MA-105 Information Booklet and Problem Sheets Autumn 2019

INSTRUCTORS:

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Name:

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Division and Tut Batch:

Contents

Basic Information

Course contents

Sequences of real numbers, Review of limit, Continuity and differentiability of functions, Rolle's theorem, Mean value theorems and Taylor's theorem, Maxima, minima and curve sketching, Riemann integral, Fundamental theorem of calculus, Applications to length, area, volume, surface area of revolution.

Functions of several variables, Limit, Continuity and partial derivatives, Chain rule, Gradient, Directional Derivative and differentiability, Tangent planes and normals, Maxima, minima, saddle points, Lagrange multipliers, Double and triple integrals, Change of variables.

Vector fields, Gradient, Curl and Divergence, Curves, Line integrals and their applications, Green's theorem and applications, Surfaces, Surface area, Surface integrals, Divergence theorem, Stokes' theorem and applications.

Text/References

[TF] G. B. Thomas and R. L. Finney, *Calculus and Analytic Geometry*, 9th ed., Addison-Wesley/Narosa, 1998.

[GL-1] S. R. Ghorpade and B. V. Limaye, *A Course in Calculus and Real Analysis*, 2nd Ed., Springer, 2018. [`http://www.math.iitb.ac.in/~srg/acicara2/`]

[GL-2] S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis*, Springer, 2010 (First Indian Reprint, Springer (India), 2010). [`http://www.math.iitb.ac.in/~srg/acicmc/`]

[S] James Stewart, *Calculus: Early Transcendentals*, 5th Ed., Thompson Press, 2003 (Second Indian Reprint, 2007).

[HH] D. Hughes-Hallett et al, *Calculus: Single and Multivariable*, 4th Ed., John Wiley, 2005.

[A] T. M. Apostol, *Calculus, Volume-I,II*, Wiley Eastern, 1980.

Course Plan (roughly till the Mid-sem exam)

Sr. No.	Topic	Sections from Text [TF]	Sections from [GL-1]	No. of Lectures
1.	Real Numbers, Functions	P.1, P.3	1.1–1.3	1
2.	Sequences	8.1, 8.2	2.1	1
3.	Limits and Continuity	1.1-1.5	3.1, 3.3	2
4.	Differentiation	2.1, 2.2, 2.5, 2.6	4.1	1
5.	Rolle's, Mean Value, and Taylor's Theorems	3.1, 3.2	4.2	2
6.	Maxima/Minima and Curve Sketching	3.1–3.3, 3.5	5.1, 5.2, 4.3	2
7.	Riemann Integral and the Funda. Theorem of Calculus	4.5-4.8	6.4, 6.2, 6.3	2
8.	Natural Logarithm and the Exponential Function	6.2-6.4, 6.7	7.1	2
9.	Applications of Integrals	5.1-5.6	8.1–8.4	2
10.	Functions of Several Variables: Limits, Continuity	12.1, 12.2	Sections from [GL-2] 1.2, 2.1–2.3	2
	TOTAL			17 Lectures

Course Plan (roughly after the Mid-sem exam)

Sr. No.	Topic	Sections from Text [TF]	Sections from [GL-2]	No. of Lectures
11.	Partial and Total Differentiation	12.3-12.7	3.1–3.3	2
12.	Maxima, Minima	12.8, 12.9	4.1–4.3	2
13.	Multiple Integrals	13.1-13.3	5.1–5.3	2
14.	Scalar and vector fields, Line Integrals	14.1, 14.2		1.5
15.	Path independence and conservative vector fields	14.3		1.5
16.	Green's theorem and its applications	14.4		2
17.	Surfaces area and surface integrals	14.5		2
18.	Parametrized surfaces	14.6		1
19.	Stokes' theorem and its applications	14.7		2
20.	Divergence theorem and its applications	14.8		2
	TOTAL			18 Lectures

Lectures and Tutorials

Every week we have either three lectures of about one hour duration each or two lectures of about one and half hour duration each. In addition, there will be a tutorial of one hour duration. The mode of lectures will be new to you and puts more responsibility on you. It may not be possible for you to take down notes of each lecture fully. At the same time, the course will be fast paced. Thus it is extremely important that you remain attentive in the class and do not miss a lecture. Consult the text book (and if you wish, the reference books) regularly. Sufficient copies of these have been kept in the Central Library. Also, be sure to consult the Moodle (`http://moodle.iitb.ac.in/`) page of the course regularly, at least once a week.

For the purpose of tutorials, each division will be divided into 6 batches. Each batch will be assigned to a "course associate". The aim of the tutorials is to clear your doubts and to give you practice for problem solving. Based on the material covered, certain problems from the tutorial sheets in this booklet will be assigned to you each week. You are expected to try the problems before coming to the tutorial class. In case you have doubts, please seek the help of your course associate.

Policy for Attendance

Attendance in lectures and tutorials is compulsory. Please ensure that your attendance is marked on the biometric machines located in the lecture hall, and that this is done no later than the first five minutes of the class. Students who do not meet 80% attendance requirement may be automatically given a failing grade.

In case you miss lectures for valid (medical) reasons, get a medical certificate from the IIT Hospital and keep it with you. You can produce it if you fall short of attendance.

Chronological Day versus Teaching Day

Please note that as per the academic calender, on certain chronological days, time-table of another teaching days will be followed. This is partly to take into account the holidays falling more frequently on certain weekdays. The special days where there will be a change are as follows.

Chronological Day	Instructional Timetable to be followed
6 September 2019 (Friday)	Monday
28 October 2019 (Monday)	Tuesday

Extra lectures and tutorials

We are tentatively planning one extra class in the week of August 12 - 18, 2019 for Divisions I and II, and an extra tutorial on October 5 or October 6, 2019. Further details will be communicated during the class.

Evaluation Plan

1. There will be **two quizzes** common for all the four divisions. Each quiz will be of 40 minutes duration and will carry **10% weightage**. Syllabi for Quiz I and Quiz 2 will be announced later in the class. Tentative dates for the two quizzes are September 6 and October 25, 2019.
2. The **Mid-Semester examination**, scheduled to be held during the week of **September 16-20, 2019** will be of **30% weightage**. The portion for Mid-Sem examination will be announced later in the class. The **End-Semester Examination, scheduled to be held during 11-22 November 2019** will be of **40% weightage**, and will cover all the topics.
3. **At the beginning of almost every tutorial**, there will be a short quiz based on the material covered in the previous tutorial hour. These quizzes together will have a **10% weightage**. Approximately 12 such quizzes will be conducted during the semester of which we will consider best 10, each of weight 1%. There will be no make-up quiz for these under any circumstances.

