability theory, covering a number of bases: It is a very thorough discussion of many of the pillars of the subject, showing in particular how "measure theory with total measure one" is just the tip of the iceberg, and it is a voyage from a relatively prosaic point of departure (a beginning graduate student might book passage) to some pretty sophisticated destinations replete with visits to Kolmogorov. It's quite a book.

9. Albert N. Shiryaev, *Probability-1*, Third Edition, 2016, and *Probability-2*, Third Edition, 2019. (Translated from Russian.)

10. Galen R. Shorack, *Probability for Statisticians*, second edition, 2017. Based on the preview available at Amazon, this is a highly useful, very attractive, and very high level book, requiring a solid previous exposure to, and quite some fluency in, measure-theoretic probability theory.

I end this section by mentioning two more books. The first is *Foundations of Modern Probability*, by Olav Kallenberg, now in its third edition, 2021. The previous editions have been highly praised—so why is it not in the above list? Because its level is yet higher, and I mention it simply for readers who (unlike myself) are comfortable with all the primary material in the above books, and are eager and ready for the next level. The same can be said about *Real Analysis and Probability*, by R. M. Dudley, second edition, 2002. This book receives mixed reviews at Amazon. For example, one reviewer is not so positive, writing (in 2008):

> I have been teaching a one semester course of Real Analysis (measure and integration) from this book. The students have already been through a course based on Rudin's Principles of Mathematical Analysis though not the Lebesgue integral there, and pretty comfortable with metric spaces and such and the standards of mathematical proof. So as the next step in analysis this book seems to be in the right place esp. because the book advertises itself as self-contained.
>
> While I appreciate the wonderful integration of Real Analysis and Probability and short proofs, the brevity is often achieved by omitting details rather than choosing a simpler argument and so the book is a bit too hard on the students. Many proofs are too terse and have significant gaps which often take a lot of classroom time to get over, unless you are willing to leave them puzzled. The wording in the proofs is often counterintuitive, in particular it is usually not clear if the sentence continues the line of argument or starts a new one. This is an unnecessary hiccup for the reader and it would cost just few friendly words here and there to fix. Overall the book is harder to follow than Royden's Real Analysis. Many of the exercises are great and illuminative but many are just impossibly hard.

Another defends the book (against presumably a one-star review in 2004), writing (in 2005):

> Finally, despite the complaint of some reviewers, the book is extremely well written and amazingly comprehensive. The sole prerequisite to reading it is a certain amount of "mathematical maturity" which perhaps these reviewers lack.

8.5 A Next^3 Course: Stochastic Limit Theory

Brainstorming about courses (as opposed to implementing them) is cheap, easy, and fun, so allow me to indulge. A PhD-level course could be envisioned based on James Davidson's 1994 book *Stochastic Limit Theory: An Introduction for Econometricians.*

Coincidentally and conveniently, Amazon states that a second edition is expected in 2022. The second part of the title, with "introduction", is absurd: This is a serious and heavy theorem-proof mathematics book that nicely starts with some essentials (sets, limits, continuity), but quickly gets to measure spaces, Lebesgue-Stieltjes integral, metric spaces, and topology. That gets us to page 110, where part II starts, on probability, and then part III, on the theory of stochastic processes, followed by part IV on the law of large numbers, and part V, the central limit theorem. Finally, part VI is the functional central limit theorem.

The typesetting (of the 1994 book) is good, but not latex, and not great, and the pdf file is not searchable or markable.

To get an idea of the level of the book, in the preface, we read:

> ...the following titles were, for one reason or another, the most frequently consulted in the course of writing this book. T. M. Apostol's *Mathematical Analysis* (2nd edition) hits just the right note for the basic bread-and-butter results. For more advanced material, Dieudonné's *Foundations of Modern Analysis* and Royden's *Real Analysis* are well-known references, the latter being the more user-friendly although the treatment is often fairly concise. Halmos's classic *Measure Theory* and Kingman and Taylor's *Introduction to Measure and Probability* are worth having access to.
>
> ...
>
> Pollard's book takes up the weak convergence story more or less where Billingsley leaves off, and much of the material complements the coverage of the present volume.

Many more books are cited besides these (in)famous ones, and the idea that this book essentially takes off where Billingsley ends gives an indication of its level.

References

Abbott, S. (2016). *Understanding Analysis.* Springer, New York City, New York.

Adams, C. and Franzosa, R. (2008). *Introduction to Topology: Pure and Applied.* Pearson, Upper Saddle River, New Jersey.

Adams, H. and Moy, M. (2021). Topology Applied to Machine Learning: From Global to Local. *Frontiers in Artificial Intelligence*, 4:54.

Agarwal, R. P. and Flaut, E. C. (2017). *An Introduction to Linear Algebra.* CRC Press, Boca Raton, Florida.

Aggarwal, C. (2020). *Linear Algebra and Optimization for Machine Learning.* Springer, Cham, Switzerland.

Akcoglu, M. A., Bartha, P. F., and Ha, D. M. (2009). *Analysis in Vector Spaces.* John Wiley & Sons, Hoboken, New Jersey.

Aksoy, A. G. and Khamsi, M. (2010). *A Problem Book in Real Analysis.* Springer, New York City, New York.

Alcock, L. (2014). *How to Think About Analysis.* Oxford University Press, Oxford, the United Kingdom.

Aliprantis, C. D. and Burkinshaw, O. (1998). *Principles of Real Analysis.* Academic Press, San Diego, California.

Aliprantis, C. D. and Burkinshaw, O. (1999). *Problems in Real Analysis: A Workbook with Solutions.* Academic Press, San Diego, California.

Alsamraee, H. E. (2019). Advanced Calculus Explored. CuriousMathPublications,https://www.curiousmath.com/.

Amann, H. and Escher, J. (2005). *Analysis I.* Birkhäuser, Basel, Switzerland.

Amann, H. and Escher, J. (2008). *Analysis II.* Birkhäuser, Basel, Switzerland.

Amann, H. and Escher, J. (2009). *Analysis III.* Birkhäuser, Basel, Switzerland.

Andreescu, T. (2014). *Essential Linear Algebra with Applications: A Problem-Solving Approach.* Birkhäuser, Basel, Switzerland.

Andrilli, S. and Hecker, D. (2016). *Elementary Linear Algebra.* Academic Press, Boston, Massachusetts.

Anthony, M. and Harvey, M. (2012). *Linear Algebra: Concepts and Methods.* Cambridge University Press, Cambridge, the United Kingdom.

Anton, H. and Rorres, C. (2013). *Elementary Linear Algebra: Applications Version.* John Wiley & Sons, Hoboken, New Jersey.

Apostol, T. M. (1974). *Mathematical Analysis.* Pearson, Harlow, Essex.

Ash, R. B. and Doléans-Dade, C. A. (2000). *Probability and Measure Theory.* Academic Press, San Diego, California.

Athreya, K. B. and Lahiri, S. N. (2006). *Measure Theory and Probability Theory.* Springer, New York City, New York.

Ault, S. V. (2018). *Understanding Topology: A Practical Introduction.* JHU Press, Baltimore, Maryland.

Axler, S. (2015). *Linear Algebra Done Right*. Springer, Cham, Switzerland.

Axler, S. (2020). *Measure, Integration, and Real Analysis*. Springer, Cham, Switzerland.

Bachman, D. (2012). *A Geometric Approach to Differential Forms*. Birkhäuser, Basel, Switzerland.

Ballmann, W. (2018). *Introduction to Geometry and Topology*. Birkhäuser, Basel, Switzerland.

Banchoff, T. and Wermer, J. (1992). *Linear Algebra Through Geometry*. Springer, New York City, New York.

Banerjee, S. and Roy, A. (2014). *Linear Algebra and Matrix Analysis for Statistics*. Chapman and Hall/CRC, New York City, New York.

Bapat, R. B. (2012). *Linear Algebra and Linear Models*. Springer, London, the United Kingdom.

Bartle, R. G. (1966). *The Elements of Integration and Lebesgue Measure*. John Wiley & Sons, Hoboken, New Jersey.

Bartle, R. G. (1996). Return to the Riemann integral. *The American Mathematical Monthly*, 103(8):625–632.

Bartle, R. G. and Sherbert, D. R. (2011). *Introduction To Real Analysis*. John Wiley & Sons, Hoboken, New Jersey.

