Additions to A Course in Multivariable Calculus and Analysis (Corrected Publication)

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The "additions" listed below may be incorporated if and when a new edition of *A Course in Multivariable Calculus and Analysis* is brought out. This is a work in progress and the file will be updated as and when a new version is ready. Comments and suggestions are welcome and may be communicated to the authors by e-mail. In the following, **p. i, +j** means the jth line from the top on page i, whereas **p. i, −j** means the jth line from the bottom on page i. Also, ACICARA 2Ed will stand for the authors' book *A Course in Calculus and Real Analysis*, Second edition, Springer, New York, 2018.

Chapter 1

p. 4, +7

Two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are said to be **orthogonal** (to each other) if $\mathbf{x} \cdot \mathbf{y} = 0$. In case both $\mathbf{x}$ and $\mathbf{y}$ are nonzero, this means that the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\pi/2$. Given any $D \subseteq \mathbb{R}^n$, a vector $\mathbf{x} \in \mathbb{R}^n$ is said to be **orthogonal** to D if $\mathbf{x} \cdot \mathbf{y} = 0$ for every $\mathbf{y} \in D$.

p. 9, +7

We say that Γ is a **closed path** if the initial point of Γ and the terminal point of Γ are the same, that is, if $\mathbf{c} = \mathbf{d}$.

p. 10, −8

Let $D \subseteq \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a function. Given a subset C of D, the **restriction** of f to C is the function $f_{|C} : C \to \mathbb{R}$ defined by $f_{|C}(x, y) = f(x, y)$ for $(x, y) \in C$.

p. 11, before Fig. 1.3

Note that the level curve of f corresponding to c is the projection of the contour line of f corresponding to c on the plane given by $z = 0$.

p. 26, −4

If Γ is a closed path, that is, if $(x(\alpha), y(\alpha)) = (x(\beta), y(\beta))$, then we say that Γ passes through $(x(\alpha), y(\alpha))$, and further, we say that a tangent to Γ at the point $(x(\alpha), y(\alpha))$ exists if x and y are differentiable at α and at β, and if $(x'(\alpha), y'(\alpha)) = (x'(\beta), y'(\beta)) \neq (0, 0)$.

p. 27, +1 to +3

Replace the sentence "In general, . . . (α, β)." by "In general, we say that Γ is a **regular path** if a tangent to Γ is defined at every point through which Γ passes."

p. 27, +11

We say that f has a **local extremum at** (x_0, y_0) **along** Γ if f has a local maximum at (x_0, y_0) along Γ or if f has a local minimum at (x_0, y_0) along Γ.

p. 28, +5

a **local extremum at** (x_0, y_0) if f has a local maximum at (x_0, y_0) or if f has a local minimum at (x_0, y_0)..

Chapter 2

p. 45, +2

The result in part (iii) of Fact 2.3 can be improved as follows. If $a_n \to a$ and $a \neq 0$, then there is $m \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq m$ and $(1/a_n) \to 1/a$. (See Part (iv) of Proposition 2.3 in ACICARA 2Ed.)

p. 46, −1

Proposition 2.7 can be improved as follows: Let $D \subseteq \mathbb{R}^2$. Then $\overline{D} = D \cup \partial D$. Consequently, D is closed if and only if $\partial D \subseteq D$.

p. 65, +10

Remark. In the notation of Proposition 2.40 (Implicit Function Theorem), the solution set $\{(x, y) \in \mathbb{S}_r(x_0, y_0) : x \in (x_0-\delta, x_0+\delta) \text{ and } f(x, y) = 0\}$ of the equation $f(x, y) = 0$ is the graph $\{(x, \eta(x)) : x \in (x_0 - \delta, x_0 + \delta)\}$ of the function given by $y = \eta(x)$. This follows by noting that given $x \in (x_0 - \delta, x_0 + \delta)$, the function $y \longmapsto f(x, y)$ is strictly monotonic and hence one-one on $(y_0 - r, y_0 + r)$.

p. 67, +13

Remark. In the notation of Proposition 2.46 (Trivariate Implicit Function Theorem), the solution set $\{(x, y, z) \in \mathbb{S}_r(x_0, y_0, z_0) : (x, y) \in \mathbb{S}_\delta(x_0, y_0) \text{ and } f(x, y, z) = 0\}$ of the equation $f(x, y, z) = 0$ is the graph $\{(x, y, \zeta(x, y)) : (x, y) \in \mathbb{S}_\delta(x_0, y_0)\}$ of the function given by $z = \zeta(x, y)$. This follows by noting that given $(x, y) \in \mathbb{S}_\delta(x_0, y_0)$, the function $z \longmapsto f(x, y, z)$ is strictly monotonic, and so one-one on $(z_0 - r, z_0 + r)$.

pp. 68-69

The result in Corollary 2.49 can be proved independently, and Proposition 2.48 can be deduced from it.

p. 77, after Exercise 1

New Exercise: If $(x_n, y_n) \to (x_0, y_0)$, then show that $\big(\max\{x_n, y_n\}, \min\{x_n, y_n\}\big)$ converges to $\big(\max\{x_0, y_0\}, \min\{x_0, y_0\}\big)$.

p. 77, after Exercise 4

New Exercise: Show that if (x_0, y_0) is a cluster point of $\big((x_n, y_n)\big)$, then x_0 is a cluster point of (x_n) and y_0 is a cluster point of (y_n), but the converse does not hold.

p. 78, after Exercise 9

New Exercise: Let $D \subseteq \mathbb{R}^2$, and let f and g be real-valued continuous functions on D. Show that the real-valued functions $\max\{f, g\}$ and $\min\{f, g\}$ are also continuous on D.

Chapter 3

p. 87, after Fact 3.2 (MVT)

Remark. Fact 3.2 shows that if $a, b \in \mathbb{R}$ with $a < b$, and $f : (a, b) \to \mathbb{R}$ is any function, then f is a constant function on (a, b) if and only if f' exists and is identically zero on (a, b). (See Corollary 4.23 of ACICARA 2Ed.) Now let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $D := (a, b)\times(c, d)$. Suppose $f : D \to \mathbb{R}$ is any function. Then f is a constant function on D if and only if ∇f exists and is identically zero on D. To see this, note that the 'only if' part is obvious. To prove the 'if' part, assume that ∇f exists and is identically zero on D. Let $(x_1, y_1), (x_2, y_2) \in D$. Then there is $x_0 \in (x_1, x_2)$ such that $f(x_2, y_1) - f(x_1, y_1) = f_x(x_0, y_1)(x_2 - x_1)$ and there is $y_0 \in (y_1, y_2)$ such that $f(x_2, y_2) - f(x_2, y_1) = f_y(x_2, y_0)(y_2 - y_1)$. Since $\nabla f(x_2, y_0) = 0 = \nabla f(x_0, y_1)$, we obtain $f_y(x_2, y_0) = 0 = f_x(x_0, y_1)$, and hence $f(x_2, y_2) = f(x_2, y_1) = f(x_1, y_1)$. Thus f is constant on D.

