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Book reviews

In Issue 35(1) of the *Gazette*, N.J. Wildberger reviewed *A course in calculus and real analysis*. Unfortunately, due to a technical problem, the review was not published in full. The editors sincerely apologise to Professor Wildberger for this error. Below is the full review as it should have appeared in 35(1).

A course in calculus and real analysis

S.R. Ghorpade and B.V. Limaye
Springer, 2006, ISBN 978-0-387-30530-1

The important new text *A Course in Calculus and Real Analysis* by S.R. Ghorpade and B.V. Limaye (Springer UTM) is a rigorous, well-presented and original introduction to the core of undergraduate mathematics — *first-year calculus*. It develops this subject carefully from a foundation of high-school algebra, with interesting improvements and insights rarely found in other books. Its intended audience includes mathematics majors who have already taken some calculus, and now wish to understand the subject more carefully and deeply, as well as those who teach calculus at any level. Because of the high standard, only very motivated and capable students can expect to learn the subject for the first time using this text, which is comparable to Spivak's *Calculus* [1], or perhaps Rudin's *Principles of Mathematical Analysis* [2].

The book strives to be precise yet informative at all times, even in traditional 'hand waving' areas, and strikes a good balance between theory and applications for a mathematics major. It has a voluminous and interesting collection of exercises, conveniently divided into two groups. The first group is more routine, but still often challenging, and the second group is more theoretical and adds considerable detail to the coverage. However no solutions are presented. There are a goodly number of figures, and each chapter has an informative Notes and Comments section that makes historical points or otherwise illuminates the material.

The authors based the book on some earlier printed teaching notes and then spent seven years putting it all together. The extensive attention to detail shows, and an honest comparison with the calculus notes generally used in Australian universities would be a humbling exercise. Here is a brief indication of the contents of the book by chapter.

Chapter 1 (Numbers and Functions) introduces integers, rational numbers and the basic properties of real numbers, without defining exactly what real numbers are — this difficulty seems unavoidable in an elementary text. Inequalities and basic facts about functions are introduced, and the main examples are *polynomials* and their quotients, the *rational functions*. This book is notable for not using the log, exponential and circular functions until they are properly defined: these appear roughly half-way through the book.

Boundedness, *convexity* and *local extrema* of functions are defined, and a function $f(x)$ is said to have the *intermediate value property* (IVP) if r between $f(a)$ and

$f(b)$ implies $r = f(x)$ for some x in $[a,b]$. The geometric nature of these notions is thus brought to the fore, before the corresponding criteria for them in terms of continuity and differentiability are introduced.

Chapter 2 (Sequences) introduces the basic idea of convergence used in the text: a sequence a_n of real numbers *converges to* a if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq n_0$. Convergence of bounded monotonic sequences is shown, and the *little-oh* and *big-oh* notations are briefly introduced. Every real sequence is shown to have a monotonic subsequence, and from this follows the Bolzano–Weierstrass theorem that every bounded sequence has a convergent subsequence. Convergent sequences are shown to be Cauchy and conversely.

In Chapter 3 (Continuity and Limits), the book strikes out into less-charted waters. A function $f(x)$ defined on a domain D, which is allowed to be an arbitrary subset of $\mathbb{R}$, is *continuous at* $c \in D$ if for any sequence x_n in D converging to c, $f(x_n)$ converges to $f(c)$. So continuity, defined in terms of sequences, occurs before the notion of the limit of a function, and works for more general domains than the usual setup. This approach is probably more intuitive for students, and has been used in other texts, for example, Goffman's *Introduction to Real Analysis* [3].

The usual $\varepsilon - \delta$ formulation is shown to be equivalent to the above condition. A strictly monotonic function f defined on an interval I has an inverse function f^{-1} defined on $f(I)$ which is continuous. A continuous function defined on an interval is shown to have the IVP. Uniform continuity is also defined in terms of sequences: if f is defined on D, then it is *uniformly continuous* on D if x_n and y_n sequences in D with $x_n - y_n \to 0$ implies that $f(x_n) - f(y_n) \to 0$. If D is closed and bounded then a continuous function on D is shown to be uniformly continuous on D. If D is a set which contains open intervals around c, then a function f defined on D has *limit* l as x approaches c if for any sequence x_n in $D\backslash\{c\}$ converging to c, the sequence $f(x_n)$ converges to l. So again the notion of the limit of a function comes down to the concept of limits of sequences. The usual $\varepsilon - \delta$ formulation is shown to be equivalent to this definition. Relative notions of little-oh and big-oh between two functions as $x \to \infty$ are introduced, and more generally there is a careful discussion of infinite limits of functions and *asymptotes*, including oblique asymptotes.

Chapter 4 (Differentiation) introduces the *derivative* of a function $f(x)$ at a point c as the usual limit of a quotient. Then it proves the Lemma of Carathéodory, which becomes crucial in what follows: that f is differentiable at c if and only if there is an *increment function* f_1 such that $f(x) - f(c) = (x - c)f_1(x)$ for all x in the domain D of f, and f_1 is continuous at c. By replacing differentiability of f at c with the continuity of f_1 (which depends on c) at c, routine properties of derivatives — continuity, sums, products, quotients and especially the Chain rule and the derivative of an inverse function — have more direct proofs which no longer require mention of limits.