Bass, R. F. (2013). *Real Analysis for Graduate Students*. CreateSpace, Scotts Valley, California.

Bass, R. F. (2020). Real Analysis for Graduate Students. http://bass.math.uconn.edu/books.html.

Bauer, H. (2001). *Measure and Integration Theory*. De Gruyter, Berlin, Germany.

Bauldry, W. C. (2009). *Introduction to Real Analysis: An Educational Approach*. John Wiley & Sons, Hoboken, New Jersey.

Beals, R. (2004). *Analysis: An Introduction*. Cambridge University Press, Cambridge, the United Kingdom.

Bear, H. S. (2001). *A Primer of Lebesgue Integration*. Academic Press, San Diego, California.

Beardon, A. F. (1997). *Limits: A New Approach to Real Analysis*. Springer, New York City, New York.

Beezer, R. A. (2015). *A First Course in Linear Algebra*. Congruent Press.

Beilina, L., Karchevskii, E., and Karchevskii, M. (2017). *Numerical Linear Algebra: Theory and Applications*. Springer, Cham, Switzerland.

Ben-Israel, A. and Greville, T. N. E. (2003). *Generalized Inverses: Theory and Applications*. Springer, New York City, New York.

Billingsley, P. (1995). *Probability and Measure, Third Edition*. John Wiley & Sons, New York City, New York.

Bisgard, J. (2021). *Analysis and Linear Algebra: The Singular Value Decomposition and Applications*. American Mathematical Society, Providence, Rhode Island.

Blitzstein, J. K. and Hwang, J. (2014). *Introduction to Probability*. Chapman and Hall/CRC, London, the United Kingdom.

Bloch, E. D. (2011). *The Real Numbers and Real Analysis.* Springer, New York City, New York.

Blyth, S. (2014). *An Introduction to Quantitative Finance.* Oxford University Press, Oxford, the United Kingdom.

Blyth, T. S. and Robertson, E. F. (2002a). *Basic Linear Algebra.* Springer, London, the United Kingdom.

Blyth, T. S. and Robertson, E. F. (2002b). *Further Linear Algebra.* Springer, London, the United Kingdom.

Boas, R. P. and Boas, H. P. (1996). *A Primer of Real Functions.* Mathematical Association of America, Washington, D.C, fourth edition.

Boelkins, M. R., Goldberg, J. L., and Potter, M. C. (2009). *Differential Equations with Linear Algebra.* Oxford University Press, Oxford, the United Kingdom.

Bogachev, V. I. and Smolyanov, O. G. (2020). *Real and Functional Analysis.* Springer, Cham, Switzerland.

Bogacki, P. (2019). *Linear Algebra: Concepts and Applications.* American Mathematical Society, Providence, Rhode Island.

Bornemann, F. (2018). *Numerical Linear Algebra: A Concise Introduction with MATLAB and Julia.* Springer, Cham, Switzerland.

Bourchtein, A. and Bourchtein, L. (2015). *CounterExamples: From Elementary Calculus to the Beginnings of Analysis.* CRC Press, Boca Raton, Florida.

Boyd, S. and Vandenberghe, L. (2018). *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least Squares.* Cambridge University Press, Cambridge, the United Kingdom.

Bradie, B. (2006). *A Friendly Introduction to Numerical Analysis.* Prentice Hall, Upper Saddle River, New Jersey.

Brannan, D. A. (2006). *A First Course in Mathematical Analysis.* Cambridge University Press, Cambridge, the United Kingdom.

Breiman, L. (2001). Statistical Modeling: The Two Cultures (with comments and a rejoinder by the author). *Statistical science*, 16(3):199–231.

Bressoud, D. M. (1991). *Second Year Calculus: From Celestial Mechanics to Special Relativity.* Springer, New York City, New York.

Bressoud, D. M. (2008a). *A radical approach to Lebesgue's theory of integration.* Cambridge University Press, Cambridge, the United Kingdom.

Bressoud, D. M. (2008b). *A Radical Approach to Real Analysis.* Mathematical Association of America, Washington, D.C, second edition.

Bretscher, O. (2013a). *Linear Algebra: Theory, Intuition, Code.* Pearson, Harlow, Essex.

Bretscher, O. (2013b). *Linear Algebra with Applications.* Pearson, Harlow, Essex.

Bridger, M. (2007). *Real analysis : A Constructive Approach.* John Wiley & Sons, Hoboken, New Jersey.

Bridger, M. (2019). *Real analysis : A Constructive Approach Through Interval Arithmetic.* American Mathematical Society, Providence, Rhode Island.

Bronson, R., Saccoman, J. T., and Costa, G. B. (2007). *Linear Algebra: Algorithms, Applications, and Techniques*. Academic Press, San Diego, California.

Browder, A. (1995). *Mathematical Analysis: An Introduction*. Springer, Cham, Switzerland.

Bruckner, J. B., Bruckner, A. M., and Thomson, B. S. (2008a). *Elementary Real Analysis*. CreateSpace, Scotts Valley, California.

Bruckner, J. B., Bruckner, A. M., and Thomson, B. S. (2008b). *Real Analysis*. CreateSpace, Scotts Valley, California.

Bryant, V. (1985). *Metric Spaces: Iteration and Application*. Cambridge University Press, Cambridge, the United Kingdom.

Bryant, V. (1990). *Yet Another Introduction to Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Bryant, V. (1994). *Metric Spaces: Iteration and Application*. Cambridge University Press, Cambridge, the United Kingdom.

Buchanan, J. R. (2012). *Undergraduate Introduction To Financial Mathematics*. World Scientific, Singapore.

Burden, R. and Faires, J. D. (2011). *Numerical Analysis*. Brooks Cole, Pacific Grove, California.

Burstein, S., Lax, P. D., and Lax, A. (1976). *Calculus with Applications*. Springer, New York City, New York.

Calin, O. (2015). *An Informal Introduction to Stochastic Calculus with Applications*. World Scientific, Singapore.

Callahan, J. J. (2010). *Advanced Calculus: A Geometric View*. Springer, New York City, New York.

Calvis, D. T., Edwards, C. H., and Penney, D. E. (2018). *Differential Equations and Linear Algebra*. Pearson, Upper Saddle River, New Jersey.

Campolieti, G. and Makarov, R. N. (2014). *Financial Mathematics: A Comprehensive Treatment*. CRC Press, Boca Raton, Florida.

Campolieti, G. and Makarov, R. N. (2021). *Financial Mathematics: A Comprehensive Treatment in Discrete Time*. CRC Press, Boca Raton, Florida.

Cannarsa, P. and D'Aprile, T. (2015). *Introduction to Measure Theory and Functional Analysis*. Springer, Cham, Switzerland.

Cannon, J. W. (2017). *Two-Dimensional Spaces, Volume 1: Geometry of Lengths, Areas, and Volumes*. American Mathematical Society, Providence, Rhode Island.

Canuto, C. and Tabacco, A. (2008). *Mathematical Analysis I*. Springer, Cham, Switzerland.

Canuto, C. and Tabacco, A. (2010). *Mathematical Analysis II*. Springer, Cham, Switzerland.

Carr, G. S. (1886). *A Synopsis of Elementary Results in Pure Mathematics: Containing Propositions, Formulæ, and Methods of Analysis, with Abridged Demonstrations*. F. Hodgson, London, the United Kingdom.

Carrell, J. B. (2005). The Basics of Linear Algebra. https://secure.math.ubc.ca/~carrell/NB.pdf.

Carter, M. and Brunt, B. v. (2000). *The Lebesgue-Stieltjes Integral: A Practical Introduction*. Springer, New York City, New York.

Chacón, G., Rafeiro, H., and Vallejo, J. C. (2017). *Functional Analysis: A Terse Introduction*. De Gruyter, Berlin, Germany.

Chahal, J. S. (2019). *Fundamentals of Linear Algebra*. CRC Press, Boca Raton, Florida.

Chen, H. (2010). *Excursions in Classical Analysis: Pathways to Advanced Problem Solving and Undergraduate Research*. Mathematical Association of America, Washington, D.C.

Cheney, E. W. and Kincaid, D. R. (2008). *Numerical Mathematics and Computing*. Brooks Cole, Pacific Grove, California.