Instructors and their coordinates

Instructor for Division I and III: Prof. Sudhir R. Ghorpade (Instructor-in-charge)
Room No. 106-B, Department of Mathematics, Internal Phone : 7470

Instructor for Division II: Prof. Mayukh Mukherjee
Room No. B1-B, Department of Mathematics, Internal Phone: 9472

Instructor for Divison IV: Prof. Madhusudan Manjunath
Room No. 204-A, Department of Mathematics, Internal Phone: 9473

Timings for Lectures and Tutorials

Lectures of Division I and Division II are scheduled on Mondays and Thursdays during 2.00 pm to 3.25 pm in LA 001 for Division I and LA 002 for Division II.

Lecture of Divisions III and IV are scheduled on Mondays, Tuesdays, and Thursdays during 8.30-9.25 am, 9.30-10.25 am, and 10.35–11.30 am, respectively, in LA 001 for Division III, and LA 002 for Division IV.

Tutorials of all the Divisions are scheduled on Wednesdays during 2.00 pm to 3.00 pm. The venues for the tutorials will be LT 001 - 006, LT 101 - 106, LT 201 - 206 and LT 301 - 306.
Refer the notice boards and Moodle (`http://moodle.iitb.ac.in/`) page of the course for any changes or modifications.

Tutorial Sheets: 0-7

Tutorial sheet No. 0: Revision material on Real numbers

Mark the following statements as True/False:

(1) $+\infty$ and $-\infty$ are both real numbers.

(2) The set of all even natural numbers is bounded.

(3) The set $\{x\}$ is an open interval for every $x \in \mathbb{R}$.

(4) The set $\{2/m : m \in \mathbb{N}\}$ is bounded above.

(5) The set $\{2/m : m \in \mathbb{N}\}$ is bounded below.

(6) Union of intervals is also an interval.

(7) Nonempty intersection of intervals is also an interval.

(8) Nonempty intersection of open intervals is also an open interval.

(9) Nonempty intersection of closed intervals is also a closed interval.

(10) Nonempty finite intersection of closed intervals is also a closed interval.

(11) For every $x \in \mathbb{R}$, there exists a rational $r \in \mathbb{Q}$, such that $r > x$.

(12) Between any two rational numbers there lies an irrational number.

Tutorial Sheet No.1: Sequences

1. Using (ϵ-n_0) definition prove the following:
   (i) $\lim\limits_{n\to\infty} \frac{10}{n} = 0$
   (ii) $\lim\limits_{n\to\infty} \frac{5}{3n+1} = 0$
   (iii) $\lim\limits_{n\to\infty} \frac{n^{2/3}\sin(n!)}{n+1} = 0$
   (iv) $\lim\limits_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

2. Show that the following limits exist and find them :
   (i) $\lim\limits_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$
   (ii) $\lim\limits_{n\to\infty} \left(\frac{n!}{n^n}\right)$
   (iii) $\lim\limits_{n\to\infty} \left(\frac{n^3+3n^2+1}{n^4+8n^2+2}\right)$
   (iv) $\lim\limits_{n\to\infty} (n)^{1/n}$
   (v) $\lim\limits_{n\to\infty} \left(\frac{\cos\pi\sqrt{n}}{n^2}\right)$
   (vi) $\lim\limits_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\right)$

3. Show that the following sequences are not convergent :
   (i) $\left\{\frac{n^2}{n+1}\right\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$

4. Determine whether the sequences are increasing or decreasing :
   (i) $\left\{\frac{n}{n^2+1}\right\}_{n\geq 1}$
   (ii) $\left\{\frac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$
   (iii) $\left\{\frac{1-n}{n^2}\right\}_{n\geq 2}$

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits :
   (i) $a_1 = 1, a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \forall\, n \geq 1$
   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n} \;\forall\; n \geq 1$
   (iii) $a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2} \;\forall\, n \geq 1$

6. If $\lim\limits_{n\to\infty} a_n = L$, find the following : $\lim\limits_{n\to\infty} a_{n+1}$, $\lim\limits_{n\to\infty} |a_n|$

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that
$$|a_n| \geq \frac{|L|}{2} \qquad \text{for all } n \geq n_0.$$
8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$.
Optional: State and prove a corresponding result if $a_n \to L > 0$.
9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following :
(i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.
(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.
10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Supplement

1. A sequence $\{a_n\}_{n\geq 1}$ is said to be **Cauchy** if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $m, n \geq n_0$.
In other words, the elements of a Cauchy sequence come arbitrarily close to each other after some stage. One can show that *every convergent sequence is also Cauchy and conversely, every Cauchy sequence in* $\mathbb{R}$ *is also convergent.* This is an equivalent way of stating the **Completeness property of real numbers**.)
2. To prove that a sequence $\{a_n\}_{n\geq 1}$ is convergent to L, one needs to find a real number L (not given by the sequences) and verify the required property. However the concept of 'Cauchyness' of a sequence is purely an 'intrinsic' property of the given sequence. Nonetheless a sequence of real numbers is Cauchy if and only if it is convergent.
3. In problem 5(i) we defined
$$a_0 = 1, \;\; a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \;\; \forall\, n \geq 1.$$
The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find rational approximations (in computing machines) of $\sqrt{2}$.

Tutorial Sheet No. 2:
Limits, Continuity and Differentiability

1. Let $a, b, c \in \mathbb{R}$ with $a < c < b$ and let $f, g : (a, b) \to \mathbb{R}$ be such that $\lim_{x\to c} f(x) = 0$. Prove or disprove the following statements.
   (i) $\lim_{x\to c} [f(x)g(x)] = 0$.
   (ii) $\lim_{x\to c} [f(x)g(x)] = 0$, if g is bounded.
   (iii) $\lim_{x\to c} [f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that
$$\lim_{h\to 0} [f(\alpha + h) - f(\alpha - h)] = 0.$$
   Analyze the converse.
3. Discuss the continuity of the following functions :
   (i) $f(x) = \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (ii) $f(x) = x \sin \frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$
   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad x = 2 \\ \sqrt{6-x} & \text{if} \quad 2 < x \le 3 \end{cases}$
4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.
   (Optional) Show that the function f satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.
5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?
6. Let $f : (a, b) \to \mathbb{R}$ be a function such that
$$|f(x + h) - f(x)| \le C|h|^\alpha \text{ for all } x, x + h \in (a, b),$$
   where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f^{'}(x)$ for $x \in (a, b)$.
7. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that
$$\lim_{h\to 0^+} \frac{f(c + h) - f(c - h)}{2h}$$
   exists and equals $f^{'}(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]
8. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy
$$f(x + y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.
(Optional) Show that f has a derivative of every order on $\mathbb{R}$.

9. Using the Theorem on derivative of inverse function. Compute the derivative of
(i) $\cos^{-1} x,\ -1 < x < 1$. (ii) $\operatorname{cosec}^{-1}x,\ |x| > 1$.

10. Compute $\dfrac{dy}{dx}$, given
$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f'(x) = \sin(x^2).$$

Optional Exercises:

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

12. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$
Show that f is discontinuous at every $c \in \mathbb{R}$.

13. **(Optional)**
Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$
Show that g is continuous only at $c = 1/2$.