There is then a discussion of normals and implicit differentiation and the *mean value theorem* (MVT), which is used to prove Taylor's theorem: we can express an n-times differentiable function $f(x)$ on $[a,b]$ by an n degree *Taylor polynomial* with an error term involving the $(n+1)$th derivative at some interior point c.

The connection between derivatives and monotonicity, convexity and concavity are discussed, and the chapter ends with an unusually careful and thorough treatment of L'Hôpital's rule, treating both $0/0$ and ∞/∞ forms with some care.

Chapter 5 (Applications of Differentiation) begins with a discussion of *maxima* and *minima*, *local extrema* and *inflexion points.* Then the *linear* and *quadratic approximations* to a function f at a point c given by Taylor's theorem are studied in more detail, including explicit bounds on the errors as x approaches c. The most novel parts of this chapter are a thorough treatment of *Picard's method* for finding a fixed point of a function $f\colon [a,b] \to [a,b]$ provided $|f'(x)| < 1$, and *Newton's method* for finding the zeros of a function $f(x)$. Conditions are given that ensure that the latter converges, one such condition uses Picard's method, the other assumes the monotonicity of $f'(x)$.

Chapter 6 (Integration) is the heart of the subject. Many calculus texts introduce the integral of a function as some kind of 'limit of Riemann sums', even though this kind of limit has not been defined, as it ranges over a net of partitions, not a set of numbers. Ghorpade and Limaye choose another standard approach: to define the *Riemann integral* using supremums and infimums of sets of real numbers. Given $f(x)$ on $[a,b]$, they define the lower sum $L(P,f)$ and the upper sum $U(P,f)$ of f with respect to a partition P of $[a,b]$ in terms of minima and maxima of f on the various subintervals, then set

$$L(f) \equiv \sup\{L(P,F)\colon P \text{ is a partition of } [a,b]\}$$
$$U(f) \equiv \inf\{U(P,F)\colon P \text{ is a partition of } [a,b]\}$$

and declare f to be *integrable* on $[a,b]$ if $L(f) = U(f)$, in which case this common value is the definite integral $\int_a^b f(x)\,\mathrm{d}x$. This is a definition which is reasonably intuitive, and respects Archimedes' understanding that one ought to estimate an area from both the inside and outside to get proper control of it. Nevertheless one must make the point that no good examples of using this definition to compute an integral are given — while the book shows that $f(x) = x^n$ is integrable, an evaluation of the integral must wait for the Fundamental theorem.

A key technical tool is the *Riemann condition*: that a bounded function is integrable if and only if for any $\varepsilon > 0$ there is a partition P for which the difference between the lower and upper sums is less than ε. The *Fundamental theorem* is established, in both forms: that the integral of a function may be found by evaluating an antiderivative, and that the indefinite integral of a function f is differentiable and has derivative f. Integration by parts and substitution are derived, and then the idea of a Riemann sum is introduced both as a tool to evaluate integrals, and to allow integration theory to evaluate certain series.

Chapter 7 (Elementary Transcendental Functions) introduces the *logarithm*, the *exponential function*, and the *circular functions* and their *inverses.* The book defines $\ln x$ as the integral of $1/x$ and the exponential function as its inverse, and develops more general power functions using the exponential function and the log function. The number e is defined by the condition $\ln e = 1$. This is familiar territory. Defining $\sin x$, $\cos x$ and $\tan x$ is less familiar, but a crucial point for calculus. Most texts are sadly lacking, pretending that these functions are somehow part of the background 'ether' of mathematical understanding, and so exempt from requiring proper definitions. More than fifty years ago, G.H. Hardy spelled out the problem quite clearly in his *A Course in Pure Mathematics* [4], stating 'The whole difficulty lies in the question, what is the x which occurs in $\cos x$ and $\sin x$'. He described four different approaches to the definition of the circular functions.

The one taken by Ghorpade and Limaye is to start with an *inverse circular function*. There are several good reasons to justify this choice. Historically the inverse circular functions were understood analytically before the circular functions themselves; Newton obtained the power series for $\sin x$ by first finding the power series for $\arcsin x$ and then inverting it, and indeed the $\arcsin x$ series was discovered several centuries earlier by Indian mathematicians in Kerala. In addition, the theory of elliptic functions is arguably easier to understand if it proceeds by analogy with the circular functions, and starts with the inverse functions — the elliptic integrals.