Cheney, W. and Kincaid, D. (2009). *Linear Algebra: Theory and Applications*. Jones and Bartlett Publishers, Sudbury, Massachusetts.

Cheney, W. and Kincaid, D. (2012). *Numerical Mathematics and Computing*. Cengage Learning, Boston, Massachusetts.

Cheng, S. (2004). A Short Course on the Lebesgue Integral and Measure Theory. http://webeducation.com/wp-content/uploads/2021/05/Steve-Cheng-A-Short-Course-on-the-Lebesgue-Integral-and-Measure-Theory-Perseus-Books-Sd-1990.pdf.

Cherney, D., Denton, T., Thomas, R., and Waldron, A. (2016). Linear Algebra. https://www.math.ucdavis.edu/~linear/.

Cherney, D., Denton, T., and Waldron, A. (2021). Linear Algebra. https://math.libretexts.org/Bookshelves/Linear_Algebra/Map%3A_Linear_Algebra_(Waldron_Cherney_and_Denton).

Chung, K. L. (2001). *A Course in Probability Theory*. Academic Press, San Diego, California.

Chung, K. L. and AitSahlia, F. (2012). *Elementary Probability Theory: With Stochastic Processes and an Introduction to Mathematical Finance*. Springer, New York City, New York.

Cohen, M. (2021). *Linear Algebra: Theory, Intuition, Code*. Sincxpress BV, Amsterdam, Netherlands.

Collet, J.-F. (2018). *Discrete Stochastic Processes and Applications*. Springer, Cham, Switzerland.

Conway, J. B. (2012). *A Course in Abstract Analysis*. American Mathematical Society, Providence, Rhode Island.

Conway, J. B. (2014). *A Course in Point Set Topology*. Springer, Cham, Switzerland.

Conway, J. B. (2018). *A First Course in Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Cooperstein, B. (2016). Elementary Linear Algebra. https://people.ucsc.edu/~morozco7/Books/cooperstein2016elementary.pdf.

Cooperstein, B. N. (2015). *Advanced Linear Algebra*. Chapman and Hall/CRC, London, the United Kingdom.

Cooperstein, B. N. (2019). Elementary Linear Algebra. https://www.centerofmath.org/textbooks/ela/index.html.

Copson, E. T. (1968). *Metric Spaces.* Cambridge University Press, Cambridge, the United Kingdom.

Corbae, D., Stinchcombe, M. B., and Zeman, J. (2009). *An Introduction to Mathematical Analysis for Economic Theory and Econometrics.* Princeton University Press, Princeton, New Jersey.

Cover, T. M. and Thomas, J. A. (2006). *Elements of Information Theory.* John Wiley & Sons, Hoboken, New Jersey.

Croom, F. H. (2016). *Principles of Topology.* Dover Publications, Mineola, New York.

Crossley, M. D. (2005). *Essential Topology.* Springer, London, the United Kingdom.

Cull, P., Flahive, M., and Robson, R. (2005). *Difference Equations From Rabbits to Chaos.* Springer, New York City, New York.

Cullen, C. G. (1972). *Matrices and Linear Transformations.* Dover, New York, second edition. This Dover edition, first published in 1990, is an unabridged, corrected republication of the second edition (1972) of the work originally published in 1966 by Addison-Wesley Publishing Company, Reading, Massachusetts.

Cummings, J. (2019). *Real Analysis : A Long-Form Mathematics Textbook.* CreateSpace, Scotts Valley, California.

Cunningham, D. W. (2020). *Real Analysis: With Proof Strategies.* Chapman and Hall/CRC, London, the United Kingdom.

Cuoco, A., Waterman, K., Kerins, B., Kaczorowski, E., and Manes, M. (2019). *Linear Algebra and Geometry.* MAA Press, Providence, Rhode Island.

Curtis, C. W. (1984). *Linear Algebra: An Introductory Approach.* Springer, New York City, New York.

Damiano, D. B. and Little, J. B. (1988). *A Course in Linear Algebra.* Harcourt Brace Jovanovich, San Diego, CA.

D'Angelo, J. P. and West, D. B. (1997). *Mathematical Thinking: Problem-solving and Proofs.* Prentice Hall, Upper Saddle River, New Jersey.

Davidson, J. (1994). *Stochastic Limit Theory: An Introduction for Econometricians.* Oxford University Press, Oxford, the United Kingdom.

Davidson, K. R. and Donsig, A. P. (2010). *Real Analysis and Applications: Theory in Practice.* Springer, New York City, New York.

DeFranza, J. and Gagliardi, D. (2009). *Introduction to Linear Algebra with Applications.* McGraw-Hill, New York.

Deisenroth, M. P., Faisal, A. A., and Ong, C. S. (2020). *Mathematics for Machine Learning.* Cambridge University Press, Cambridge, the United Kingdom.

DePree, J. and Swartz, C. (1988). *Introduction to Real Analysis.* John Wiley & Sons, Hoboken, New Jersey.

Dieudonné, J. (1960). *Foundations of Modern Analysis.* Academic Press, San Diego, California.

Dixmier, J. (1984). *General Topology.* Springer, New York City, New York.

Dudley, R. M. (2002). *Real Analysis and Probability.* Cambridge University Press, Cambridge, the United Kingdom.

Duren, P. (2012). *Invitation to Classical Analysis*. American Mathematical Society, Providence, Rhode Island.

Dym, H. (2013). *Linear Algebra in Action*. American Mathematical Society, Providence, Rhode Island.

Eberlein, E. and Kallsen, J. (2019). *Mathematical Finance*. Springer, Cham, Switzerland.

Edwards, H. M. (1995). *Linear Algebra*. Birkhäuser, Basel, Switzerland.

Edwards, H. M. (2005). *Essays in Constructive Mathematics*. Springer, New York City, New York.

Elaydi, S. (2005). *An Introduction to Difference Equations*. Springer, New York City, New York.

Ellenberg, J. (2021). *Shape: The Hidden Geometry of Information, Biology, Strategy, Democracy, and Everything Else*. Penguin Random House, London, the United Kingdom.

Epperson, J. F. (2013). An Introduction to Numerical Methods and Analysis.

Erdman, H. M. (1995). *Linear Algebra*. American Mathematical Society, Providence, Rhode Island.

Erdman, J. M. (2018). *A Problems Based Course in Advanced Calculus*. American Mathematical Society, Providence, Rhode Island.

Estep, D. (2002). *Practical Analysis in One Variable*. Springer, New York City, New York.

Eves, H. (1968). *Elementary Matrix Theory*. Allyn and Bacon, Boston. Second Printing.

Exner, G. R. (2012). *An Accompaniment to Higher Mathematics*. Springer, New York City, New York.

Farin, G. and Hansford, D. (2005). *Practical Linear Algebra: A Geometry Toolbox*. A K Peters, Natick, Massachusetts.

Faul, A. C. (2016). *A Concise Introduction to Numerical Analysis*. CRC Press, Boca Raton, Florida.

Field, M. (2017). *Essential Real Analysis*. Springer, Cham, Switzerland.

Finan, Marcel, B. (2001). Fundamentals Of Linear Algebra. https://faculty.atu.edu/mfinan/algebra2.pdf.

Finan, Marcel, B. (2015). The Basics of Linear Algebra. https://faculty.atu.edu/mfinan/4003/LINENG.pdf.

Fischer, E. (1983). *Intermediate Real Analysis*. Springer, New York City, New York.

Fitzpatrick, P. (2009). *Advanced Calculus*. American Mathematical Society, Providence, Rhode Island.

Flanigan, F. J. and Kazdan, J. L. (1971). *Calculus Two: Linear and Nonlinear Functions, 1st Edition*. Prentice Hall, Upper Saddle River, New Jersey.

Folland, G. (2007). *Real Analysis: Modern Techniques and Their Applications*. John Wiley & Sons, Hoboken, New Jersey.

Folland, G. (2009). *A Guide to Advanced Real Analysis*. American Mathematical Society, Providence, Rhode Island.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance: An Introduction in Discrete Time*. De Gruyter, Berlin, Germany.

Fonda, A. (2018). *Kurzweil-Henstock Integral for Undergraduates: A Promenade Along the Marvelous Theory of Integration*. Birkhäuser, Basel, Switzerland.

Ford, W. (2015). *Numerical Linear Algebra with Applications: Using MATLAB*. Academic Press, Cambridge, Massachusetts.