14. **(Optional)**
Let $f : (a,b) \to \mathbb{R}$, $\alpha \in \mathbb{R}$ and $c \in (a,b)$ be such that $\lim_{x\to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c+h) > \alpha \text{ for all } 0 < |h| < \delta.$$
(See also question 7 of Tutorial Sheet 1.

15. **(Optional)** Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following are equivalent :
(i) f is differentiable at c.
(ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta,\delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and
$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta,\delta).$$
(iii) There exists $\alpha \in \mathbb{R}$ such that
$$\lim_{h\to 0}\left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|}\right) = 0.$$

Tutorial Sheet No. 3: Rolle's and Mean Value Theorems, Maximum/Minimum

1. Show that the cubic $x^3 - 6x + 3$ has all roots real.
2. Let p and q be two real numbers with $p > 0$. Show that the cubic x^3+px+q has exactly one real root.
3. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)$ and $f(b)$ are of different signs and $f^{'}(x) \neq 0$ for all $x \in (a, b)$, show that there is a unique $x_0 \in (a, b)$ such that $f(x_0) = 0$.
4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If $f(x)$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:
   (i) $p < 0$.
   (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.
   (iii) The maximum/minimum values are of opposite signs.
5. Use the MVT to prove $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f^{'}(c_1) + f^{'}(c_2) = 2$.
7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f^{'}(x)$ exists and $f^{'}(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.
   Optional: Show that under the given conditions, in fact $f(x) = x$ for every x.
8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.
   (i) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 1$
   (ii) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f^{'}(1) = 2$
   (iii) $f^{''}(x) \geq 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 100$ for all $x > 0$
   (iv) $f^{''}(x) > 0$ for all $x \in \mathbb{R}$, $f^{'}(0) = 1$, $f(x) \leq 1$ for all $x < 0$
9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on $[-2, 5]$. Verify it from the sketch of the curve $y = f(x)$ on $[-2, 5]$.
10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

Tutorial Sheet No. 4:
Curve Sketching, Riemann Integration

1. Sketch the following curves after locating intervals of increase/decrease, intervals of concavity upward/downward, points of local maxima/minima, points of inflection and asymptotes. How many times and approximately where does the curve cross the x-axis?

   (i) $y = 2x^3 + 2x^2 - 2x - 1$

   (ii) $y = \dfrac{x^2}{x^2+1}$

   (iii) $y = 1 + 12|x| - 3x^2,\ x \in [-2, 5]$

2. Sketch a continuous curve $y = f(x)$ having all the following properties:

   $f(-2) = 8,\ f(0) = 4,\ f(2) = 0;\ \ f^{'}(2) = f^{'}(-2) = 0;$

   $f^{'}(x) > 0$ for $|x| > 2,\ f^{'}(x) < 0$ for $|x| < 2;$

   $f^{''}(x) < 0$ for $x < 0$ and $f^{''}(x) > 0$ for $x > 0$.

3. Give an example of $f : (0, 1) \to \mathbb{R}$ such that f is

   (i) strictly increasing and convex.

   (ii) strictly increasing and concave.

   (iii) strictly decreasing and convex.

   (iv) strictly decreasing and concave.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \mathbb{R}$. Define $h(x) = f(x)g(x)$ for $x \in \mathbb{R}$. Which of the following statements are true? Why?

   (i) If f and g have a local maximum at $x = c$, then so does h.

   (ii) If f and g have a point of inflection at $x = c$, then so does h.

5. Let $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 2$ if $x \in (1, 2]$. Show from the first principles that f is Riemann integrable on $[0, 2]$ and find $\displaystyle\int_0^2 f(x)dx$.

6. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\displaystyle\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\displaystyle\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

   (b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

7. Evaluate $\lim\limits_{n\to\infty} S_n$ by showing that S_n is an approximate Riemann sum for a suitable function over a suitable interval:

   (i) $S_n = \dfrac{1}{n^{5/2}} \displaystyle\sum_{i=1}^{n} i^{3/2}$

(ii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{n}{i^2 + n^2}$

(iii) $S_n = \displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

(iv) $S_n = \displaystyle\frac{1}{n} \sum_{i=1}^{n} \cos\frac{i\pi}{n}$

(v) $S_n = \displaystyle\frac{1}{n} \left\{ \sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^{2} \right\}$

8. Compute

   (a) $\displaystyle\frac{d^2y}{dx^2}$, if $x = \displaystyle\int_0^y \frac{dt}{\sqrt{1+t^2}}$

   (b) $\displaystyle\frac{dF}{dx}$, if for $x \in \mathbb{R}$ (i) $F(x) = \displaystyle\int_1^{2x} \cos(t^2)dt$ (ii) $F(x) = \displaystyle\int_0^{x^2} \cos(t)dt.$

9. Let p be a real number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\displaystyle\int_a^{a+p} f(t)dt$ has the same value for every real number a. (Hint : Consider $F(a) = \displaystyle\int_a^{a+p} f(t)dt,\ a \in \mathbb{R}$.)

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R},\ \lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x-t)dt.$$

   Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = 0 = g'(0)$.

Tutorial Sheet No. 5:
Applications of Integration

(1) Find the area of the region bounded by the given curves in each of the following cases.
 (i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$
 (ii) $y = x^4 - 2x^2$ and $y = 2x^2$.
 (iii) $x = 3y - y^2$ and $x + y = 3$

(2) Let $f(x) = x - x^2$ and $g(x) = ax$. Determine a so that the region above the graph of g and below the graph of f has area 4.5

(3) Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1 + \cos\theta)$.

(4) Find the arc length of the each of the curves described below.
 (i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$
 (ii) $y = \displaystyle\int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4$.

(5) For the following curve, find the arc length as well as the the area of the surface generated by revolving it about the line $y = -1$.
$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$$

(6) The cross sections of a certain solid by planes perpendicular to the x-axis are circles with diameters extending from the curve $y = x^2$ to the curve $y = 8 - x^2$. The solid lies between the points of intersection of these two curves. Find its volume.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.

(8) A fixed line L in 3-space and a square of side r in a plane perpendicular to L are given. One vertex of the square is on L. As this vertex moves a distance h along L, the square turns through a full revolution with L as the axis. Find the volume of the solid generated by this motion.

(9) Find the volume of the solid generated when the region bounded by the curves $y = 3 - x^2$ and $y = -1$ is revolved about the line $y = -1$, by both the Washer Method and the Shell Method.

(10) A round hole of radius $\sqrt{3}$ cms is bored through the center of a solid ball of radius 2 cms. Find the volume cut out.