Note that the key idea in the above proof is that any two points in $(x_1, y_1), (x_2, y_2)$ in $(a, b)\times(c, d)$ can be joined by a horizontal and a vertical line segment lying in $(a, b)\times(c, d)$, namely, a horizontal line segment from (x_1, y_1) to (x_2, y_1) followed by a vertical line segment from (x_2, y_1) to (x_2, y_2). In general, suppose $D \subseteq \mathbb{R}^2$ is nonempty, open and path-connected. Then any two points of D can be joined by a finite number of horizontal and vertical line segments (which are themselves joined end to end) lying in D. To see this, fix $(u_0, v_0) \in D$. Let D_0 be the set of all $(x, y) \in D$ which can be joined to (u_0, v_0) by a finite number of horizontal and vertical line segments lying in D, and let $D_1 := D \setminus D_0$. Consider $\phi : D \to \mathbb{R}$ defined by $\phi(x, y) := 0$ if $(x, y) \in D_0$ and $\phi(x, y) := 1$ if $(x, y) \in D_1$. We claim that ϕ is a continuous function. To prove the claim, let $(x_0, y_0) \in D$. Since D is open, there is $r > 0$ such that $\mathbb{S}_r(x_0, y_0) \subseteq D$. Clearly, every element of $\mathbb{S}_r(x_0, y_0)$ can be joined to (x_0, y_0) by a horizontal and a vertical line segment lying in $\mathbb{S}_r(x_0, y_0)$ and hence in D. It follows that if $(x_0, y_0) \in D_0$, then $\mathbb{S}_r(x_0, y_0) \subseteq D_0$, whereas if $(x_0, y_0) \in D_1$, then $\mathbb{S}_r(x_0, y_0) \subseteq D_1$. Let $\big((x_n, y_n)\big)$ be a sequence in D such that $(x_n, y_n) \to (x_0, y_0)$. Then there is $n_0 \in \mathbb{N}$ such that $(x_n, y_n) \in \mathbb{S}_r(x_0, y_0)$ for all $n \geq n_0$. Consequently, $\phi(x_n, y_n) = 0$ for all $n \geq n_0$ in case $(x, y) \in D_0$, whereas $\phi(x_n, y_n) = 1$ for all $n \geq n_0$ in case $(x_0, y_0) \in D_1$. In any case, we see that $\phi(x_n, y_n) \to \phi(x, y)$, and so the claim is proved. Assume for a moment that $D_0 \neq D$. Then there is $(u_1, v_1) \in D_1$, and since D is path-connected, there is a (continuous) path $(x(t), y(t))$, $t \in [\alpha, \beta]$, joining (u_0, v_0) to (u_1, v_1) and lying in D. Now by part (ii) of Proposition 2.17, the function $F : [\alpha, \beta] \to \mathbb{R}$ given by $F(t) := \phi\left(x(t), y(t)\right)$, $t \in [\alpha, \beta]$, is continuous. Moreover, $F(\alpha) = 0$ and $F(\beta) = 1$, but there is no $t \in [\alpha, \beta]$ with $F(t) = 1/2$. This contradicts the IVP (Proposition 3.16 in ACICARA 2Ed). Thus $D_0 = D$, and so any two points of D can be joined by a finite number of horizontal and vertical line segments lying in D. As a consequence, if $D \subseteq \mathbb{R}^2$ is nonempty, open and path-connected, and if $f : D \to \mathbb{R}$ is any function, then f is a constant function on D if and only if ∇f exists and is identically zero on D.

If $D \subseteq \mathbb{R}^2$ is not path-connected, then there may exist a nonconstant function on D whose gradient vanishes identically on D. For example, if $D := \mathbb{S}_1(0,0) \cup \mathbb{S}_1(2,2)$ is a disjoint union of two open squares and $f : D \to \mathbb{R}$ is defined by $f(x,y) := 1$ if $x \in \mathbb{S}_1(0,0)$ and $f(x,y) := 2$ if $x \in \mathbb{S}_1(2,2)$, then clearly D is nonempty and open, and $f_x = f_y = 0$ on D, but f is not a constant function.

p. 88, before Examples 3.4

With notations as above, if we let $F : D_0 \to \mathbb{R}$ be the univariate function defined by $F(t) := f(x_0 + tu_1, y_0 + tu_2)$ for $t \in D_0$, then it is clear that

$$f \text{ has a directional derivative at } (x_0, y_0) \text{ along } \mathbf{u} \iff F \text{ is differentiable at } 0.$$

Moreover, in this case, $\mathbf{D_u} f(x_0, y_0) = F'(0)$. Consequently, the Carathéodory Lemma (Proposition 4.2 of ACICARA 2Ed) applied to F yields the following.

Carathéodory Lemma for Directional Derivatives: *Let $D \subseteq \mathbb{R}^2$ and let $(x_0, y_0) \in D$. Suppose $\mathbf{u} := (u_1, u_2)$ is a unit vector in $\mathbb{R}^2$ such that D contains a segment of the line passing through (x_0, y_0) in the direction of $\mathbf{u}$. Let $E := \{(x_0 + tu_1, y_0 + tu_2) : t \in \mathbb{R}\} \cap D$ and let $f : E \to \mathbb{R}$ be any function. Then $(\mathbf{D_u} f)(x_0, y_0)$ exists if and only if there is a function $f_1 : E \to \mathbb{R}$ such that $f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0) = t\, f_1(x_0 + tu_1, y_0 + tu_2)$ for all $t \in \mathbb{R}$ satisfying $(x_0 + tu_1, y_0 + tu_2) \in D$, and f_1 is continuous at (x_0, y_0). Moreover, if these conditions hold, then $(\mathbf{D_u} f)(x_0, y_0) = f_1(x_0, y_0)$.*

Using the Carathéodory Lemma for Directional Derivatives given above (or alternatively, applying Proposition 4.6 of ACICARA 2Ed to suitable univariate functions), we readily see that directional derivatives of sums, scalar multiples, products, reciprocals, and radicals possess exactly the same properties as derivatives of functions of one variable.

p. 90, $+11$

Proposition 3.5 (Bivariate Mean Value Theorem) can be improved as follows:

Suppose (x_0, y_0) and (x_1, y_1) are distinct points in $\mathbb{R}^2$. For $t \in [0,1]$, let $x(t) := x_0 + t(x_1 - x_0)$ and $y(t) := y_0 + t(y_1 - y_0)$. Let

$$L := \{\big(x(t), y(t)\big) \in \mathbb{R}^2 : t \in (0,1)\} \quad \textit{and} \quad E := \{\big(x(t), y(t)\big) \in \mathbb{R}^2 : t \in [0,1]\}$$

denote the (open and closed) line segments joining (x_0, y_0) to (x_1, y_1). Let r be the length of L, and let $\mathbf{u}$ be the unit vector $(x_1 - x_0,\, y_1 - y_0)/r$. Suppose $f : E \to \mathbb{R}$ is a continuous function such that $\mathbf{D_u} f$ exists at each point of L. Then there is $(c, d) \in L$ such that

$$f(x_1, y_1) - f(x_0, y_0) = r(\mathbf{D_u} f)(c, d).$$

We remark that an alternative nomenclature for the Bivariate Mean Value Theorem would be Mean Value Theorem for a Line Segment in $\mathbb{R}^2$. A version of Mean Value Theorem for a Parallelogram in $\mathbb{R}^2$ is given in the Exercises (as a new exercise on p. 153 mentioned below).

p. 91, +1

The hypotheses of Corollary 3.6 can be weakened as follows.

Let the notations and hypotheses be as in Proposition 3.5. In addition, assume that $\mathbf{D_u} f(x(t), y(t)) = \nabla f(x(t), y(t)) \cdot \mathbf{u}$ *for all* $t \in (0, 1)$.

p. 94, after the proof of Proposition 3.11

Remark. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : [a, b] \times [c, d] \to \mathbb{R}$. Suppose $f(x, y) = \phi(x) + \psi(y)$ for all $(x, y) \in [a, b] \times [c, d]$, where $\phi : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and $\psi : [c, d] \to \mathbb{R}$ is any function. Then it is easy to see that $f_{xy} = 0$ on $(a, b) \times (c, d)$. Conversely, suppose f satisfies the hypotheses of Proposition 3.11, and in addition, $f_{xy} = 0$ on $(a, b) \times (c, d)$. Let $x_0 \in (a, b]$ and $y_0 \in (c, d]$. Applying the Rectangular Mean Value Theorem to the restriction of f to $[a, x_0] \times [c, y_0]$, it follows that $f(x_0, y_0) + f(a, c) - f(x_0, c) - f(a, y_0) = 0$, that is, $f(x_0, y_0) = f(x_0, c) + f(a, y_0) - f(a, c)$. This equality also holds if $x_0 = a$ or $y_0 = c$. Thus we see that $f(x, y) = f(x, c) + f(a, y) - f(a, c)$ for all $(x, y) \in [a, b] \times [c, d]$. Now define $\phi : [a, b] \to \mathbb{R}$ by $\phi(x) := f(x, c)$ for $x \in [a, b]$ and $\psi : [c, d] \to \mathbb{R}$ by $\psi(y) := f(a, y) - f(a, c)$ for $y \in [c, d]$. Then ϕ is continuous on $[a, b]$ and differentiable on (a, b), and $f(x, y) = \phi(x) + \psi(y)$ for all $(x, y) \in [a, b] \times [c, d]$.