Fortney, J. P. (2018). *A Visual Introduction to Differential Forms and Calculus on Manifolds*. Birkhäuser, Basel, Switzerland.

Fraleigh, J. B. and Beauregard, R. A. (1995). *Linear Algebra*. Addison-Wesley Publishing Company, Reading, Massachusetts, third edition.

Franks, J. (2000). *A (Terse) Introduction to Lebesgue Integration*. American Mathematical Society, Providence, Rhode Island.

Freedman, D. A. (2009). *Statistical Models: Theory and Practice*. Cambridge University Press, Cambridge, second edition.

Friedberg, S. H., Insel, A. J., and Spence, L. E. (2002). *Linear Algebra*. Pearson, Upper Saddle River, New Jersey, fourth edition.

Friedberg, S. H., Insel, A. J., and Spence, L. E. (2019). *Linear Algebra*. Pearson, Upper Saddle River, New Jersey, fifth edition.

Friedman, J., Hastie, T., Tibshirani, R., et al. (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. Springer, New York City, New York.

Fristedt, B. E. and Gray, L. F. (1996). *A Modern Approach to Probability Theory*. Birkhäuser, Basel, Switzerland.

Gallier, J. (2015). Fundamentals of Linear Algebra and Optimization. https://www.cis.upenn.edu/~cis515/cis515-13-sl1-a.pdf.

Gallier, J. and Quaintance, J. (2020a). *Linear Algebra and Optimization with Applications to Machine Learning: Volume I: Linear Algebra for Computer Vision, Robotics, and Machine Learning*. World Scientific, Singapore.

Gallier, J. and Quaintance, J. (2020b). *Linear Algebra And Optimization With Applications To Machine Learning-Volume II: Fundamentals Of Optimization Theory With Applications To Machine Learning*. World Scientific, Singapore.

Garcia, S. R. and Horn, R. A. (2017). *A Second Course in Linear Algebra*. Cambridge University Press, Cambridge, the United Kingdom.

Garling, D. J. (2013). *A Course in Mathematical Analysis: Volume 1: Foundations and Elementary Real Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Garling, D. J. (2014a). *A Course in Mathematical Analysis: Volume 2, Metric and Topological Spaces, Functions of a Vector Variable*. Cambridge University Press, Cambridge, the United Kingdom.

Garling, D. J. (2014b). *A Course in Mathematical Analysis: Volume 3, Complex Analysis, Measure and Integration*. Cambridge University Press, Cambridge, the United Kingdom.

Gautschi, W. (2012). *Numerical Analysis, 2nd Edition*. Birkhäuser, Basel, Switzerland.

Gentle, J. E. (2017). *Matrix Algebra: Theory, Computations, and Applications in Statistics*. Springer, New York City, New York.

Ghorpade, S. R. and Limaye, B. V. (2010). *A Course in Multivariable Calculus and Analysis.* Springer, New York City, New York.

Ghorpade, S. R. and Limaye, B. V. (2018). *A Course in Calculus and Real Analysis.* Springer, Cham, Switzerland.

Giaquinta, M. and Modica, G. (2003). *Mathematical Analysis: Functions of One Variable.* Birkhäuser, Basel, Switzerland.

Giaquinta, M. and Modica, G. (2007). *Mathematical Analysis: Linear and Metric Structures and Continuity.* Birkhäuser, Basel, Switzerland.

Giaquinta, M. and Modica, G. (2009). *Mathematical Analysis: An Introduction to Functions of Several Variables.* Birkhäuser, Basel, Switzerland.

Giaquinta, M. and Modica, G. (2012). *Mathematical Analysis: Foundations and Advanced Techniques for Functions of Several Variables.* Birkhäuser, Basel, Switzerland.

Giné, E. and Nickl, R. (2016). *Mathematical Foundations of Infinite-Dimensional Statistical Models.* Cambridge University Press, Cambridge, the United Kingdom.

Giraud, C. (2015). *Introduction to High-Dimensional Statistics.* CRC Press, Boca Raton, Florida.

Godement, R. (2004). *Analysis I: Convergence, Elementary functions.* Springer, Berlin Heidelberg, Germany.

Golan, J. S. (2012). *The Linear Algebra a Beginning Graduate Student Ought to Know.* Springer, Dordrecht, Netherlands.

Goldberg, R. R. (1976). *Methods of Real Analysis.* John Wiley & Sons, Hoboken, New Jersey.

Golubitsky, M. and Dellnitz, M. (1999). *Linear Algebra and Differential Equations Using MATLAB.* Brooks Cole, Pacific Grove, California.

Goodaire, E. G. (2014). *Linear Algebra: Pure and Applied.* World Scientific, New Jersey.

Goode, S. W. and Annin, S. (2016). *Differential Equations and Linear Algebra.* Pearson, Boston, Massachusetts.

Gopal, D., Deshmukh, A., Ranadive, A. S., and Yadav, S. (2020). *An Introduction to Metric Spaces.* CRC Press, Boca Raton, Florida.

Graybill, F. A. (1983). *Matrices with Applications in Statistics.* Brooks Cole, Pacific Grove, California.

Grimmett, G. and Stirzaker, D. (2020a). *One Thousand Exercises in Probability.* Oxford University Press, Oxford, the United Kingdom.

Grimmett, G. and Stirzaker, D. (2020b). *Probability and Random Processes.* Oxford University Press, Oxford, the United Kingdom.

Grimmett, G. and Welsh, D. (2014). *Probability: An Introduction.* Oxford University Press, Oxford, the United Kingdom.

Ha, D. M. (2006). *Functional Analysis: A Gentle Introduction.* Matrix Editions, Ithaca, New York.

Haase, M. (2014). *Functional Analysis: An Elementary Introduction.* American Mathematical Society, Providence, Rhode Island.

Harari, Y. N. (2016). *Homo Deus: A Brief History of Tomorrow.* Harvill Secker, London, the United Kingdom.

Harari, Y. N. (2018). *21 Lessons for the 21st Century.* Harvill Secker, London, the United Kingdom.

Harville, D. A. (2008). *Matrix Algebra from a Statistician's Perspective.* Springer, New York City, New York.

Harville, D. A. (2018). *Linear Models and the Relevant Distributions and Matrix Algebra.* Chapman and Hall/CRC, Boca Raton, Florida.

Hassler, U. (2016). *Stochastic Processes and Calculus. An Elementary Introduction with Applications.* Springer, Cham, Switzerland.

Hastie, T., Tibshirani, R., and Friedman, J. (2001). *The Elements of Statistical Learning.* Springer, New York City, New York.

Hastie, T., Tibshirani, R., and Friedman, J. (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction.* Springer, New York City, New York.

Hata, M. (2007). *Problems and Solutions in Real Analysis.* World Scientific, Hackensack, New Jersey.

Hawkins, T. (2001). *Lebesgue's Theory of Integration: Its Origins and Development.* American Mathematical Society, Providence, Rhode Island.

Healy, M. J. R. (2000). *Matrices for Statistics.* Oxford University Press, Oxford, the United Kingdom.

Hefferon, J. (2020). *Linear Algebra.* Orthogonal Publishing L3c, Ann Arbor, Michigan.

Heil, C. (2018). *Metrics, Norms, Inner Products, and Operator Theory.* Birkhäuser, Basel, Switzerland.

Heil, C. (2019). *Introduction to Real Analysis.* Springer, Cham, Switzerland.

Heister, T., Rebholz, L. G., and Xue, F. (2019). *Numerical Analysis: An Introduction.* De Gruyter, Berlin, Germany.

Hijab, O. (2016). *Introduction to Calculus and Classical Analysis.* Springer, Cham, Switzerland.

Hoffman, K. (2007). *Analysis in Euclidean Space.* Dover Publications, Mineola, New York.

Horn, R. A. and Johnson, C. R. (2013). *Matrix Analysis.* Cambridge University Press, Cambridge, the United Kingdom, second edition.

Hsing, T. and Eubank, R. (2015). *Theoretical Foundations of Functional Data Analysis.* John Wiley & Sons, Hoboken, New Jersey.

Hubbard, J. H. and Hubbard, B. B. (2015). *Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach.* Matrix Editions, Ithaca, New York.

Jacob, N. and Evans, K. P. (2016a). *A Course in Analysis: Vol. I: Introductory Calculus Analysis of Functions of One Real Variable.* World Scientific, Singapore.