Tutorial Sheet No. 6:
Functions of two variables, Limits, Continuity

(1) Find the natural domains of the following functions of two variables:

$$(i)\ \frac{xy}{x^2-y^2} \qquad (ii)\ \ln(x^2+y^2)$$

(2) Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:
(i) $f(x, y) = x - y$ (ii) $f(x, y) = x^2 + y^2$ (iii) $f(x, y) = xy$

(3) Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

$$(i)\ \frac{x^3y}{x^6+y^2} \quad (ii)\ xy\frac{x^2-y^2}{x^2+y^2} \quad (iii)\ ||x|-|y||-|x|-|y|.$$

(4) Suppose $f,\, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. Show that each of the following functions of $(x, y) \in \mathbb{R}^2$ are continuous:
(i) $f(x) \pm g(y)$ (ii) $f(x)g(y)$ (iii) $\max\{f(x),\, g(y)\}$
(iv) $\min\{f(x),\, g(y)\}$.

(5) Let

$$f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2} \text{ for } (x,y) \neq (0,0).$$

Show that the iterated limits

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \text{ and } \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

exist and both are equal to 0, but $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(6) Examine the following functions for the existence of partial derivatives at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero.
(i) $xy\dfrac{x^2-y^2}{x^2+y^2}$
(ii) $\dfrac{\sin^2(x+y)}{|x|+|y|}$

(7) Let $f(0, 0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0, 0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0, 0)$.

(8) Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y), & \text{if } x \neq 0,\ y \neq 0 \\ x \sin 1/x, & \text{if } x \neq 0,\ y = 0 \\ y \sin 1/y, & \text{if } y \neq 0,\ x = 0. \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

(9) Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

$$(i)\ \ xy\frac{x^2-y^2}{x^2+y^2} \qquad (ii)\ \ \frac{x^3}{x^2+y^2} \qquad (iii)\ \ (x^2+y^2)\sin\frac{1}{x^2+y^2}$$

(10) Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $D_{\underline{u}}f(0,0)$ exists for every vector $\underline{u}$, yet f is not differentiable at $(0,0)$.

Tutorial Sheet No. 7: Maxima, Minima, Saddle Points

(1) Let $F(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $F(x,y,z) = 7$ at $(1,-1,3)$.

(2) Find $D_{\underline{u}}F(2,2,1)$, where $F(x,y,z) = 3x-5y+2z$, and $\underline{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

(3) Given $\sin(x+y) + \sin(y+z) = 1$, find $\dfrac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

(4) If $f(0,0) = 0$ and

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x,y) \neq (0,0),$$

show that both f_{xy} and f_{yx} exist at $(0,0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0,0)$?

(5) Show that the following functions have local minima at the indicated points.

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7, \quad (x_0, y_0) = (-1, 2)$

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3, \quad (x_0, y_0) = (0,0)$

(6) Analyze the following functions for local maxima, local minima and saddle points :

(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$ (ii) $f(x,y) = x^3 - 3xy^2$

(7) Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2 - 4x)\cos y \text{ for } 1 \leq x \leq 3,\ -\pi/4 \leq y \leq \pi/4.$$

(8) The temperature at a point (x,y,z) in 3-space is given by $T(x,y,z) = 400xyz$. Find the highest temperature on the unit sphere $x^2+y^2+z^2 = 1$.

(9) Maximize the $f(x,y,z) = xyz$ subject to the constraints

$$x + y + z = 40 \text{ and } x + y = z.$$

(10) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraints

$$x + 2y + 3z = 6 \text{ and } x + 3y + 4z = 9.$$

Answers: Tutorial Sheets 0-7

Tutorial sheet No. 0:

(1) False

(2) False

(3) False

(4) True

(5) True.

(6) False

(7) True

(8) False

(9) True

(10) True.

(11) True

(12) True

Tutorial sheet No. 1

(1) For a given $\epsilon > 0$, select $n_0 \in \mathbb{N}$ satisfying:

$$\text{(i) } n_0 > \frac{10}{\epsilon}, \quad \text{(ii) } n_0 > \frac{5-\epsilon}{3\epsilon}, \quad \text{(iii) } n_0 > \frac{1}{\epsilon^3}\,, \quad \text{(iv) } n_0 > \frac{2}{\epsilon}.$$

(2) The limits of the sequences are as follows:

(i) 1, (ii) 0, (iii) 0, (iv) 1, (v) 0, (vi) 1/2.

(3) (i) Not convergent, (ii) Not convergent.

(4) (i) Decreasing, (ii) Increasing, (iii) Increasing.

(5) Hint: In each case, use induction on n to show that $\{a_n\}$ is bounded and monotonic. The limits are:
(i) $\sqrt{2}$, (ii) 2, (iii) 6.

(7) Hint: Consider $\epsilon = L/2$.

(9) Both the statements are **False**.

Tutorial sheet No. 2

(1) (i) False, (ii) True, (iii) True.

(2) Yes. The converse is **False**.

(3) (i) Not continuous at $\alpha = 0$
(ii) Continuous
(iii) Continuous everywhere except at $x = 2$

(5) Continuous for $x \neq 0$, not continuous at $x = 0$.

(7) The converse is **False** .

(9) (i) $\dfrac{-1}{\sqrt{1-\cos^2(x)}} = \dfrac{-1}{\sqrt{1-y^2}}$, (ii) $\dfrac{1}{\sqrt{(1-\frac{1}{x^2})}}\left(\dfrac{-1}{x^2}\right)$, $|x| > 1$.

(10) $\dfrac{3}{(x+1)^2}\sin\left(\dfrac{2x-1}{x+1}\right)^2$.

Tutorial sheet No. 3

(8) (i) Not possible
(ii) Possible
(iii) Not possible
(iv) Possible

(9) The absolute maximum is 13 which is attained at $x = \pm 2$ and the absolute minimum is -14 which is attained at $x = 5$.

(10) $h = \dfrac{p(4+\pi)}{2(8+3\pi)}$.

Tutorial sheet No. 4

(4) (i) **True** .
(ii) **False**; consider $f(x) = g(x) = 1 + \sin(x), c = 0$.

(7) (i) $\dfrac{2}{5}$, (ii) $\dfrac{\pi}{4}$, (iii) $2(\sqrt{2}-1)$, (iv) 0, (v) $\dfrac{1}{2} + \dfrac{2}{5}(4\sqrt{2}-1) + \dfrac{19}{3}$.

(8) (a) $\dfrac{y}{\sqrt{1+y^2}}\dfrac{dy}{dx} = y$
(b) (i) $F'(x) = 2\cos(4x^2)$, (ii) $F'(x) = 2x\cos(x^2)$.

Tutorial Sheet No. 5

(1) (i) $\frac{1}{6}$, (ii) $\frac{128}{15}$, (iii) $\frac{4}{3}$.

(2) $a = -2$

(3) $4\pi a^2$

(4) (i) 8, (ii) 1.

(5) $\frac{dy}{dx} = x^2 + \left(-\frac{1}{4x^2}\right)$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \frac{1}{16x^4} - \frac{1}{2}} = x^2 + \frac{1}{4x^2}$$

Therefore $L = \frac{53}{6}$. The surface area is $\left(101\frac{5}{18}\right)\pi$.