p. 95, after Example 3.16

Another example to illustrate $f_{xy} \neq f_{yx}$: Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x, y) := \begin{cases} \dfrac{x^3 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

It is easy to see that $f_x(0, y_0) = 0$ for any $y_0 \in \mathbb{R}$ and $f_y(x_0, 0) = x_0$ for any $x_0 \in \mathbb{R}$. Hence $f_{xy}(0, 0) = 0 \neq 1 = f_{yx}(0, 0)$.

p. 105, at the end of part (ii) of Remarks 3.26

More generally, suppose $g_1, g_2 : D \to \mathbb{R}$ are functions such that g_1 and g_2 are continuous at (x_0, y_0). It is easy to see that (g_1, g_2) is a pair of increment functions associated with f and (x_0, y_0) if and only if

$$(x - x_0)\big(f_1(x, y) - g_1(x, y)\big) + (y - y_0)\big(f_2(x, y) - g_2(x, y)\big) = 0 \quad \text{for all } (x, y) \in D.$$

Moreover, in that case, $f_1(x, y_0) = g_1(x, y_0)$ for all $x \in \mathbb{R}$ such that $(x, y_0) \in D$, and $f_2(x_0, y) = g_2(x_0, y)$ for all $y \in \mathbb{R}$ such that $(x_0, y) \in D$. This follows by first putting $y = y_0$ and using the continuity of the functions f_1 and g_1 at (x_0, y_0), and then putting $x = x_0$, and using the continuity of the functions f_2 and g_2 at (x_0, y_0). Thus if (f_1, f_2) and (g_1, g_2) are two pairs of increment functions associated with f and (x_0, y_0), and if $D_1 := \{(x, y) \in D : y = y_0\}$

and $D_2 := \{(x,y) \in D : x = x_0\}$, then $(f_1)_{|D_1} = (g_1)_{|D_1}$ and $(f_2)_{|D_2} = (g_2)_{|D_2}$.

p. 108, +8

Proposition 3.33 can be improved as follows: Let $D \subseteq \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Let $f : D \to \mathbb{R}$ be such that one of f_x and f_y exists on $D \cap \mathbb{S}_\delta(x_0, y_0)$ for some $\delta > 0$ and is continuous at (x_0, y_0), while the other exists at (x_0, y_0). Then f is differentiable at (x_0, y_0).

p. 116, +18

Proposition 3.43 (Classical Version of Bivariate Mean Value Theorem) can be improved as follows: Let D be a convex subset of $\mathbb{R}^2$, and let D° denote its interior. Suppose $f : D \to \mathbb{R}$ is a continuous function that is differentiable on D°. Given any distinct points (x_0, y_0) and (x_1, y_1) in D such that the open line segment L joining them lies in D°, there is $(c, d) \in L$ such that

$$\begin{aligned} f(x_1, y_1) - f(x_0, y_0) &= (x_1 - x_0) f_x(c, d) + (y_1 - y_0) f_y(c, d) \\ &= (x_1 - x_0,\, y_1 - y_0) \cdot \nabla f(c, d). \end{aligned}$$

Proof. Since f is differentiable on D°, by Proposition 3.35 we see that $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$ for all $(x,y) \in D^\circ$ and all unit vectors $\mathbf{u}$ in $\mathbb{R}^2$. Thus the desired result follows from Corollary 3.6.

p. 116, −3 to p. 117, +14

Corollary 3.45 and Remark 3.46 are subsumed by the Remark added on page 87, after Fact 3.2 (MVT).

p. 122, −7

The displayed identities in (i) above can be written as follows:

$$\left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right] = \frac{dw}{dz}\left[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right].$$

p. 151, −1

Additional item in Exercise 7: (viii) $\dfrac{x(x^2 - y^2)}{x^2 + y^2}$.

p. 152, after Exercise 8

New Exercise: Let $D \subseteq \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Show that f is differentiable at (x_0, y_0) if and only if there are $\alpha, \beta \in \mathbb{R}$ and a function $\phi : D \to \mathbb{R}$ such that $\phi(x_0, y_0) = 0$, ϕ is continuous at (x_0, y_0) and

$$f(x,y) = f(x_0, y_0) + \alpha(x - x_0) + \beta(y - y_0) + \phi(x,y)\sqrt{(x - x_0)^2 + (y - y_0)^2}$$

for all $(x,y) \in D$. Moreover, if these conditions hold, then $\nabla f(x_0, y_0) = (\alpha, \beta)$. Use this result to prove Propositions 3.28, 3.30, 3.33 and 3.35.

p. 153, before Exercise 29

New Exercise: Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Let $D := (a,b) \times (c,d)$ and let $(x_0, y_0) \in D$. Consider $\phi : (a,b) \to \mathbb{R}$ and $\psi : (c,d) \to \mathbb{R}$, and define $f, g : D \to \mathbb{R}$ by $f(x,y) := \phi(x) + \psi(y)$ and $g(x,y) := \phi(x)\psi(y)$ for $(x,y) \in D$. Prove the following.

(i) If ϕ is differentiable at x_0 and ψ is differentiable at y_0, and then f and g are differentiable at (x_0, y_0), and $\nabla f(x_0, y_0) = \big(\phi'(x_0), \psi'(y_0)\big)$ and $\nabla g(x_0, y_0) = \big(\phi'(x_0)\psi(y_0), \phi(x_0)\psi'(y_0)\big)$.

(ii) If f is differentiable at (x_0, y_0), then ϕ is differentiable at x_0 and ψ is differentiable at y_0.

(iii) If g is differentiable at (x_0, y_0), then ϕ is differentiable at x_0 provided $\psi(y_0) \neq 0$, in which case $\phi'(x_0) = g_x(x_0, y_0)/\psi(y_0)$, while ψ is differentiable at y_0 provided $\phi(x_0) \neq 0$, in which case $\psi'(y_0) = g_y(x_0, y_0)/\phi(x_0)$. Neither of the conditions $\psi(y_0) \neq 0$ and $\phi(x_0) \neq 0$ can be dropped.
(Hint: Use Fact 3.24 and Proposition 3.25.)

p. 153, before Exercise 29

New Exercise: **(Mean Value Theorem for a Parallelogram in $\mathbb{R}^2$)**. Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be noncollinear points in $\mathbb{R}^2$. For $s, t \in [0, 1]$, let $x(s, t) := x_0 + s(x_1 - x_0) + t(x_2 - x_0)$ and $y(s, t) := y_0 + s(y_1 - y_0) + t(y_2 - y_0)$. Also let $x_3 := x_1 + x_2 - x_0$ and $y_3 := y_1 + y_2 - y_0$. Consider the parallelogram $E := \{\big(x(s, t), y(s, t)\big) : s, t \in [0, 1]\}$ with vertices at (x_i, y_i) for $i = 0, 1, 2, 3$, and let $P := \{\big(x(s, t), y(s, t)\big) \in \mathbb{R}^2 : s, t \in (0, 1)\}$ be the interior of E. Let $\mathbf{u} := (x_1 - x_0,\ y_1 - y_0)/r_1$ and $\mathbf{v} := (x_2 - x_0,\ y_2 - y_0)/r_2$ be unit vectors along two nonparallel sides of E, where r_i denotes the length of the line segment joining (x_0, y_0) to (x_i, y_i), for $i = 1, 2$. Suppose $f : E \to \mathbb{R}$ is continuous and moreover, $\mathbf{D_u} f$ and $\mathbf{D}^2_{\mathbf{uv}} f$ exist at each point of P. Show that there is $(x^*, y^*) \in P$ such that $f(x_3, y_3) + f(x_0, y_0) - f(x_1, y_1) - f(x_2, y_2) = r_1 r_2 (\mathbf{D}^2_{\mathbf{uv}} f)(x^*, y^*)$. (Hint: Define $F : [0, 1] \times [0, 1] \to \mathbb{R}$ by $F(s, t) := f(x(s, t), y(s, t))$ for s, t in $[0, 1]$, and apply Proposition 3.11 to F. Compare Proposition 3.5 and its proof.)

p. 153, after Exercise 29

New Exercise: **(Extended Rectangular Mean Value Theorem)**.
Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. Suppose $f : [a, b] \times [c, d] \to \mathbb{R}$ is a function having continuous partial derivatives of orders 1, 2, 3 and 4. Show that there is $(x_1, y_1) \in (a, b) \times (c, d)$ such that $f(b, d) + f(a, c) - f(b, c) - f(a, d)$ $+ (b - a)\big(f_x(a, c) - f_x(a, d)\big) + (d - c)\big(f_y(a, c) - f_y(b, c)\big) + (b - a)(d - c) f_{xy}(a, c)$ is equal to $\dfrac{(b - a)^2 (d - c)^2}{4} f_{xxyy}(x_1, y_1)$.