Jacob, N. and Evans, K. P. (2016b). *A Course in Analysis: Vol. II: Differentiation and Integration of Functions of Several Variables, Vector Calculus.* World Scientific, Singapore.

Jacob, N. and Evans, K. P. (2018). *A Course in Analysis: Vol. III: Measure and Integration Theory, Complex-Valued Functions of a Complex Variable.* World Scientific, Singapore.

James, G., Witten, D., Hastie, T., and Tibshirani, R. (2013). *An Introduction to Statistical Learning*. Springer, New York City, New York.

Jänich, K. (1984). *Topology*. Springer, New York City, New York.

Jänich, K. (1994). *Lineare Algebra*. Springer, New York City, New York.

Jaynes, E. T. (2003). *Probability Theory: The Logic of Science*. Cambridge University Press, Cambridge, the United Kingdom.

Jeanblanc, M., Yor, M., and Chesney, M. (2009). *Mathematical Methods for Financial Markets*. Springer, London, the United Kingdom.

Johnston, N. (2021a). *Advanced Linear and Matrix Algebra*. Springer, Cham, Switzerland.

Johnston, N. (2021b). *Introduction to Linear and Matrix Algebra*. Springer, Cham, Switzerland.

Johnston, W. (2015). *The Lebesgue Integral for Undergraduates*. Mathematical Association of America, Washington, D.C.

Jones, F. (1993). *Lebesgue Integration on Euclidean Space*. Jones and Bartlett Publishers, Sudbury, Massachusetts.

Junghenn, H. D. (2018). *Principles of Analysis: Measure, Integration, Functional Analysis, and Applications*. CRC Press, Boca Raton, Florida.

Kaczor, W. J. and Nowak, M. T. (2000). *Problems in Mathematical Analysis I: Real Numbers, Sequences and Series*. American Mathematical Society, Providence, Rhode Island.

Kaczor, W. J. and Nowak, M. T. (2001). *Problems in Mathematical Analysis II: Continuity and Differentiation*. American Mathematical Society, Providence, Rhode Island.

Kaczor, W. J. and Nowak, M. T. (2003). *Problems in Mathematical Analysis III: Integration*. American Mathematical Society, Providence, Rhode Island.

Kallenberg, O. (2021). *Foundations of Modern Probability*. Springer, Cham, Switzerland.

Kalman, D. (1996). A Singularly Valuable Decomposition: The SVD of a Matrix. *The College Mathematics Journal*, 27(1):2–23.

Kantorovitz, S. (2016). *Several Real Variables*. Springer, Cham, Switzerland.

Katznelson, Y. and Katznelson, Y. R. (2008). *A (terse) Introduction to Linear Algebra*. American Mathematical Society, Providence, Rhode Island.

Kazdan, J. L. (1966). Intermediate Calculus And Linear Algebra Part I. https://www2.math.upenn.edu/~kazdan/425S11/math21-2011.pdf.

Khamsi, M. A. and Kirk, W. A. (2011). *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley & Sons, Hoboken, New Jersey.

Kijima, M. (2013). *Stochastic Processes with Applications to Finance*. CRC Press, Boca Raton, Florida.

Kincaid, D. and Cheney, W. (2009). *Numerical Analysis: Mathematics of Scientific Computing*. American Mathematical Society, Providence, Rhode Island.

Klebaner, F. C. (2012). *Introduction to Stochastic Calculus with Applications*. World Scientific, Singapore.

Klymchuk, S. (2010). *CounterExamples in Calculus.* Mathematical Association of America, Washington, D.C.

Knapp, A. W. (2005a). *Advanced Real Analysis.* Birkhäuser, Basel, Switzerland.

Knapp, A. W. (2005b). *Basic Real Analysis.* Birkhäuser, Basel, Switzerland.

Koch, I. (2013). *Analysis of Multivariate and High-Dimensional Data.* Cambridge University Press, New York City, New York.

Koch-Medina, P. and Merino, S. (2003). *Mathematical Finance and Probability: A Discrete Introduction.* Birkhäuser, Basel, Switzerland.

Koch-Medina, P. and Munari, C. (2020). *Market-Consistent Prices: An Introduction to Arbitrage Theory.* Birkhäuser, Basel, Switzerland.

Kokoszka, P. and Reimherr, M. (2017). *Introduction to Functional Data Analysis.* CRC Press, Boca Raton, Florida.

Königsberger, K. (2004). *Analysis I.* Springer, Berlin Heidelberg, Germany.

Körner, T. W. (2003). *A Companion to Analysis: A Second First and First Second Course in Analysis.* American Mathematical Society, Providence, Rhode Island.

Körner, T. W. (2013). *Vectors, Pure and Applied : A General Introduction to Linear Algebra.* Cambridge University Press, Cambridge, the United Kingdom.

Kosmala, W. A. (2004). *A Friendly Introduction to Analysis.* Prentice Hall, Upper Saddle River, New Jersey.

Krantz, S. G. (2015). *Convex Analysis.* CRC Press, Boca Raton, Florida.

Krantz, S. G. (2017a). *Harmonic and Complex Analysis in Several Variables.* Springer, Cham, Switzerland.

Krantz, S. G. (2017b). *Real Analysis and Foundations.* CRC Press, Boca Raton, Florida.

Krantz, S. G. (2018). *Elementary Introduction to the Lebesgue Integral.* Chapman and Hall/CRC, Boca Raton, Florida.

Krantz, S. G. (2020). *Differential Equations: A Modern Approach with Wavelets.* Chapman and Hall/CRC, Boca Raton, Florida.

Kreyszig, E. (1978). *Introductory Functional Analysis with Applications.* John Wiley & Sons, Hoboken, New Jersey.

Lambers, J. V., Mooney, A. S., and Montiforte, V. A. (2018). *Explorations in Numerical Analysis.* World Scientific, Singapore.

Lanani, A. (2018). Relationship Between the Fixed Point Theorem and the EM Algorithm. *Journal of Informatics and Mathematical Sciences*, 10(4):697–702.

Lang, S. (1986). *Introduction to Linear Algebra.* Springer, New York City, New York.

Lang, S. (1987a). *Calculus of Several Variables.* Springer, New York City, New York.

Lang, S. (1987b). *Linear Algebra.* Springer, New York City, New York.

Lang, S. (1997). *Undergraduate Analysis.* Springer, New York City, New York.

Lankham, I., Nachtergaele, B., and Schilling, A. (2016). *Linear Algebra as An Introduction to Abstract Mathematics.* World Scientific, Singapore.

Lawler, G. F. (2006). *Introduction to Stochastic Processes.* CRC Press, Boca Raton, Florida.

Lax, P. D. (2007). *Linear Algebra and Its Applications.* John Wiley & Sons, New York City, New York, second edition.

Lax, P. D. and Terrell, M. S. (2014). *Calculus with Applications.* Springer, Cham, Switzerland.

Lax, P. D. and Terrell, M. S. (2017). *Multivariable Calculus with Applications.* Springer, Cham, Switzerland.

Lay, D. C., Lay, S. R., and McDonald, J. J. (2015). *Linear Algebra and Its Applications.* Pearson, London, the United Kingdom.

Lebl, J. (2020a). *Introduction to Real Analysis, Volume I.* CreateSpace, Scotts Valley, California.

Lebl, J. (2020b). *Introduction to Real Analysis, Volume II.* CreateSpace, Scotts Valley, California.

Lehmann, E. L. (1993). The Fisher, Neyman-Pearson Theories of Testing Hypotheses: One Theory or Two? *Journal of the American statistical Association*, 88(424):1242–1249.

Leon, S. J. (2015). *Linear Algebra with Applications.* Pearson, Harlow, Essex.

Liesen, J. and Mehrmann, V. (2015). *Linear Algebra.* Springer, Cham, Switzerland.

Limaye, B. V. (2016). *Linear Functional Analysis for Scientists and Engineers.* Springer, Singapore.

Lin, I.-H. (2005). *Geometric Linear Algebra 1.* World Scientific, Singapore.

Lin, I.-H. (2008). *Geometric Linear Algebra 2.* World Scientific, Singapore.

Lin, X. S. (2006). *Introductory Stochastic Analysis for Finance and Insurance.* John Wiley & Sons, Hoboken, New Jersey.