(6) $\frac{512\pi}{15}$

(7) Volume is $\frac{16a^3}{3}$

(8) Volume is r^2h

(9) $2^9\pi/15$

(10) $\frac{28\pi}{3}$.

Tutorial Sheet No. 6

(1) (i) $\{(x, y) \in \mathbb{R}^2 \mid x \neq \pm y\}$.
(ii) $\mathbb{R}^2 - \{(0, 0)\}$

(2) (i) Level curves are parallel lines $x - y = c$. Contours are the same lines shifted to $z = c$. (same c)

(ii) Level curves do not exist for $c \leq -1$. It is just a point for $c = 0$ and are concentric circles for $c = 1, 2, 3, 4$. Contours are the sections of paraboloid of revolution $z = x^2 + y^2$ by $z = c$, i.e., concentric circles in the plane $z = c$.

(iii) Level curves are rectangular hyperbolas. Branches are in first and third quadrant for for $c > 0$ and in second and fourth quadrant for $c < 0$. For $c = 0$ it is the union of x-axis and y-axis.

(3) (i) Discontinuous at $(0, 0)$
(ii) Continuous at $(0, 0)$
(iii) Continuous at $(0, 0)$

(6) (i) $f_x(0, 0) = 0 = f_y(0, 0)$.
(ii) f is continuous at $(0, 0)$. Both $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

(9) (i) $(D_{\vec{v}}f)(0,0)$ exists and equals 0 for every $\vec{v} \in \mathbb{R}^2$;
f is also differentiable at $(0,0)$.
(ii) It is not differentiable, but for every vector $\vec{v} = (a,b)$, $D_{\vec{v}}f(0,0)$ exists.
(iii) $(D_{\vec{v}}f)(0,0) = 0$; f is differentiable at $(0,0)$.

Tutotial Sheet No. 7

(1) Tangent plane:

$$0.(x-1) + 4(y+1) + 6(z-3) = 0, \text{ i.e., } 2y + 3z = 7$$

Normal line: $x = 1,\ 3y - 2z + 9 = 0$

(2) $-\dfrac{2}{3}$

(3) $\dfrac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\dfrac{\cos^2(x+y)}{\cos^2(y+z)}$.

(6) (i) $(0,0)$ is a saddle point;
$(\pm\sqrt{2},0)$ are local maxima;
$((0,\pm\sqrt{2})$ are local minima.
(ii) $(0,0)$ is a saddle point.

(7) $f_{\min} = -4$ at $(2,0)$ and $f_{\max} = -3/\sqrt{2}$ at $(3,\pm\pi/4)$

(8) $T_{\max} = \dfrac{400}{3\sqrt{3}}$ at

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\ \left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),$$
$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\ \left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

(9) $F(10,10,20) = 2000$ is the maximum value.

(10) At $(x,y,z) = (-1,2,1)$, $f(-1,2,1) = 6$ is the minimum value.

Tutorial sheets 8-14

Tutorial Sheet No. 8:
Multiple Integrals

(1) For the following, write an equivalent iterated integral with the order of integration reversed:

(i) $\displaystyle\int_0^1 \left[\int_1^{e^x} dy\right] dx$

(ii) $\displaystyle\int_0^1 \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)dx\right] dy$

(2) Evaluate the following integrals

(i) $\displaystyle\int_0^\pi \left[\int_x^\pi \frac{\sin y}{y} dy\right] dx$

(ii) $\displaystyle\int_0^1 \left[\int_y^1 x^2 e^{xy} dx\right] dy$

(iii) $\displaystyle\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x) dx.$

(3) Find $\displaystyle\iint_D f(x,y)d(x,y)$, where $f(x,y) = e^{x^2}$ and D is the region bounded by the lines $y = 0,\ x = 1$ and $y = 2x$.

(4) Evaluate the integral

$$\iint_D (x-y)^2 \sin^2(x+y) d(x,y),$$

where D is the parallelogram with vertices at $(\pi, 0),\ (2\pi, \pi),\ (\pi, 2\pi)$ and $(0, \pi)$.

(5) Let D be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1,\ xy = 9$ and the lines $y = x,\ y = 4x$. Find $\iint_D d(x,y)$ by transforming it to $\iint_E d(u,v)$, where $x = \dfrac{u}{v},\ y = uv,\ v > 0$.

(6) Find

$$\lim_{r\to\infty} \iint_{D(r)} e^{-(x^2+y^2)} d(x,y),$$

where $D(r)$ equals:

(i) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2\}$.

(ii) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le r^2,\ x \ge 0,\ y \ge 0\}$.

(iii) $\{(x,y) \in \mathbb{R}^2 : |x| \le r,\ |y| \le r\}$.

(iv) $\{(x,y) \in \mathbb{R}^2 : 0 \le x \le r,\ 0 \le y \le r\}$.

(7) Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ using double integral over a region in the plane. (Hint: Consider the part in the first octant.)

(8) Express the solid $D = \{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \le z \le 1\}$ as

$$\{(x,y,z) \in \mathbb{R}^3 : a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x),\ \xi_1(x,y) \le z \le \xi_2(x,y)\}.$$

for suitable functions $\phi_1, \phi_2, \xi_1, \xi_2$

(9) Evaluate

$$I = \int_0^{\sqrt{2}} \left(\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x dz \right) dy \right) dx.$$

Sketch the region of integration and evaluate the integral by expressing the order of integration as $dxdydz$.

(10) Using suitable change of variables, evaluate the following:

(i) The triple integral

$$I = \iiint_D (z^2x^2 + z^2y^2) dxdydz$$

where D is the cylindrical region $x^2+y^2 \le 1$ bounded by $-1 \le z \le 1$.

(ii) The triple integral

$$I = \iiint_D \exp(x^2 + y^2 + z^2)^{3/2} dxdydz$$

where D is the region enclosed by the unit sphere in $\mathbb{R}^3$.

Tutorial Sheet No. 9:
Scalar and Vector fields, Line Integrals

(1) Let $\mathbf{a}, \mathbf{b}$ be two fixed vectors, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r^2 = x^2 + y^2 + z^2$. Prove the following:
  (i) $\nabla(r^n) = nr^{n-2}\mathbf{r}$ for any integer n.
  (ii) $\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right) = -\left(\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$.
  (iii) $\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \left(\frac{1}{r}\right)\right) = \frac{3(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$.

(2) For any two scalar functions f, g on $\mathbb{R}^m$ establish the relations:
  (i) $\nabla(fg) = f\nabla g + g\nabla f$;
  (ii) $\nabla f^n = nf^{n-1}\nabla f$;
  (iii) $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$ whenever $g \neq 0$.

(3) Prove the following:
  (i) $\nabla \cdot (f\mathbf{v}) = f\nabla \cdot \mathbf{v} + (\nabla f) \cdot \mathbf{v}$
  (ii) $\nabla \times (f\mathbf{v}) = f(\nabla \times \mathbf{v}) + \nabla f \times \mathbf{v}$
  (iii) $\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v}$,
  where $\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.
  (iv) $\nabla \cdot (f\nabla g) - \nabla \cdot (g\nabla f) = f\nabla^2 g - g\nabla^2 f$
  (v) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$
  (vi) $\nabla \times (\nabla f) = 0$
  (vii) $\nabla \cdot (g\nabla f \times f\nabla g) = 0$.