Chapter 4

p. 157, −2 and p. 158, +6

Absolute minimum, absolute maximum, and absolute extremum are also known as global minimum, global maximum, and global extremum, respectively. This nomenclature is perhaps more appropriate since it juxtaposes with the usual nomenclature local minimum, local maximum, and local extremum.

p. 158, −3 to −8

The proof of Lemma 4.2 can be improved as follows.

Suppose $(\mathbf{D_u} f)(x_0, y_0)$ exists. Let $E := \{(x_0 + t u_1, y_0 + t u_2) : t \in \mathbb{R}\} \cap D$. By the Carathéodory Lemma for Directional Derivatives, there is $f_1 : E \to \mathbb{R}$

such that $f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0) = t\, f_1(x_0 + tu_1, y_0 + tu_2)$ for all $t \in \mathbb{R}$ satisfying $(x_0 + tu_1, y_0 + tu_2) \in D$, and f_1 is continuous at (x_0, y_0). Suppose f has a local minimum at the interior point (x_0, y_0) of D. Then there is $\delta > 0$ such that $f(x_0 + tu_1, y_0 + tu_2) \geq f(x_0, y_0)$ for all $t \in (-\delta, \delta)$. Hence $f_1(x_0 + tu_1, y_0 + tu_2) \leq 0$ for all $t \in (-\delta, 0)$ and $f_1(x_0 + tu_1, y_0 + tu_2) \geq 0$ for all $t \in (0, \delta)$. The continuity of f_1 at (x_0, y_0) shows that $f_1(x_0, y_0) = \lim_{t \to 0^+} f_1(x_0 + tu_1, y_0 + tu_2) \geq 0$ and also $f_1(x_0, y_0) = \lim_{t \to 0^-} f_1(x_0 + tu_1, y_0 + tu_2) \leq 0$. Thus $(\mathbf{D_u} f)(x_0, y_0) = f_1(x_0, y_0) = 0$.

p. 159, −3

These one-variable methods are useful if the boundary of D consists of line segments. In general, the boundary of D may be determined by one or more paths in $\mathbb{R}^2$, and we need to find the local extrema of f along each such path. It turns out that the gradient of f is orthogonal to the tangent vectors of paths at points of local extema of f along them. Here is a more precise result.

Orthogonal Gradient Theorem. *Let $D_0 \subseteq \mathbb{R}^2$ and (x_0, y_0) be an interior point of D_0. Let Γ be a path lying in D_0 given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, such that $(x_0, y_0) = (x(t_0), y(t_0))$ for some $t_0 \in (\alpha, \beta)$, and the functions x, y are differentiable at t_0. Suppose $f_0 : D_0 \to \mathbb{R}$ is differentiable at (x_0, y_0) and has a local extremum at (x_0, y_0) along Γ. Then $(\nabla f_0)(x_0, y_0) \cdot (x'(t_0), y'(t_0)) = 0$.*

Proof. Define $F : [\alpha, \beta] \to \mathbb{R}$ by $F(t) := f_0(x(t), y(t))$. Clearly, F has a local extremum at t_0. Also, by the Chain Rule (part (ii) of Proposition 3.51), F is differentiable at t_0 and $F'(t_0) = (f_0)_x(x_0, y_0)x'(t_0) + (f_0)_y(x_0, y_0)y'(t_0)$. Now, by Fact 4.1, $F'(t_0) = 0$. This yields $(\nabla f_0)(x_0, y_0) \cdot (x'(t_0), y'(t_0)) = 0$.

In view of the Orthogonal Gradient Theorem, to find the absolute extrema $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$ is closed and bounded, and the boundary of D is given by one or more paths in $\mathbb{R}^2$, we can proceed as follows. Let Γ be such a path. Consider an open subset D_0 of $\mathbb{R}^2$ containing D and a function $f_0 : D_0 \to \mathbb{R}$ such that $(f_0)_{|D} = f$. Find points at which f_0 is differentiable, a tangent vector to Γ is defined and is orthogonal to ∇f_0 at that point. Consider the values of f at all such points and also at the initial point as well as the terminal point of Γ, and at points where a tangent vector to Γ is not defined or f_0 is not diffentiable, and moreover, at the critical points of f. Then compare these values. The maximum among these will give the absolute maximum of f, whereas the minimum among these will give the absolute minimum of f.

p. 161, +14

A part of Example 4.5 (iii) can be reworked using the Orthogonal Gradient Theorem as follows. Let $f(x, y) := x^2 - y^2$ for $(x, y) \in \mathbb{R}^2$. The boundary of D corresponds to the path Γ given by $(x(t), y(t))$, where $x(t) := a \cos t$ and $y(t) := b \sin t$, $t \in [0, 2\pi]$. Now for $t \in (0, 2\pi)$, $(\nabla f)(x(t), y(t)) \cdot (x'(t), y'(t)) = (2a \cos t,\ -2b \sin t) \cdot (-a \sin t,\ b \cos t) = -2(a^2 + b^2) \cos t \sin t$, and this equals 0 if and only if $\cos t = 0$ or $\sin t = 0$, that is, if and only if $t \in \{\pi/2, \pi, 3\pi/2\}$. The initial point as well as the terminal point of Γ is $(a, 0)$, which corresponds to $t = 0$ and $t = 2\pi$. Thus the boundary points of D at which the absolute extrema of f on D can possibly be attained are $(a, 0), (0, b), (-a, 0)$, and $(0, -b)$. Since the value of f at any of these points is either a^2 or $-b^2$, and also since the value of f at its only critical point $(0, 0)$ is 0, it follows that the absolute maximum of f is a^2, which is attained at $(\pm a, 0)$, while the absolute minimum of f is $-b^2$, which is attained at $(0, \pm b)$.

p. 161, before Remark 4.6

Additional example: (iv) Let $a, b \in \mathbb{R}$ be positive and as in (iii) above, let $D := \{(x, y) \in \mathbb{R}^2 : b^2x^2 + a^2y^2 \leq a^2b^2\}$. Let $f(x, y) := x\,y$ for $(x, y) \in \mathbb{R}^2$. Let us find the absolute extrema of the continuous function f on the closed and bounded subset D of $\mathbb{R}^2$. To begin with, $\nabla f(x, y) = (y, x)$ for $(x, y) \in \mathbb{R}^2$, and thus $(0, 0)$ is the only critical point of f in D. As before, the boundary of D corresponds to the path Γ given by $(x(t), y(t))$, where $x(t) := a \cos t$ and $y(t) := b \sin t$, $t \in [0, 2\pi]$. Now for $t \in (0, 2\pi)$, $(\nabla f)(x(t), y(t)) \cdot (x'(t), y'(t)) = (b \sin t, a \cos t) \cdot (-a \sin t, b \cos t) = ab(\cos^2 t - \sin^2 t)$, and this equals 0 if and only if $\cos t = \pm \sin t$, that is, if and only if $t \in \{\pi/4, \pi, 3\pi/4, 5\pi/4, 7\pi/4\}$. The initial point as well as the terminal point of Γ is $(a, 0)$, which corresponds to $t = 0$ and $t = 2\pi$. The Orthogonal Gradient Theorem shows that the boundary points of D at which the absolute extrema of f on D can possibly be attained are $(a/\sqrt{2}, b/\sqrt{2}), (-a/\sqrt{2}, b/\sqrt{2}), (-a/\sqrt{2}, -b/\sqrt{2}), (a/\sqrt{2}, -b/\sqrt{2})$ and $(a, 0)$. The value of f at any of these points is $ab/2$ or $-ab/2$ or 0. Since $f(0, 0) = 0$, we see that the absolute maximum of f is $ab/2$, which is attained at $(a/\sqrt{2}, b/\sqrt{2})$ as well as at $(-a/\sqrt{2}, -b/\sqrt{2})$, while the absolute minimum of f is $-ab/2$, which is attained at $(-a/\sqrt{2}, b/\sqrt{2})$ as well as at $(a/\sqrt{2}, -b/\sqrt{2})$.