Lindstrøm, T. L. (2017). *Spaces: An Introduction to Real Analysis.* American Mathematical Society, Providence, Rhode Island.

Little, C. H., Teo, K. L., and Van Brunt, B. (2015). *Real Analysis via Sequences and Series.* Springer, New York City, New York.

Loehr, N. (2014). *Advanced Linear Algebra.* CRC Press, Boca Raton, Florida.

Lord, N. (2005). Analysis I: convergence, elementary functions, by R. Godement. Pp. 430.£ 34.50. 2004. ISBN 3 540 05923 7 (Springer). *The Mathematical Gazette*, 89(514):152–153.

Lyche, T. (2020). *Numerical Linear Algebra and Matrix Factorizations.* Springer, Cham, Switzerland.

Lyche, T., Muntingh, G., and Ryan, Ø. (2020). *Exercises in Numerical Linear Algebra and Matrix Factorizations.* Springer, Cham, Switzerland.

MacCluer, B. (2008). *Elementary Functional Analysis.* Springer, New York City, New York.

Macdonald, A. (2011). *Linear and Geometric Algebra.* CreateSpace, Scotts Valley, California.

MacKay, D. J. (2003). *Information theory, inference and learning algorithms.* Cambridge University Press, Cambridge, the United Kingdom.

Magnus, J. R. and Neudecker, H. (2019). *Matrix Differential Calculus with Applications in Statistics and Econometrics.* John Wiley & Sons, Hoboken, New Jersey.

Magnus, R. (2020). *Fundamental Mathematical Analysis*. Springer, Cham, Switzerland.

Maligranda, L. (2012). The AM-GM inequality is equivalent to the Bernoulli inequality. *The Mathematical Intelligencer*, 34(1):1–2.

Manetti, M. (2015). *Topology*. Springer, New York City, New York.

Margalit, D. and Rabinoff, J. (2019). Interactive Linear Algebra. https://textbooks.math.gatech.edu/ila/ila.pdf.

Marrison, C. (2002). *The Fundamentals of Risk Measurement*. McGraw-hill, New York City, New York.

Mathai, A. M. and Haubold, H. J. (2017). *Linear Algebra : a Course for Physicists and Engineers*. De Gruyter, Berlin, Germany.

Matoušek, J. and Gärtner, B. (2007). *Understanding and Using Linear Programming*. Springer, Berlin.

Mazzoni, T. (2018). *A First Course in Quantitative Finance*. Cambridge University Press, Cambridge, the United Kingdom.

Meckes, E. S. and Meckes, M. W. (2018). *Linear Algebra*. Cambridge University Press, Cambridge, the United Kingdom.

Mendelson, B. (1990). *Introduction to Topology*. Dover Publications, Mineola, New York.

Meyer, C. (2010). *Matrix Analysis and Applied Linear Algebra*. SIAM: Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

Mickens, R. E. (2015). *Difference Equations: Theory, Applications and Advanced Topics*. CRC Press.

Miklavcic, S. J. (2020). *An Illustrative Guide to Multivariable and Vector Calculus*. Springer, Cham, Switzerland.

Mikosch, T. (1998). *Elementary Stochastic Calculus with Finance in View*. World Scientific, Singapore.

Mohri, M., Rostamizadeh, A., and Talwalkar, A. (2018). *Foundations of Machine Learning*. MIT Press, Cambridge, Massachusetts.

Montesinos, V., Zizler, P., and Zizler, V. (2015). *An Introduction to Modern Analysis*. Springer, Cham, Switzerland.

Munkres, J. R. (2000). *Topology*. Pearson, Harlow, Essex.

Munkres, J. R. (2018). *Analysis On Manifolds*. CRC Press, Boca Raton, Florida.

Murphy, I. S. (1982). Karl R. Stromberg, An Introduction to Classical Real Analysis (Wadsworth International Group, 1981). *Proceedings of the Edinburgh Mathematical Society*, 25(1):105–106.

Muscat, J. (2014). *Functional Analysis: An Introduction to Metric Spaces, Hilbert Spaces, and Banach Algebras*. Springer, Cham, Switzerland.

Musiela, M. and Rutkowski, M. (2005). *Martingale Methods in Financial Modelling*. Springer, Berlin Heidelberg, Germany.

Nair, M. T. (2019). *Measure and Integration: A First Course*. CRC Press, Boca Raton, Florida.

Nair, M. T. and Singh, A. (2018). *Linear Algebra.* Springer, Singapore.

Nelson, G. S. (2015). *A User-Friendly Introduction to Lebesgue Measure and Integration.* American Mathematical Society, Providence, Rhode Island.

Neri, F. (2019). *Linear Algebra for Computational Sciences and Engineering.* Springer, Cham, Switzerland.

Nicholson, W. K. (2021). *Linear Algebra with Applications.* Lyryx Learning, Calgary, Alberta.

Norman, D. and Wolczuk, D. (2020). *Introduction to Linear Algebra for Science and Engineering.* Pearson, London, the United Kingdom.

Ó Searcóid, M. (2006). *Metric Spaces.* Springer, London, the United Kingdom.

Oden, J. T. and Demkowicz, L. F. (2017). *Applied Functional Analysis.* CRC Press, Boca Raton, Florida.

Ok, E. A. (2007). *Real Analysis with Economic Applications.* Princeton University Press, Princeton, New Jersey.

Olver, P. J. and Shakiban, C. (2018). *Applied Linear Algebra.* Springer, Cham, Switzerland.

Ovchinnikov, S. (2013). *Measure, Integral, Derivative: A Course on Lebesgue's Theory.* Springer, New York City, New York.

Paolella, M. S. (2006). *Fundamental Probability: A Computational Approach.* John Wiley & Sons, Hoboken, New Jersey.

Paolella, M. S. (2007). *Intermediate Probability: A Computational Approach.* John Wiley & Sons, Hoboken, New Jersey.

Paolella, M. S. (2018). *Fundamental Statistical Inference: A Computational Approach.* John Wiley & Sons, Hoboken, New Jersey.

Paolella, M. S. (2019). *Linear Models and Time-series Analysis: Regression, ANOVA, ARMA and GARCH.* John Wiley & Sons, Hoboken, New Jersey.

Pardoux, É. (2008). *Markov Processes and Applications: Algorithms, Networks, Genome and Finance,.* John Wiley & Sons, Hoboken, New Jersey.

Petersen, P. (2012). *Linear Algebra.* Springer, New York City, New York.

Petters, A. O. and Dong, X. (2016). *An Introduction to Mathematical Finance with Applications: Understanding and Building Financial Intuition.* Springer, New York City, New York.

Piziak, R. and Odell, P. L. (2007). *Matrix Theory: From Generalized Inverses to Jordan Form.* Chapman and Hall/CRC, London, the United Kingdom.

Pollard, D. (2001). *A User's Guide of Measure Theoretic Probability.* Cambridge University Press, Cambridge, the United Kingdom.

Pólya, G. and Szegö, G. (1998a). *Problems and Theorems in Analysis II: Theory of Functions. Zeros. Polynomials. Determinants. Number Theory. Geometry.* Springer, Berlin Heidelberg, Germany.

Pólya, G. and Szegö, G. (1998b). *Problems and Theorems in Analysis I:Series. Integral Calculus. Theory of Functions.* Springer, Berlin Heidelberg, Germany.

Poole, D. (2011). *Linear Algebra: A Modern Introduction.* Cengage Learning, Boston, Massachusetts.

Pop, V. and Furdui, O. (2017). *Square Matrices of Order 2: Theory, Applications, and Problems*. Springer, Cham, Switzerland.

Potters, M. and Bouchaud, J.-P. (2021). *A First Course in Random Matrix Theory: For Physicists, Engineers and Data Scientists*. Cambridge University Press, Cambridge, the United Kingdom.

Pryce, J. (2014). *Basic Methods of Linear Functional Analysis*. Dover Publications, Mineola, New York.

Pugh, C. C. (2015). *Real Mathematical Analysis*. Springer, Cham, Switzerland.

Radin, M. A. (2019). *Difference Equations for Scientists and Engineering: Interdisciplinary Difference Equations*. World Scientific, Singapore.

Radulescu, T.-L., Radulescu, V. D., and Andreescu, T. (2009). *Problems in Real Analysis : Advanced Calculus on the Real Axis*. Springer, New York City, New York.