The book begins with $\arctan x$, the integral of $1/(1+x^2)$ which, after $1/x$, is the last serious barrier to integrating general rational functions. Defining $\tan x$ as the inverse of $\arctan x$ only defines it in the range $(-\pi/2, \pi/2)$, where π is introduced as twice the supremum of the values of $\int_0^a 1/(1+x^2)\,\mathrm{d}x$. Then the circular functions

$$\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}} \quad \text{and} \quad \cos x = \frac{1}{\sqrt{1+\tan^2 x}}$$

are defined on $(-\pi/2, \pi/2)$ as suggested by Hardy, extended by continuity to the closed interval, and then to all of $\mathbb{R}$ by the rules

$$\sin(x+\pi) = -\sin x \quad \text{and} \quad \cos(x+\pi) = -\cos x.$$

I would suggest an alternative: to define

$$\sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)} \quad \text{and} \quad \cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$$

on $(-\pi, \pi)$ and then extend by continuity and 2π periodicity. This more algebraic approach connects to the rational parametrisation of the circle, Pythagorean triples, and the well-known half-angle substitution.

After $\sin x$ and $\cos x$ are pinned down, the book finds their nth Taylor polynomials and derives the main algebraic relations, namely $\cos^2 x + \sin^2 x = 1$ and the addition laws. After defining the *reciprocal* and the (other) *inverse circular functions* the book establishes their derivatives. It then discusses a good source of counterexamples: the function $\sin(1/x)$.

Having defined the circular functions precisely, the authors define the *polar coordinates* r and θ of a point $(x, y) \neq (0, 0)$ in the Cartesian plane precisely: $r = \sqrt{x^2+y^2}$ as usual, while

$$\theta = \begin{cases} \cos^{-1}\left(\dfrac{x}{r}\right) & \text{if } y \geq 0 \\ -\cos^{-1}\left(\dfrac{x}{r}\right) & \text{if } y < 0. \end{cases}$$

This insures that θ lies in $(-\pi, \pi]$, and is single valued.

The book augments the discussion of polar coordinates by giving precise definitions of an *angle* φ in various contexts. An angle is not some God-given notion bestowed on each of us at birth. As Euclid well realised, it is rather a problematic concept, and must be correctly defined. The essential formula, here shortened by using linear algebra language, is that the angle between vectors u and v is

$$\varphi = \cos^{-1} \frac{u \cdot v}{|u||v|} \,.$$

Having gone through considerable care in defining the elementary functions $\ln x$, $\exp x$, $\sin x$ and $\cos x$, Ghorpade and Limaye then devote a section to proving these really are *transcendental functions*; that is, they are not algebraic functions. This is a very nice section that would capture the interest of many students.

The exercises at the end of this chapter establish many of the usual identities for the elementary transcendental functions. There is also a separate set of revision exercises that revisit the topics of the previous chapters with the help of the new functions now at our disposal.

Chapter 8 (Applications and Approximations of Riemann Integrals) looks at defining *areas* of regions, *volumes* of solids, *arclengths* of curves and *centroids* of regions, as well as establishing *quadrature rules*. The area of a region bounded by two functions $f_1(x) < f_2(x)$ on $[a, b]$ is defined to be $\int_a^b (f_2(x) - f_1(x))\,\mathrm{d}x$. While this corresponds to our intuition, it still remains to be proven that this notion of area behaves as we expect, in particular that it is invariant under isometries of the plane, and that this notion of area of regions is additive. For a pleasant introduction to area, see another novel book in the UTM series: Hijab's *Introduction to Calculus and Classical Analysis*.

The authors establish that the area of a circle is indeed πr^2 and rightfully point out that the dependence on r is proven, not assumed as it is in most high-school treatments of area. Assuming additivity, they develop formulas for area in polar coordinates, and there is a section on volumes, involving slicing, decomposition by cylinders and solids of revolution.

The subject of arclength is often a stumbling block in traditional texts. The authors motivate the usual definition

$$L = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t$$

by discussing tangent lines and the linear approximation to a curve, and show that L is indeed independent of parametrisation. They also prove the formula for the arclength of an arc of a circle in terms of the notion of angle which they have established in the previous section. Again they observe that the linear dependence of the circumference of a circle on its radius is thus a result, not an assumption. While they do make some computations of arclengths, for example that of the *cardioid* and *helix*, they point out that for an ellipse and a lemniscate the integrals are not easily evaluated. Many texts present artificial examples of computations of arclengths of curves, without making clear to the student how atypical such computations are. Most curves have arclengths about which we know very little.

A section on areas of surfaces features *Pappus' theorem*. The section on quadrature rules gives the *trapezoidal* and *Simpson rules* for estimating integrals, and has a detailed discussion of error estimates.

The last chapter (Chapter 9: Infinite Series and Improper Integrals) follows Apostal's *Mathematical Analysis* [5] in defining an *infinite series* and presents the familiar tests for the convergence of series and *power series*, with more advanced ones such as Abel's test relegated to the exercises. *Taylor series* for the common functions are given, and *improper integrals* are also given a thorough treatment which mirrors that of infinite series. The chapter ends with a brief introduction to the *beta* and *gamma functions*.

This book is a tour de force, and a necessary addition to the library of anyone involved in teaching calculus, or studying it seriously. It also raises some interesting questions. How easy is it for us to change our courses if it becomes clear that substantial improvements can be made? Are we aware that most of the top 100 universities in the world rely much more extensively on high quality textbooks written by dedicated and knowledgeable authors? And where in Australia is that elite first-year mathematics course, where students *learn calculus rigorously*?

References

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