Chapter 5

p. 193, +13

In general, if $f : [a, b] \times [c, d] \to \mathbb{R}$ is any integrable function, then it will be seen later (in Remark 5.35) that the double integral of f can be interpreted as the "signed volume" delineated by the surface $z = f(x, y)$, $(x, y) \in [a, b] \times [c, d]$.

p. 193, before Basic Inequality and Criterion for Integrability

Proposition *Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Given any $\epsilon > 0$, there is $\delta > 0$ such that for every partition P of $[a, b] \times [c, d]$ with $\mu(P) < \delta$,*

$$L(f) - \epsilon < L(P, f) \leq U(P, f) < U(f) + \epsilon$$

Proof. Let $\epsilon > 0$ be given. Since $U(f)$ is the infimum of the set of all upper sums for f and $L(f)$ is the supremum of the set of all lower sums for f, there are partitions P_1 and P_2 of $[a, b] \times [c, d]$ such that $U(P_1, f) < U(f) + \epsilon/2$ and $L(P_2, f) > L(f) - \epsilon/2$. Let P_0 denote the common refinement of P_1 and P_2. Then by part (i) of Proposition 5.3,

$$U(P_0, f) < U(f) + \frac{\epsilon}{2} \quad \text{and} \quad L(P_0, f) > L(f) - \frac{\epsilon}{2}.$$

Let $\alpha > 0$ be such that $|f(x,y)| \le \alpha$ for all $(x,y) \in [a,b] \times [c,d]$. Also, let m_0 be the number of grid points of P_0, and let $\ell := 2\,(b-a+d-c)$ be the perimeter of $[a,b]\times[c,d]$. Define $\delta := \epsilon/2\alpha\ell m_0$. Suppose P is any partition of $[a,b]\times[c,d]$ such that $\mu := \mu(P) < \delta$. Let P^* denote the common refinement of P and P_0. Then P^* is obtained from P by successive one-step refinements by points of P_0 that are not in P. Since the number of such points is atmost m_0, successive applications of Lemma 5.2 shows that

$$U(P,f) \le U(P^*,f) + m_0\alpha\mu\ell \quad \text{and} \quad L(P,f) \ge L(P^*,f) - m_0\alpha\mu\ell.$$

Further, in view of part (i) of Proposition 5.3,

$$U(P^*,f) \le U(P_0,f) < U(f) + \frac{\epsilon}{2} \quad \text{and} \quad L(P^*,f) \ge L(P_0,f) > L(f) - \frac{\epsilon}{2}.$$

Combining the last two sets of inequalities displayed above and noting that $m_0\alpha\mu\ell < (\epsilon/2)$, thanks to our choice of δ, we see that

$$L(f) - \epsilon < L(P,f) \le U(P,f) < U(f) + \epsilon. \qquad \square$$

p. 194, −12

It may be better to write (in the usual order of factors of the summands)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} m_{i,j}(f)(x_i - x_{i-1})(y_j - y_{j-1}) = (d-c)\sum_{i=1}^{n} m_i(\phi)(x_i - x_{i-1}).$$

p. 195, add to Proposition 5.6 (Riemann Condition)

Furthermore, if f is integrable, then for every $\epsilon > 0$, there is $\delta > 0$ such that

$$U(P,f) - L(P,f) < \epsilon \quad \textit{for every partition } P \textit{ of } [a,b]\times[c,d] \textit{ with } \mu(P) < \delta.$$

Proof. Let $\epsilon > 0$ be given. By the Proposition added on page 193, there is $\delta > 0$ such that $L(f) - (\epsilon/2) < L(P,f) \le U(P,f) < U(f) + (\epsilon/2)$ for every partition P of $[a,b]\times[c,d]$ with $\mu(P) < \delta$. This implies the desired result since $L(f) = U(f)$ if f is integrable.

p. 196, after Example 5.7

An immediate consequence of the Riemann Condition (Proposition 5.6) is the following. Here and hereinafter, by a sequence (P_n) of partitions of $[a,b]\times[c,d]$ we mean a map that associates to each $n \in \mathbb{N}$, a partition P_n of $[a,b]\times[c,d]$.

Corollary *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. If (P_n) is a sequence of partitions of $[a,b]\times[c,d]$ such that $\mu(P_n) \to 0$, then $L(P_n,f) \to L(f)$ and $U(P_n,f) \to U(f)$.*

Moreover, we have the following sequential characterization of integrability: f is integrable if and only if there is a sequence (P_n) of partitions of $[a,b]\times[c,d]$ such that $U(P_n,f) - L(P_n,f) \to 0$. In this case,

$$L(P_n,f) \to \iint_{[a,b]\times[c,d]} f \quad \textit{and} \quad U(P_n,f) \to \iint_{[a,b]\times[c,d]} f.$$

Proof. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ satisfying the conclusion of the Proposition added on page 193. Now, let (P_n) be a sequence of partitions of $[a,b]\times[c,d]$ such that $\mu(P_n) \to 0$. Then there is $n_0 \in \mathbb{N}$ such that $0 \le \mu(P_n) < \delta$ for all $n \ge n_0$. It follows that

$$0 \le L(f) - L(P_n, f) < \epsilon \quad \text{and} \quad 0 \le U(P_n, f) - U(f) < \epsilon \quad \text{for all } n \ge n_0,$$

Hence $L(P_n, f) \to L(f)$ and $U(P_n, f) \to U(f)$.

To prove the sequential characterization, first suppose f is integrable. Then we can use the Riemann condition with $\epsilon = 1/n$ for each $n \in \mathbb{N}$ to obtain a desired sequence of partitions. Conversely, suppose there is a sequence (P_n) of partitions of $[a,b]\times[c,d]$ such that $U(P_n, f) - L(P_n, f) \to 0$. Then for every $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \ge n_0$. So the Riemann condition is satisfied with $P_\epsilon = P_{n_0}$. Hence f is integrable.

Finally, suppose f is integrable and (P_n) is a sequence of partitions such that $U(P_n, f) - L(P_n, f) \to 0$. Let $I(f) := \iint_{[a,b]\times[c,d]} f$ and for $n \in \mathbb{N}$, let $\delta_n := U(P_n, f) - L(P_n, f)$. Then

$$0 \le I(f) - L(P_n, f) \le \delta_n \quad \text{and} \quad 0 \le U(P_n, f) - I(f) \le \delta_n \text{ for all } n \in \mathbb{N}.$$

Since $\delta_n \to 0$, it follows that $L(P_n, f) \to I(f)$ and $U(P_n, f) \to I(f)$. □

p. 201, +1

The proof of parts (i) and (ii) of Proposition 5.12 can be simplified by using the Corollary added on page 196.

p. 211, before Remark 5.21

While part (i) of the FTC (Fact 5.18) for Riemann integrable functions on $[a,b]$ provides the most widely used method of evaluating Riemann integrals, this is not the case for the analogue of the FTC for double integrals given in part (i) of Proposition 5.20. This is partly because the analogue itself is not widely known. Also, while in the case of a Riemann integrable function on $[a,b]$, one needs to think of a function whose derivative is the given function on (a,b), in the case of an integrable function f on $[a,b]\times[c,d]$, one needs to conjure up a function F such that $F_{xy} = f$ on $(a,b)\times(c,d)$. The latter is admittedly a more challenging task. Nonetheless, if one can come up with such a function F, then the evaluation of the double integral of f is extremely easy. The following example illustrates this method.