Ramsay, J. and Silverman, B. W. (2002). *Applied Functional Data Analysis: Methods and Case Studies*. Springer, New York City, New York.

Ramsay, J. and Silverman, B. W. (2005). *Functional Data Analysis*. Springer, New York City, New York.

Reitano, R. R. (2010). *Introduction to Quantitative Finance: A Math Tool Kit*. MIT Press, Cambridge, Massachusetts.

Ricardo, H. J. (2009). *A Modern Introduction to Linear Algebra*. CRC Press, Boca Raton, Florida, first edition.

Ricardo, H. J. (2020). *A Modern Introduction to Differential Equations*. Academic Press, San Diego, California, third edition.

Richardson, L. (2008). *Advanced Calculus: An Introduction to Linear Analysis*. John Wiley & Sons, Hoboken, New Jersey.

Roberts, A. (2020). *Linear Algebra for the 21st Century*. Oxford University Press, Oxford.

Robinson, D. J. S. (2006). *A Course in Linear Algebra with Applications*. World Scientific, New Jersey, second edition.

Robinson, J. C. (2020). *An Introduction to Functional Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Roman, S. (2007). *Advanced Linear Algebra*. Springer, New York City, New York.

Ross, K. A. (2013). *Elementary Analysis: The Theory of Calculus*. Springer, New York City, New York.

Ross, S. (2020). *A First Course in Probability*. Pearson, Harlow, Essex, tenth edition.

Ross, S. M. (2011). *An Elementary Introduction to Mathematical Finance*. Cambridge University Press, Cambridge, the United Kingdom.

Rotman, J. J. (2007). *Journey into Mathematics: An Introduction to Proofs*. Dover Publications, Mineola, New York.

Roussas, G. G. (2014). *An Introduction to Measure-Theoretic Probability*. Academic Press, San Diego, California.

Royden, H. L. (1988). *Real Analysis*. Pearson, London, the United Kingdom.

Royden, H. L. and Fitzpatrick, P. (2010). *Real Analysis*. Pearson, London, the United Kingdom.

Rudin, W. (1964). *Principles of Mathematical Analysis*. McGraw-hill, New York City, New York.

Rynne, B. P. and Youngson, M. A. (2008). *Linear Functional Analysis*. Springer, London, the United Kingdom.

Saari, D. G. (2019). *Mathematics of Finance: An Intuitive Introduction*. Springer, Cham, Switzerland.

Said Houari, B. (2017). *Linear Algebra*. Springer, Cham, Switzerland.

Salsburg, D. (2002). *The Lady Tasting Tea: How Statistics Revolutionized Science in the Twentieth Century*. W. H. Freeman and Company, New York City, New York.

Sasane, A. (2015). *The How and Why of One Variable Calculus*. John Wiley & Sons, Hoboken, New Jersey.

Sasane, A. (2017). *A Friendly Approach to Functional Analysis*. World Scientific, Singapore.

Sauer, T. (2018). *Numerical Analysis*. Pearson, Hoboken, New Jersey.

Saveliev, P. (2020). *Linear Algebra Illustrated*. Independently published.

Saxe, K. (2001). *Beginning Functional Analysis*. Springer, New York City, New York.

Schatzman, M. (2002). *Numerical Analysis: A Mathematical Introduction*. Oxford University Press, Oxford, the United Kingdom.

Schay, G. (2012). *A Concise Introduction to Linear Algebra*. Birkhäuser, Basel, Switzerland.

Schechter, M. (2002). *Principles of Functional Analysis*. American Mathematical Society, Providence, Rhode Island.

Schilling, R. L. (2017). *Measures, Integrals and Martingales*. Cambridge University Press, Cambridge, the United Kingdom.

Schilling, R. L. and Partzsch, L. (2012). *Brownian Motion : An Introduction to Stochastic Processes*. De Gruyter, Berlin, Germany.

Schneider, H. and Barker, G. P. (1973). *Matrices and Linear Algebra*. Dover, Mineola, New York, second edition. This Dover edition, first published in 1989, is an unabridged, slightly corrected republication of the second edition (1973) of the work originally published by Holt, Rinehart and Winston, Inc., New York, in 1968.

Schott, J. R. (2016). *Matrix Analysis for Statistics*. John Wiley & Sons, Hoboken, New Jersey.

Schröder, B. S. (2008). *Mathematical Analysis: A Concise Introduction*. John Wiley & Sons, Hoboken, New Jersey.

Schwartz, B. (2016). *The Paradox of Choice: Why More is Less*. Ecco, New York City, New York.

Searle, S. R. and Khuri, A. I. (2017). *Matrix Algebra Useful for Statistics*. John Wiley & Sons, Hoboken, New Jersey.

Selinger, P. (2020). Matrix Theory and Linear Algebra. `https://www.mathstat.dal.ca/~selinger/linear-algebra/downloads/LinearAlgebra.pdf`.

Severini, T. A. (2011). *Elements of Distribution Theory.* Cambridge University Press, Cambridge, the United Kingdom.

Severini, T. A. (2018). *Introduction to Statistical Methods for Financial Models.* CRC Press, Boca Raton, Florida.

Sewell, G. (2014). *Computational Methods of Linear Algebra.* World Scientific, Hackensack, New Jersey.

Shakarchi, R. (1998). *Problems and Solutions for Undergraduate Analysis.* Springer, New York.

Shalev-Shwartz, S. and Ben-David, S. (2014). *Understanding Machine Learning: From Theory to Algorithms.* Cambridge University Press, New York City, New York.

Shapiro, H. (2015). *Linear Algebra and Matrices: Topics for a Second Course.* American Mathematical Society, Providence, Rhode Island.

Shen, L.-Y., Wang, H., and Wojdylo, J. (2019). *Linear Algebra.* Mercury Learning and Information, Dulles, Virginia.

Shick, P. L. (2007). *Topology: Point-Set and Geometric.* John Wiley & Sons, Hoboken, New Jersey.

Shifrin, T. (2005). *Multivariable Mathematics: Linear Algebra, Multivariable, Calculus, and Manifolds.* John Wiley & Sons, Hoboken, New Jersey.

Shifrin, T. and Adams, M. R. (2011). *Linear Algebra: A Geometric Approach.* W. H. Freeman and Company, New York City, New York.

Shirali, S. (2018). *A Concise Introduction to Measure Theory.* Springer, Cham, Switzerland.

Shirali, S. and Vasudeva, H. (2006). *Metric Spaces.* Springer, London, the United Kingdom.

Shirali, S. and Vasudeva, H. L. (2010). *Multivariable Analysis.* Springer, London, the United Kingdom.

Shirali, S. and Vasudeva, H. L. (2019). *Measure and Integration.* Springer, Cham, Switzerland.

Shiryaev, A. N. (2016). *Probability-1.* Springer, New York City, New York.

Shmueli, G. (2010). To Explain or to Predict? *Statistical science*, 25(3):289–310.

Shorack, G. R. (2017). *Probability for Statisticians.* Springer, Cham, Switzerland.

Shores, T. S. (2018). *Applied Linear Algebra and Matrix Analysis.* Springer, New York City, New York.

Shreve, S. (2005). *Stochastic Calculus for Finance I: The Binomial Asset Pricing Model.* Springer, New York City, New York.

Shreve, S. E. (2010). *Stochastic Calculus for Finance II: Continuous-Time Models.* Springer, New York City, New York.

Shurman, J. (2019). *Calculus and Analysis in Euclidean Space.* Springer, Cham, Switzerland.

Simmons, G. F. (1963). *Introduction to Topology and Modern Analysis.* McGraw-hill, New York City, New York.

Singh, T. B. (2019). *Introduction to Topology.* Springer, Singapore.

Sjamaar, R. (2017). Manifolds and Differential forms. http://pi.math.cornell.edu/~sjamaar/manifolds/manifold.pdf.

Stahl, S. (1999). *Real Analysis: A Historical Approach*. John Wiley & Sons, Hoboken, New Jersey, first edition.

Stahl, S. (2012). *Real Analysis: A Historical Approach*. John Wiley & Sons, Hoboken, New Jersey, second edition.

Steele, J. M. (2004). *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge University Press, Cambridge, the United Kingdom.