(4) Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Show that
  (i) $\nabla^2 f = \text{div } (\nabla f(r)) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$
  (ii) $\text{div } (r^n\mathbf{r}) = (n+3)r^n$
  (iii) $\text{curl}(r^n\mathbf{r}) = 0$
  (iv) $\text{div } (\nabla\frac{1}{r}) = 0$ for $r \neq 0$.

(5) Prove that
  (i) $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$
  Hence, if $\mathbf{u}, \mathbf{v}$ are irrotational, $\mathbf{u} \times \mathbf{v}$ is solenoidal.
  (ii) $\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v}$.
  (iii) $\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v})$.
  Hint: Write $\nabla = \sum \mathbf{i}\frac{\partial}{\partial x}$, $\nabla \times \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \times \mathbf{v}$ and $\nabla \cdot \mathbf{v} = \sum \mathbf{i}\frac{\partial}{\partial x} \cdot \mathbf{v}$

(6) (i) If $\mathbf{w}$ is a vector field of constant direction and $\nabla \times \mathbf{w} \neq 0$, prove that $\nabla \times \mathbf{w}$ is always orthogonal to $\mathbf{w}$.
  (ii) If $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ for a constant vector $\mathbf{w}$, prove that $\nabla \times \mathbf{v} = 2\mathbf{w}$.
  (iii) If $\rho\mathbf{v} = \nabla p$ where $\rho(\neq 0)$ and p are continuously differentiable scalar functions, prove that

$$\mathbf{v} \cdot (\nabla \times \mathbf{v}) = 0.$$

(7) Calculate the line integral of the vector field

$$f(x, y) = (x^2 - 2xy)\mathbf{i} + (y^2 - 2xy)\mathbf{j}$$

from $(-1, 1)$ to $(1, 1)$ along $y = x^2$.

(8) Calculate the line integral of

$$f(x, y) = (x^2 + y^2)\mathbf{i} + (x - y)\mathbf{j}$$

once around the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ in the counter clockwise direction.

(9) Calculate the value of the line integral

$$\oint_C \frac{(x + y)dx - (x - y)dy}{x^2 + y^2}$$

where C is the curve $x^2 + y^2 = a^2$ traversed once in the counter clockwise direction.

(10) Calculate

$$\oint_C ydx + zdy + xdz$$

where C is the intersection of two surfaces $z = xy$ and $x^2+y^2 = 1$ traversed once in a direction that appears counter clockwise when viewed from high above the xy-plane.

Tutorial Sheet No. 10:
Line integrals and applications

(1) Consider the helix

$$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\,\mathbf{k} \text{ lying on } x^2 + y^2 = a^2.$$

Parameterize this in terms of arc length.

(2) Evaluate the line integral

$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2 + y^2)^2}$$

where C is the square with vertices $(\pm 1, \pm 1)$ oriented in the counterclockwise direction.

(3) Let $\mathbf{n}$ denote the outward unit normal to $C : x^2 + y^2 = 1$. Find

$$\oint_C \text{grad } (x^2 - y^2) \cdot d\mathbf{n}.$$

(4) Evaluate

$$\oint_C \text{grad } (x^2 - y^2) \cdot d\mathbf{r}$$

where C is the path along the cubic $y = x^3$ from $(0, 0)$ to $(2, 8)$.

(5) Compute the line integral

$$\oint_C \frac{dx + dy}{|x| + |y|}$$

where C is the square with vertices $(1, 0), (0, 1)$, $(-1, 0)$ and $(0, -1)$ traversed once in the counter clockwise direction.

(6) A force $F = xy\mathbf{i} + x^6 y^2 \mathbf{j}$ moves a particle from $(0, 0)$ onto the line $x = 1$ along $y = ax^b$ where $a, b > 0$. If the work done is independent of b find the value of a.

(7) Calculate the work done by the force field $F(x, y, z) = y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$ along the curve C of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$ where $z \geq 0, a > 0$ (specify the orientation of C that you use.)

(8) Determine whether or not the vector field $f(x, y) = 3xy\mathbf{i} + x^3 y\mathbf{j}$ is a gradient on any open subset of $\mathbb{R}^2$.

(9) Let $S = \mathbb{R}^2 \setminus \{(0, 0)\}$. Let

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} := f_1(x, y)\mathbf{i} + f_2(x, y)\mathbf{j}.$$

Show that $\dfrac{\partial}{\partial y} f_1(x, y) = \dfrac{\partial}{\partial x} f_2(x, y)$ on S while $\mathbf{F}$ is not the gradient of a scalar field on S.

(10) For $\mathbf{v} = (2xy + z^3)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$, show that $\nabla\phi = \mathbf{v}$ for some ϕ and hence calculate $\oint_C \mathbf{v} \cdot d\mathbf{r}$ where C is any arbitrary smooth closed curve.

(11) A radial force field is one which can be expressed as $\mathbf{F} = f(r)\mathbf{r}$ where $\mathbf{r}$ is the position vector and $r = \|\mathbf{r}\|$. Show that $\mathbf{F}$ is conservative if f is continuous.

Tutorial Sheet No. 11:
Green's theorem and its applications

(1) Verify Green's theorem in each of the following cases:
 (i) $f(x,y) = -xy^2$; $g(x,y) = x^2y$; $R:\ x \geq 0,\ 0 \leq y \leq 1 - x^2$;
 (ii) $f(x,y) = 2xy$; $g(x,y) = e^x + x^2$; where R is the triangle with vertices $(0,0), (1,0)$, and $(1,1)$.

(2) Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2\,dx + x\,dy$ where:
 (i) R is the square with vertices $(0,0), (2,0), (2,2), (0,2)$.

 (ii) R is the square with vertices $(\pm 1, \pm 1)$.

 (iii) R is the disc of radius 2 and center $(0,0)$ (specify the orientation you use for the curve.)

(3) For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2} \oint_C r^2 d\theta.$$

Use this to compute the area enclosed by the following curves:
 (i) The cardioid: $r = a(1 - \cos\theta),\ 0 \leq \theta \leq 2\pi;.$
 (ii) The lemniscate: $r^2 = a^2 \cos 2\theta,;\ -\pi/4 \leq \theta \leq \pi/4$.

(4) Find the area of the following regions:
 (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos\theta)$.
 (ii) The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j},\ 0 \leq t \leq 2\pi.$$

 (iii) The region bounded by the limaçon

$$r = 1 - 2\cos\theta,\ 0 \leq \theta \leq \pi/2$$

 and the two axes.