Example. Let $R := [0,1]\times[0,1]$ and let $f : R \to \mathbb{R}$ be defined by

$$f(x,y) := \frac{1}{(1+x+y)^{3/2}} \quad \text{for } (x,y) \in R.$$

Consider $F : R \to \mathbb{R}$ and $g : R \to \mathbb{R}$ defined by

$$F(x,y) := -4(1+x+y)^{1/2} \quad \text{and} \quad g(x,y) := \frac{-2}{(1+x+y)^{1/2}} \quad \text{for } (x,y) \in R.$$

Then it is easy to see that $F_x = g$ and $F_{xy} = g_y = f$ on $(0,1) \times (0,1)$. Consequently, by part (i) of Proposition 5.20,

$$\iint_{[0,1]\times[0,1]} f = F(1,1) - F(1,0) - F(0,1) + F(0,0) = -4(\sqrt{3} - 2\sqrt{2} + 1).$$

p. 214, +11

If $f : D \to \mathbb{R}$ is assumed to be continuous (instead of integrable), then a simple proof of part (ii) of Proposition 5.26 (Double Integration by Substitution) can be given follows. Let $R := [\phi(\alpha), \phi(\beta)] \times [\psi(\gamma), \psi(\delta)]$. Then in Case 1, $R = [a,b] \times [c,d]$ and $|J(\Phi)| = J(\Phi)$, in Case 2, $R = [a,b] \times [d,c]$ and $|J(\Phi)| = -J(\Phi)$, in Case 3, $R = [b,a] \times [c,d]$ and $|J(\Phi)| = -J(\Phi)$, and in Case 4, $R = [b,a] \times [d,c]$ and $|J(\Phi)| = J(\Phi)$. Thus, in view of our convention stated in Remark 5.11, the conclusion of part (ii) is immediate from part (i).

p. 220, −1

Additional Example: Consider the function $f : [0,1]\times[0,1] \to \mathbb{R}$ defined by $f(x,y) := xy(x^2 - y^2)/(x^2 + y^2)^3$ if $(x,y) \neq (0,0)$ and $f(0,0) := 0$. Since $f(1/n, 1/2n) = 24n^2/125 \to \infty$ and $f(1/2n, 1/n) = -24n^2/125 \to -\infty$, the function f is neither bounded above nor bounded below. Hence the double integral of f is not defined. Now, $\int_0^1 f(0,y)dy = \int_0^1 0dy = 0$, and for $x \in (0,1]$,

$$\int_0^1 f(x,y)dy = \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}dy = \frac{1}{2}\int_{x^2}^{1+x^2} \frac{x(2x^2-u)}{u^3}du = \frac{x}{2(1+x^2)^2},$$

where the second equality is obtained using the substitution $u = x^2 + y^2$. Thus

$$\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = \int_0^1 \frac{x}{2(1+x^2)^2}dx = \frac{1}{4}\int_1^2 \frac{1}{t^2}dt = \frac{1}{8}.$$

By interchanging the roles of x and y, we obtain $\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = -1/8$. So we see that both the iterated integrals exist without being equal.

p. 222, before Riemann Double Sums

Remark. A result related to Proposition 5.28 (Fubini's Theorem on Rectangles), but not involving a double integral, was proved by G. Fichtenholz and, independently, by L. Lichtenstein in 1910. It can be stated as follows.

Theorem *Let $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$, and let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function such that for each fixed $y \in [c,d]$, the function given by $x \longmapsto f(x,y)$ is integrable on $[a,b]$, and for each fixed $x \in [a,b]$, the function given by $y \longmapsto f(x,y)$ is integrable on $[c,d]$. Then the function $F : [c,d] \to \mathbb{R}$ defined by $F(y) := \int_a^b f(x,y)dx$ is integrable on $[c,d]$ and the function $G : [a,b] \to \mathbb{R}$ defined by $G(x) := \int_c^d f(x,y)dy$ is integrable on $[a,b]$, and moreover, $\int_c^d F(y)dy = \int_a^b G(x)dx$, that is,*

$$\int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

For a proof, we refer to Theorem 16.6.2 in J. Lewin's book *An Interactive Introduction to Mathematical Analysis*, third ed., Cambridge University Press, Cambridge, 2014. For a weaker version of this result (in which the integrability of the functions F and G is assumed), see Proposition 10.50 of ACICARA2Ed.

p. 222, -11 to p. 225, -1

Alternative treatment of **Riemann Double Sums**:

Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be an integrable function. As a consequence of the Riemann condition, we have seen that if (P_n) is a sequence of partitions of $[a,b]\times[c,d]$ such that $\mu(P_n) \to 0$, then $L(P_n, f) \to \iint_{[a,b]\times[c,d]} f$ and $U(P_n, f) \to \iint_{[a,b]\times[c,d]} f$. (See Corollary added on page 196.) Although we have made good use of the Riemann Condition (and its corollary) to prove several interesting results (including Proposition 5.12) earlier in this section, a major difficulty arises in calculating the approximations $L(P_n, f)$ and $U(P_n, f)$ of the double integral of f. For a given partition P, the calculation of $U(P, f)$ and $L(P, f)$ involves finding suprema and infima of f over many subrectangles of $[a,b]\times[c,d]$. This task is rarely easy, and performing it for a large number of partitions in the sequence (P_n) would be challenging. To overcome this difficulty, we observe that evaluating f at points of $[a,b]\times[c,d]$ is much easier than finding suprema and infima of f over subrectangles. With this in mind, we introduce the following variant of lower and upper double sums.

Let $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ be a partition of $[a,b]\times[c,d]$, and let $\mathcal{T}$ be a **tag set** associated with P, by which we mean a set $\mathcal{T} = \{(s_i, t_j) : i = 1, \ldots, n \text{ and } j = 1, \ldots, k\}$, where $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$ and $t_j \in [y_{j-1}, y_j]$ for $j = 1, \ldots, k$. Then

$$S(P, \mathcal{T}, f) := \sum_{i=1}^{n}\sum_{j=1}^{k} f(s_i, t_j)(x_i - x_{i-1})(y_j - y_{j-1})$$

is called a **Riemann double sum** for f corresponding to the partition P and the tag set $\mathcal{T}$. It is clear that $L(P, f) \le S(P, \mathcal{T}, f) \le U(P, f)$ for every tag set $\mathcal{T}$ associated with the partition P. In some special cases, $L(P, f)$ or $U(P, f)$ can itself be a Riemann double sum. For example, when f is monotonic or continuous, the proof of Proposition 5.12 shows that for any partition P of $[a,b]\times[c,d]$, the lower sum $L(P, f)$ as well as the upper sum $U(P, f)$ is itself a Riemann double sum $S(P, \mathcal{T}, f)$ for some tag set $\mathcal{T}$ associated with P.

It will turn out that the integrability of f can be characterized in terms of Riemann double sums. But first we show how Riemann double sums can be used to approximate the double integral of an integrable function.