Stefanica, D. (2011). *A Primer for the Mathematics of Financial Engineering*. FE Press, New York City, New York.

Stefanica, D. (2014). *A Linear Algebra Primer for Financial Engineering: Covariance Matrices, Eigenvectors, OLS, and more*. FE Press, New York City, New York.

Stein, Elias, M. and Shakarchi, R. (2005). *Real Analysis: Measure Theory, Integration and Hilbert Spaces*. Princeton University Press, Princeton, New Jersey.

Stewart, M. (2019). The Actual Difference Between Statistics and Machine Learning. *Towards Data Science*.

Stillwell, J. (2013). *The Real Numbers: An Introduction to Set Theory and Analysis*. Springer, Cham, Switzerland.

Stirzaker, D. (2003). *Elementary Probability*. Cambridge University Press, Cambridge, the United Kingdom.

Stoer, J. and Bulirsch, R. (2004). *Introduction to Numerical Analysis*. Springer, New York City, New York.

Stoll, M. (2001). *Introduction to Real Analysis*. Pearson, Boston, Massachusetts.

Stoll, R. R. (1952). *Linear Algebra and Matrix Theory*. Dover, New York. This Dover edition, first published in 1969, is an unabridged and slightly corrected republication of the work originally published in 1952 by the McGraw-Hill Book Company, Inc.

Stoll, R. R. and Wong, E. T. (1968). *Linear Algebra*. Academic Press, New York.

Strang, G. (1986). *Introduction to Applied Mathematics*. Wellesley-Cambridge Press, Wellesley, Massachusetts.

Strang, G. (1988). *Linear Algebra and Its Applications*. Harcourt Brace Jovanovich, San Diego, California.

Strang, G. (2015). *Differential Equations and Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts.

Strang, G. (2016). *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, Massachusetts.

Strang, G. (2019). *Linear Algebra and Learning from Data*. Wellesley-Cambridge Press, Wellesley, Massachusetts.

Strichartz, R. (2000). *The Way of Analysis*. Jones and Bartlett Publishers, Sudbury, Massachusetts.

Ström, J., Åström, K., and Akenine-Möller, T. (2017). Immersive Linear Algebra. http://immersivemath.com/ila/index.html.

Stromberg, K. R. (2015). *An Introduction to Classical Real Analysis*. American Mathematical Society, Providence, Rhode Island.

Süli, E. and Mayers, D. F. (2003). *An Introduction to Numerical Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Sutherland, W. A. (2009). *Introduction to Metric and Topological Spaces*. Oxford University Press, Oxford, the United Kingdom.

Swartz, C. (2001). *Introduction to Gauge Integrals*. World Scientific, Singapore.

Swartz, C. (2009). *Elementary Functional Analysis*. World Scientific, Singapore.

Tao, T. (2011). *An Introduction to Measure Theory*. American Mathematical Society, Providence, Rhode Island.

Tao, T. (2015a). *Analysis I*. Springer, Singapore.

Tao, T. (2015b). *Analysis II*. Springer, Singapore.

Taylor, J. L. (2012). *Foundations of Analysis*. American Mathematical Society, Providence, Rhode Island.

Taylor, M. E. (2020a). *Introduction to Analysis in One Variable*. American Mathematical Society, Providence, Rhode Island.

Taylor, M. E. (2020b). *Introduction to Analysis in Several Variables: Advanced Calculus*. American Mathematical Society, Providence, Rhode Island.

Taylor, M. E. (2020c). *Linear Algebra*. American Mathematical Society, Providence, Rhode Island.

Tegmark, M. (2014). *Our Mathematical Universe: My Quest for the Ultimate Nature of Reality*. Penguin Random House, New York City, New York.

Tegmark, M. (2017). *Life 3.0: Being human in the age of artificial intelligence*. Penguin Random House, New York City, New York.

Terrell, W. J. (2019). *A Passage to Modern Analysis*. American Mathematical Society, Providence, Rhode Island.

Thorp, E. O. (2011). The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market. In *The Kelly capital growth investment criterion: theory and practice*, pages 789–832. World Scientific, Singapore.

Tijms, H. (2018). *Probability: A Lively Introduction*. Cambridge University Press, Cambridge, the United Kingdom.

Torchinsky, A. (1983). An Introduction to Classical Real Analysis. By Karl R. Stromberg. *The American Mathematical Monthly*, 90(4):294–295.

Trefethen, L. N. and Bau III, D. (1997). *Numerical Linear Algebra*. SIAM: Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

Treil, S. (2017). Linear Algebra Done Wrong. http://www.math.brown.edu/streil/papers/LADW/LADW.html.

Trench, W. F. (2003). *Introduction to Real Analysis*. Pearson, Boston, Massachusetts.

Turkington, D. A. (2013). *Generalized Vectorization, Cross-products, and Matrix Calculus*. Cambridge University Press, Cambridge, the United Kingdom.

Valenza, R. J. (2012). *Linear Algebra: An Introduction to Abstract Mathematics*. Springer, New York City, New York.

Venkatesh, S. S. (2013). *The Theory of Probability: Explorations and Applications*. Cambridge University Press, Cambridge, the United Kingdom.

Vershynin, R. (2018). *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge University Press, New York City, New York.

Vestrup, E. M. (2003). *The Theory of Measures and Integration*. John Wiley & Sons, Hoboken, New Jersey.

Wade, W. R. (2009). *Introduction to Analysis*. Pearson, London, the United Kingdom.

Wainwright, M. J. (2019). *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, Cambridge, the United Kingdom.

Wasserman, L. (2012). Statistics Versus Machine Learning. https://normaldeviate.wordpress.com/2012/06/12/statistics-versus-machine-learning-5-2/.

Weintraub, S. H. (2011). *A Guide to Advanced Linear Algebra*. Mathematical Association of America, Washington, D.C.

Weintraub, S. H. (2019). *Linear Algebra for the Young Mathematician*. American Mathematical Society, Providence, Rhode Island.

Wendland, H. (2018). *Numerical Linear Algebra: An Introduction*. Cambridge University Press, Cambridge, the United Kingdom.

Wheeden, R. L. and Zygmund, A. (2015). *Measure and Integral: An Introduction to Real Analysis*. CRC Press, Boca Raton, Florida.

Whittaker, E. T. and Watson, G. N. (1996). *A Course of Modern Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Wiersema, U. F. (2008). *Brownian Motion Calculus*. John Wiley & Sons, Hoboken, New Jersey.

Williams, G. (2018). *Linear Algebra with Applications*. Jones & Bartlett Learning, Burlington, Massachusetts, 9th edition.

Woerdeman, H. (2016). *Advanced Linear Algebra*. CRC Press, Boca Raton, Florida.

Woerdeman, H. J. (2021). *Linear Algebra: What You Need to Know*. CRC Press, Boca Raton, Florida.

Yamamoto, Y. (2012). *From Vector Spaces to Function Spaces: Introduction to Functional Analysis with Applications*. SIAM: Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania.

Yanai, H., Takeuchi, K., and Takane, Y. (2007). *Projection Matrices, Generalized Inverse Matrices, and Singular Value Decomposition*. Springer, New York City, New York.

Yang, Y. (2015). *A Concise Text on Advanced Linear Algebra*. Cambridge University Press, Cambridge, the United Kingdom.

Yao, J., Zheng, S., and Bai, Z. (2015). *Large Sample Covariance Matrices and High-Dimensional Data Analysis*. Cambridge University Press, Cambridge, the United Kingdom.

Yee, L. P. and Vỳborný, R. (2000). *Integral: An Easy Approach After Kurzweil and Henstock*. Cambridge University Press, Cambridge, the United Kingdom.

Yeh, J. (2006). *Real Analysis: Theory of Measure and Integration Second Edition*. World Scientific, Singapore.

Zhan, X. (2013). *Matrix Theory.* American Mathematical Society, Providence, Rhode Island.

Zhang, F. (2009). *Linear Algebra: Challenging Problems for Students.* The Johns Hopkins University Press, Baltimore, Maryland, second edition.

Zhang, F. (2011). *Matrix theory: Basic Results and Techniques.* Springer, New York, second edition.

Zorich, V. A. (2015). *Mathematical Analysis I.* Springer, Berlin Heidelberg, Germany.

Zorich, V. A. (2016). *Mathematical Analysis II.* Springer, Berlin Heidelberg, Germany.