(5) Evaluate

$$\oint_C xe^{-y^2} dx + [-x^2ye^{-y^2} + 1/(x^2 + y^2)]dy$$

around the square determined by $|x| \leq a$, $|y| \leq a$ traced in the counter clockwise direction.

(6) Let C be a simple closed curve in the xy-plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx,$$

where I_0 is the polar moment of inertia of the region R enclosed by C.

(7) Consider $a = a(x,y)$, $b = b(x,y)$ having continuous partial derivatives on the unit disc D. If

$$a(x,y) \equiv 1,\ b(x,y) \equiv y$$

on the boundary circle C, and

$$\mathbf{u} = a\mathbf{i} + b\mathbf{j};\ \mathbf{v} = (a_x - a_y)\mathbf{i} + (b_x - b_y)\mathbf{j}, \mathbf{w} = (b_x - b_y)\mathbf{i} + (a_x - a_y)\mathbf{j},$$

find

$$\iint_D \mathbf{u}\cdot\mathbf{v}\, dxdy \ \text{ and } \iint_D \mathbf{u}\cdot\mathbf{w}\, dxdy.$$

(8) Let C be any closed curve in the plane. Compute $\displaystyle\oint_C \nabla(x^2 - y^2)\cdot \mathbf{n}ds$.

(9) Recall the Green's Identities:

(i) $\displaystyle\iint_R \nabla^2 w\, dxdy = \oint_{\partial R} \frac{\partial w}{\partial \mathbf{n}}\, ds.$

(ii) $\displaystyle\iint_R [w\nabla^2 w + \nabla w\cdot\nabla w]\, dxdy = \oint_{\partial R} w\frac{\partial w}{\partial \mathbf{n}}\, ds.$

(iii) $\displaystyle\oint_{\partial R}\left(v\frac{\partial w}{\partial \mathbf{n}} - w\frac{\partial v}{\partial \mathbf{n}}\right) ds = \iint_R (v\nabla^2 w - w\nabla^2 v)\, dxdy.$

(a) Use (i) to compute

$$\oint_C \frac{\partial w}{\partial \mathbf{n}}\, ds$$

for $w = e^x \sin y$, and R the triangle with vertices $(0,0), (4,2), (0,2)$.

(b) Let D be a plane region bounded by a simple closed curve C and let $\mathbf{F}, \mathbf{G} : U \longrightarrow \mathbb{R}^2$ be smooth functions where U is a region containing $D \cup C$ such that

$$\operatorname{curl}\mathbf{F} = \operatorname{curl}\mathbf{G},\ \operatorname{div}\,\mathbf{F} = \operatorname{div}\,\mathbf{G} \ \text{ on } \ D\cup C$$

and

$$\mathbf{F}\cdot\mathbf{N} = \mathbf{G}\cdot\mathbf{N} \ \text{ on } \ C,$$

where $\mathbf{N}$ is the unit normal to the curve. Show that $\mathbf{F} = \mathbf{G}$ on D.

(10) Evaluate the following line integrals where the loops are traced in the counter clockwise sense

(i)

$$\oint_C \frac{y\, dx - x\, dy}{x^2 + y^2}$$

where C is any simple closed curve not passing through the origin.

(ii)
$$\oint_C \frac{x^2 y dx - x^3 dy}{(x^2+y^2)^2},$$
where C is the square with vertices $(\pm 1, \pm 1)$.

(iii) Let C be a smooth simple closed curve lying in the annulus $1 < x^2 + y^2 < 2$. Find
$$\oint_C \frac{\partial(\ln r)}{\partial y} dx - \frac{\partial(\ln r)}{\partial x} dy.$$

Tutorial Sheet No. 12: Surface area and surface integrals

(1) Find a suitable parameterization $\mathbf{r}(u, v)$ and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$ for the following surface:

(i) The plane $x - y + 2z + 4 = 0$.

(ii) The right circular cylinder $y^2 + z^2 = a^2$.

(iii) The right circular cylinder of radius 1 whose axis is along the line $x = y = z$.

(2) (a) For a surface S let the unit normal $\mathbf{n}$ at every point make the same acute angle α with z-axis. Let SA_{xy} denote the area of the projection of S onto the xy plane. Show that SA, the area of the surface S satisfies the relation: $SA_{xy} = SA \cos\alpha$.

(b) Let S be a parallelogram not parallel to any of the coordinate planes. Let S_1, S_2, and S_3 denote the areas of the projections of S on the three coordinate planes. Show that the ares of S is $\sqrt{S_1^2 + S_2^2 + S_3^2}$.

(3) Compute the surface area of that portion of the sphere $x^2+y^2+z^2 = a^2$ which lies within the cylinder $x^2 + y^2 = ay$, where $a > 0$.

(4) A parametric surface S is described by the vector equation

$$\mathbf{r}(u, v) = u\cos v\ \mathbf{i} + u\sin v\ \mathbf{j} + u^2\mathbf{k},$$

where $0 \le u \le 4$ and $0 \le v \le 2\pi$.

(i) Show that S is a portion of a surface of revolution. Make a sketch and indicate the geometric meanings of the parameters u and v on the surface.

(ii) Compute the vector $\mathbf{r}_u \times \mathbf{r}_v$ in terms of u and v.

(iii) The area of S is $\dfrac{\pi}{n}(65\sqrt{65} - 1)$ where n is an integer. Compute the value of n.

(5) Compute the area of that portion of the paraboloid $x^2+z^2 = 2ay$ which is between the planes $y = 0$ and $y = a$.

(6) A sphere is inscribed in a right circular cylinder. The sphere is sliced by two parallel planes perpendicular the axis of the cylinder. Show that the portions of the sphere and the cylinder lying between these planes have equal surface areas.

(7) Let S denote the plane surface whose boundary is the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, and let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Let $\mathbf{n}$ denote the unit normal to S having a nonnegative z-component. Evaluate the surface integral $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, using

(i) The vector representation $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + (1 - 2u)\mathbf{k}$.

(ii) An explicit representation of the form $z = f(x, y)$.

(8) Let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$ and $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + x^2\,\mathbf{k}$. Compute

$$\int\int_S \mathbf{F} \cdot \mathbf{n} dS\,,$$

where $\mathbf{n}$ is the unit outward normal to S.

(9) A fluid flow has flux density vector

$$\mathbf{F}(x, y, z) = x\mathbf{i} - (2x + y)\mathbf{j} + z\mathbf{k}.$$

Let S denote the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and let $\mathbf{n}$ denote the unit normal that points out of the sphere. Calculate the mass of the fluid flowing through S in unit time in the direction of $\mathbf{n}$.

(10) Solve the previous exercise when S includes the the planar base of the hemisphere also with the outward unit normal on the base being $-\mathbf{k}$.