Proposition 1 *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. Given any $\epsilon > 0$, there is $\delta > 0$ such that for every partition P of $[a,b]\times[c,d]$ with $\mu(P) < \delta$, and for every tag set $\mathcal{T}$ associated with P,*

$$L(f) - \epsilon < S(P, \mathcal{T}, f) < U(f) + \epsilon,$$

and, in particular, if f is integrable, then

$$\left| S(P, \mathcal{T}, f) - \iint_{[a,b]\times[c,d]} f \right| < \epsilon.$$

Proof. Let $\epsilon > 0$ be given. By the Corollary added on page 196, there is $\delta > 0$ such that for every partition P of $[a,b] \times [c,d]$ with $\mu(P) < \delta$,

$$L(f) - \epsilon < L(P,f) \leq U(P,f) < U(f) + \epsilon.$$

Let P be a partition of $[a,b] \times [c,d]$ satisfying $\mu(P) < \delta$, and let $\mathcal{T}$ be a tag set associated with P. Since $L(P,f) \leq S(P,\mathcal{T},f) \leq U(P,f)$, and since $L(f) = U(f)$ when f is integrable, the desired results follow. □

Corollary 2 *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be an integrable function, and let (P_n) be a sequence of partitions of $[a,b] \times [c,d]$ such that $\mu(P_n) \to 0$. If $\mathcal{T}_n$ is any tag set associated with P_n for $n \in \mathbb{N}$, then*

$$S(P_n, \mathcal{T}_n, f) \to \iint_{[a,b]\times[c,d]} f.$$

Proof. Let $\epsilon > 0$ be given. Then there is $\delta > 0$ satisfying the conclusion of Proposition 1. Since $\mu(P_n) \to 0$, there is $n_0 \in \mathbb{N}$ such that $0 \leq \mu(P_n) < \delta$ for all $n \geq n_0$. Let $\mathcal{T}_n$ be any tag set associated with P_n for $n \in \mathbb{N}$. Since f is integrable, it follows that

$$\left| S(P_n, \mathcal{T}_n, f) - \iint_{[a,b]\times[c,d]} f \right| < \epsilon \quad \text{for all } n \geq n_0.$$

Hence the desired result follows. □

Remarks 3

(i) It may be tempting to define the mesh of a partition $P = \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ of $[a,b] \times [c,d]$ to be the maximum of the areas of subrectangles induced by P, that is, $\max\{(x_i - x_{i-1})(y_j - y_{j-1}) : i = 1, \ldots, n \text{ and } j = 1, \ldots, k\}$. However, with this definition, Corollary 2 does not hold. To see this, consider the bivariate Thomae function defined in Example 5.30 (iv). This is an integrable function $f : [0,1] \times [0,1] \to \mathbb{R}$ with the property that $f(0,y) = 1 = f(1,y)$ for $y \in \mathbb{Q} \cap [0,1]$ and $I(f) = 0$, where $I(f)$ denotes the double integral of f on $[0,1] \times [0,1]$. If for $k \in \mathbb{N}$, we let $P_k := \{(i, j/k) : i = 0, 1 \text{ and } j = 0, 1, \ldots k\}$, then P_k is a partition of $[0,1] \times [0,1]$ such that the area of each subrectangle induced by P_k is $1/k$, which tends to 0 as $k \to \infty$. Let $\mathcal{T}_k := \{(1, j/k) : j = 1, \ldots, k\}$ for $k \in \mathbb{N}$. Then the Riemann double sum $S(P_k, \mathcal{T}_k, f) = \sum_{i=1}^{1} \sum_{j=1}^{k} f(i, j/k)(1/k)$ is equal to 1 for every $k \in \mathbb{N}$. In particular, $S(P_k, \mathcal{T}_k, f) \not\to I(f)$. This example shows why it is important to define the mesh of a partition as the maximum of the lengths of sides of the subrectangles induced by it. An alternative, and essentially equivalent, definition would be to define the mesh of a partition as the maximum of the diameters of the subrectangles induced by it.

(ii) Corollary 2 overcomes the difficulty mentioned at the beginning of this subsection. For $n \in \mathbb{N}$, one chooses a partition P_n whose mesh tends to zero as $n \to \infty$ and picks a suitable tag set for each of them. It may be emphasized that the only requirement here is that $\mu(P_n) \to 0$; the actual partition points and the points in the tag sets at which f is evaluated can be chosen with sole regard to the convenience of summation. In practice, it is often convenient to use the partition $\mathsf{P}_{n,n}$ of the given rectangle into n^2 equal parts, and the tag sets consisting of the upper right corner points, or sometimes the centroids of the subrectangles corresponding to the partition. This enables us to find approximations of the double integral of f when we are not able to evaluate it exactly. For example, if we are not in a position to find a function $F : [a, b] \times [c, d] \to \mathbb{R}$ such that $F_{xy} = f$, (so that part (i) of Proposition 5.20 becomes inoperative as far as the evaluation of the double integral of f is concerned), or if the methods of Double Integration by Parts and Double Integration by Substitution are ineffective, or if Fubini's Theorem on Rectangles cannot be used, then we may resort to calculating the double integral of f approximately. On the other hand, if the double integral of f can indeed be evaluated exactly, then limits of certain Riemann double sums for f can be found. In Chapter 7, after introducing the concept of a "double sequence", we shall see that Corollary 2 can be used to find the limits of certain double sequences as well. The following examples illustrate these observations.

Examples 4

(i) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by $f(x, y) := 1/(1 + x^2 + y^2)$. Clearly, f is continuous and hence integrable on $[0, 1] \times [0, 1]$. Let $n \in \mathbb{N}$ and let $\mathsf{P}_{n,n} := \{(i/n, j/n) : i, j = 0, 1, \ldots, n\}$ be the partition of $[0, 1] \times [0, 1]$ into n^2 equal parts, and let $\mathcal{T}_{n,n} := \{(i/n, j/n) : i, j = 1, \ldots, n\}$ be the tag set consisting of the upper right corner points associated with the partition $\mathsf{P}_{n,n}$. Then $\mu(P_{n,n}) = 1/n \to 0$ and $S(\mathsf{P}_{n,n}, \mathcal{T}_{n,n}, f)$ is equal to

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{1 + (i/n)^2 + (j/n)^2}\left(\frac{i}{n} - \frac{i-1}{n}\right)\left(\frac{j}{n} - \frac{j-1}{n}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2 + i^2 + j^2}.$$

Hence by Corollary 2,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{n^2 + i^2 + j^2} \to \iint_{[0,1]\times[0,1]} \frac{1}{1 + x^2 + y^2} d(x, y).$$

Thus $\sum_{i=1}^{n}\sum_{j=1}^{n} 1/(n^2 + i^2 + j^2)$ provides an approximation of the double integral $\iint_{[0,1]\times[0,1]} d(x, y)/(1 + x^2 + y^2)$ for large $n \in \mathbb{N}$.

(ii) Consider

$$a_n := \frac{1}{\sqrt{n}} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{(n + i + j)^{3/2}} \qquad \text{for } n \in \mathbb{N}.$$

Then

$$a_n = \sum_{i=1}^{n}\sum_{i=1}^{n} \frac{1}{\big(1 + (i/n) + (j/n)\big)^{3/2}}\left(\frac{i}{n} - \frac{i-1}{n}\right)\left(\frac{j}{n} - \frac{j-1}{n}\right) \qquad \text{for } n \in \mathbb{N}.$$

We observe that $a_n = S(\mathsf{P}_{n,n}, \mathcal{T}_{n,n}, f)$, where $f(x,y) := 1/(1+x+y)^{3/2}$ for $(x,y) \in [0,1] \times [0,1]$, and $\mathsf{P}_{n,n}$, $\mathcal{T}_{n,n}$ are as in (i) above. Using part (i) of Proposition 5.20, we have seen that the double integral of f is equal to $-4(\sqrt{3}-2\sqrt{2}+1)$. (See the Addition on page 211, before Remark 5.21.) Since $\mu(\mathsf{P}_{n,n}) = 1/n \to 0$, Corollary 2 shows that $\lim_{n\to\infty} a_n = -4(\sqrt{3}-2\sqrt{2}+1)$.

(iii) Consider

$$a_n := \frac{1}{n^4} \sum_{i=1}^{n} \sum_{j=1}^{n} (i+j)^2 \quad \text{for } n \in \mathbb{N}.$$

Then

$$a_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{i}{n} + \frac{j}{n}\right)^2 \left(\frac{i}{n} - \frac{i-1}{n}\right) \left(\frac{j}{n} - \frac{j-1}{n}\right) \quad \text{for } n \in \mathbb{N}.$$

We observe that $a_n = S(\mathsf{P}_{n,n}, \mathcal{T}_{n,n}, f)$, where $f(x,y) := (x+y)^2$ for (x,y) in $[0,1] \times [0,1]$, and $\mathsf{P}_{n,n}$, $\mathcal{T}_{n,n}$ are as in (i) above. Since $\mu(\mathsf{P}_{n,n}) = 1/n \to 0$, Corollary 2 and Proposition 5.28 show that the limit of the given sequence (a_n) is equal to

$$\iint_{[0,1]\times[0,1]} f = \int_0^1 \left(\int_0^1 (x+y)^2 dx \right) dy = \frac{1}{3} \int_0^1 (1 + 3y + 3y^2) dy = \frac{7}{6}.$$

We can verify this answer by writing

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (i+j)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (i^2 + 2ij + j^2) = n \sum_{i=1}^{n} i^2 + 2 \left(\sum_{i=1}^{n} i \right) \left(\sum_{j=1}^{n} j \right) + n \sum_{j=1}^{n} j^2$$

and then using the well-known formulas for the sum of the first n positive integers and for the sum of squares of the first n positive integers, to obtain

$$a_n = \frac{2}{n^4} \left[n \frac{n(n+1)(2n+1)}{6} + \left(\frac{n(n+1)}{2} \right)^2 \right] = \frac{n^2(n+1)(7n+5)}{6n^4}. \qquad \diamond$$

We end this section with a result of theoretical interest. It is sometimes ascribed to Darboux, and it can be used to provide an alternative definition of the double integral of a bounded function as a "limit of a double sum". In particular, it gives a condition, in terms of Riemann double sums, for the integrability of a bounded real-valued function on $[a,b] \times [c,d]$. It is noteworthy that this result does not involve the notion of mesh of a partition.

Proposition 5 (**Darboux Theorem**). *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there is $r \in \mathbb{R}$ satisfying the condition that for every $\epsilon > 0$, there is a partition P of $[a,b] \times [c,d]$ such that*

$$|S(P, \mathcal{T}, f) - r| < \epsilon \quad \textit{for all tag sets } \mathcal{T} \textit{ associated with } P.$$

In this case, $r = \iint_{[a,b]\times[c,d]} f$.

Proof. If f is integrable, then by Proposition 1, the stated condition holds with $r = \iint_{[a,b]\times[c,d]} f$.

Conversely, suppose $r \in \mathbb{R}$ satisfies the stated condition. Let $\epsilon > 0$ be given. Then there is a partition $P := \{(x_i, y_j) : i = 0, 1, \dots, n \text{ and } j = 0, 1, \dots, k\}$ of $[a,b] \times [c,d]$ such that

$$|S(P, \mathcal{T}, f) - r| < \epsilon \quad \text{for all tag sets } \mathcal{T} \text{ associated with } P.$$

Now, let $A := (b-a)(d-c) = \sum_{i=1}^{n} \sum_{j=1}^{k} (x_i - x_{i-1})(y_j - y_{j-1})$. Also, for each $i = 1, \dots, n$ and $j = 1, \dots, k$, let

$$M_{i,j}(f) := \sup\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}$$

and

$$m_{i,j}(f) := \inf\{f(x,y) : (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\}.$$

Then there are (s_i, t_j) and $\left(\tilde{s}_i, \tilde{t}_j\right)$ in $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ such that

$$M_{i,j}(f) < f(s_i, t_j) + \epsilon \quad \text{and} \quad m_{i,j}(f) > f\left(\tilde{s}_i, \tilde{t}_j\right) - \epsilon.$$

If we consider the tag sets $\mathcal{T} := \{(s_i, t_j) : i = 1, \dots, n \text{ and } j = 1, \dots, k\}$ and $\widetilde{\mathcal{T}} := \{(\tilde{s}_i, \tilde{t}_j) : i = 1, \dots, n \text{ and } j = 1, \dots, k\}$, then we readily see that

$$U(P, f) < S(P, \mathcal{T}, f) + \epsilon A \quad \text{and} \quad L(P, f) > S(P, \widetilde{\mathcal{T}}, f) - \epsilon A.$$

Since $L(P, f) \le L(f) \le U(f) \le U(P, f)$, we obtain

$$S(P, \mathcal{T}, f) - \epsilon A < L(f) \le U(f) < S(P, \widetilde{\mathcal{T}}, f) + \epsilon A.$$

But $S(P, \mathcal{T}, f) > r - \epsilon$ and $S(P, \widetilde{\mathcal{T}}, f) < r + \epsilon$. It follows that

$$r - \epsilon(1 + A) < L(f) \le U(f) < r + \epsilon(1 + A).$$

Since $\epsilon > 0$ is arbitrary, we see that $r \le L(f) \le U(f) \le r$. Consequently, $L(f) = U(f) = r$. Thus f is integrable and $\iint_{[a,b]\times[c,d]} f(x,y) d(x,y) = r$.

p. 234, before Remark 5.40

Example 5.39 (iii): If a path in $\mathbb{R}^2$ is given by differentiable functions with bounded derivatives, then its image is of content zero. More precisely, consider a path in $\mathbb{R}^2$ given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, where the real-valued functions x, y are continuous on $[\alpha, \beta]$ and differentiable on (α, β), and moreover, their derivatives x', y' are bounded on (α, β). Then the set $E := \{(x(t), y(t)) \in \mathbb{R}^2 : t \in [\alpha, \beta]\}$ is of content zero. To see this, let $n \in \mathbb{N}$ and consider the partition of $[\alpha, \beta]$ into n equal parts. For $i = 1, \dots, n$, let s_i denote the mid-point of the ith subinterval of this partition. Let $t \in [\alpha, \beta]$. Then there is $i \in \{1, \dots, n\}$ such that $|t - s_i| \le (\beta - \alpha)/2n$. By Fact 3.2 (MVT), there are c_1 and c_2 between t and s_i such that

$$x(t) - x(s_i) = x'(c_1)(t - s_i) \quad \text{and} \quad y(t) - y(s_i) = y'(c_2)(t - s_i).$$

Let $M \in \mathbb{R}$ be such that $|x'(u)| \le M$ and $|y'(u)| \le M$ for all $u \in (\alpha, \beta)$. Then

$$|x(t) - x(s_i)| \le \frac{M(\beta - \alpha)}{2n} \quad \text{and} \quad |y(t) - y(s_i)| \le \frac{M(\beta - \alpha)}{2n}.$$

It follows that the point $(x(t), y(t))$ lies in the square S_i whose centroid is at $(x(s_i), y(s_i))$, whose sides are parallel to the coordinate axes and whose area is equal to $M^2(\beta - \alpha)^2/n^2$. Thus the set E is contained in the union of the n squares $S_1, \dots, S_n$, the sum of whose areas is equal to $M^2(\beta - \alpha)^2/n$. Given any $\epsilon > 0$, we may choose $n \in \mathbb{N}$ such that $n > M^2(\beta - \alpha)^2/\epsilon$. Then E is contained in the union of the n squares $S_1, \dots, S_n$, the sum of whose areas is less than ϵ. Hence the set E is of content zero.

p. 282, after Exercise 5

New Exercise: Define $f, g : [0,1] \times [0,1] \to \mathbb{R}$ by $f(x,y) := 1/(1+x+y)^{1/2}$ and $g(x,y) := 1/(1+x+y)$ for $(x,y) \in [0,1] \times [0,1]$. Find $\iint_{[0,1]\times[0,1]} f(x,y)d(x,y)$ and $\iint_{[0,1]\times[0,1]} g(x,y)d(x,y)$ by using part (i) of Proposition 5.20, and also by using Proposition 5.28.

New Exercise: Define $f : [-1,1] \times [-1,1] \to \mathbb{R}$ by $f(x,y) := e^{x^2y^2}$ for (x,y) in $[-1,1] \times [-1,1]$. Find approximations of $\iint_{[-1,1]\times[-1,1]} f(x,y)d(x,y)$ using Riemann double sums obtained by dividing the rectangle $[-1,1] \times [-1,1]$ into n^2 equal parts for $n \in \mathbb{N}$.

New Exercise: Find the limits of the sequences (a_n) and (b_n), if for $n \in \mathbb{N}$,

$$a_n := \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{(n+i+j)} \quad \text{and} \quad b_n := \frac{1}{n^9} \sum_{i=1}^{n} \sum_{j=1}^{n} i^3 j^4.$$

p. 282, before Exercise 6

New Exercise: Let D be a bounded subset of $\mathbb{R}^2$, and let f and g be real-valued integrable functions on D. Show that the real-valued functions $\max\{f, g\}$ and $\min\{f, g\}$ are also integrable on D.