Tutorial Sheet No. 13: Stoke's theorem and applications

(1) Consider the vector field $\mathbf{F} = (x-y)\mathbf{i} + (x+z)\mathbf{j} + (y+z)\mathbf{k}$. Verify Stokes theorem for $\mathbf{F}$ where S is the surface of the cone: $z^2 = x^2 + y^2$ intercepted by: (a) $x^2 + (y-a)^2 + z^2 = a^2 : z \geq 0$, (b) $x^2 + (y-a)^2 = a^2$

(2) Evaluate using Stokes Theorem, the line integral

$$\oint_C yz\,dx + xz\,dy + xy\,dz$$

where C is the curve of intersection of $x^2 + 9y^2 = 9$ and $z = y^2 + 1$ with clockwise orientation when viewed from the origin.

(3) Compute

$$\iint_S (\operatorname{curl} \mathbf{v}) \cdot \mathbf{n} dS$$

where $\mathbf{v} = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k}$ and $\mathbf{n}$ is the outward unit normal to S, the surface of the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = -3$.

(4) Compute $\oint_C \mathbf{v} \cdot d\mathbf{r}$ for

$$\mathbf{v} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2},$$

where C is the circle of unit radius in the xy plane centered at the origin and oriented clockwise. Can the above line integral be computed using Stokes Theorem?

(5) Compute

$$\oint_C (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$

where C is the curve cut out of the boundary of the cube

$$0 \leq x \leq a,\ 0 \leq y \leq a,\ 0 \leq z \leq a$$

by the plane $x + y + z = \frac{3}{2}a$ (specify the orientation of C.)

(6) Calculate $\displaystyle\oint_C ydx + zdy + xdz$, where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

(7) Consider a plane with unit normal $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. For a closed curve C lying in this plane, show that the area enclosed by C is given by

$$A(C) = \frac{1}{2}\oint_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz,$$

where C is given the anti-clockwise orientation. Compute $A(C)$ for the curve C given by $\mathbf{u}\cos t + \mathbf{v}\sin t,\ 0 \leq t \leq 2\pi$.

Tutorial Sheet No. 14: Divergence theorem and its applications

(1) Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$$

for the region

$$R : y^2 + z^2 \le x^2; 0 \le x \le 4.$$

(2) Verify the Divergence Theorem for

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

for the region in the first octant bounded by the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(3) Let R be a region bounded by a piecewise smooth closed surface S with outward unit normal

$$\mathbf{n} = n_x\,\mathbf{i} + n_y\,\mathbf{j} + n_z\,\mathbf{k}.$$

Let $u, v : R \to \mathbb{R}$ be continuously differentiable. Show that

$$\iiint_R u\,\frac{\partial v}{\partial x} dV = -\iiint_R v\,\frac{\partial u}{\partial x} dV + \int_{\partial R} u\,v\,n_x\,dS.$$

[Hint: Consider $\mathbf{F} = u\,v\,\mathbf{i}$.]

(4) Suppose a scalar field ϕ, which is never zero has the properties

$$\|\nabla\phi\|^2 = 4\phi \text{ and } \nabla\cdot(\phi\nabla\phi) = 10\phi.$$

Evaluate $\displaystyle\iint_S \frac{\partial\phi}{\partial\mathbf{n}}\,dS$, where S is the surface of the unit sphere.

(5) Let V be the volume of a region bounded by a closed surface S and $\mathbf{n} = (n_x, n_y, n_z)$ be its outer unit normal. Prove that

$$V = \iint_S x\,n_x\,dS = \iint_S y\,n_y\,dS = \iint_S z\,n_z\,dS$$

(6) Let S be the surface of the cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$. Compute

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS\,,$$

where $\mathbf{n}$ is the unit outward normal to S.

(7) Let S be the unit sphere and $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$. Compute

$$\int\int_S \mathbf{F}\cdot\mathbf{n}dS\,,$$

where $\mathbf{n}$ is the unit outward normal to S.

(8) Let $\mathbf{u} = -x^3\mathbf{i} + (y^3 + 3z^2 \sin z)\mathbf{j} + (e^y \sin z + x^4)\mathbf{k}$ and S be the portion of the sphere $x^2 + y^2 + z^2 = 1$ with $z \geq \frac{1}{2}$ and $\mathbf{n}$ is the unit normal with positive z-component. Use Divergence theorem to compute $\displaystyle\iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n}\, dS$.

(9) Let p denote the distance from the origin to the tangent plane at the point (x, y, z) to the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$. Prove that

(a) $\displaystyle\iint_S p\, dS = 4\pi abc.$

(b) $\displaystyle\iint_S \frac{1}{p}\, dS = \frac{4\pi}{3abc}(b^2c^2 + c^2a^2 + a^2b^2)$

(10) Interpret Green's theorem as a divergence theorem in the plane.

Answers to Tutorial Sheets 8-14

Tutorial Sheet No. 8

(1) (i) $\displaystyle\int_1^e \left(\int_{\ln y}^1 dx \right) dy$

(ii) $\displaystyle\int_{-1}^1 \left(\int_{x^2}^1 f(x,y)dy \right) dx$

(2) (i) 2, (ii) $\dfrac{1}{2}(\exp(-2))$,

(iii) $\dfrac{\pi-1}{2\pi}\ln 5 + 2(\tan^{-1} 2\pi - \tan^{-1} 2) - \dfrac{1}{2\pi}\left[\ln \dfrac{(4\pi^2+1)}{5}\right]$.

(3) $\exp(-1)$

(4) $\dfrac{\pi^4}{3}$

(5) $8\ln 2$

(6) (i) π, (ii) $\dfrac{\pi}{4}$, (iii) π, (iv) $\dfrac{\pi}{4}$.

(7) $\dfrac{16a^3}{3}$

(8) $\{(x,y,z) : -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ \sqrt{x^2+y^2} \le z \le 1\}$.

(9) $\dfrac{8\sqrt{2}}{15}$. We can also write D as

$$\left\{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le 2,\ 0 \le x \le \sqrt{z-y^2},\ 0 \le y \le \sqrt{z}\right\}.$$

(10) (i) $\pi/3$, (ii) $4\pi(e-1)/3$.

Tutorial Sheet No. 9

(7) $-\frac{14}{15}$.

(8) 0.

(9) -2π.

(10) $-\pi$.

Tutorial Sheet No. 10

(1) The arc length parametrization is

$$\mathbf{r}(s) = a\cos\left(\frac{s}{\sqrt{a^2+c_2}}\right)\mathbf{i} + a\sin\left(\frac{s}{\sqrt{a^2+c_2}}\right)\mathbf{j} + \frac{cs}{\sqrt{a^2+c_2}}\mathbf{k}.$$

(2) $\displaystyle\int_C = \int_{C_1} + \int_{C_2} + \int_{C_4} = -\pi.$

(3) $\oint_C \nabla(x^2-y^2)\cdot d\mathbf{n} = 0.$

(4) $(x^2-y^2)\cdot d\mathbf{r} = -60.$

(5) $\displaystyle\int_C \frac{dx+dy}{|x|+|y|} = 2-2 = 0.$

(6) $a = \sqrt{3/2}.$

(7) $-\pi a^3/4.$

Tutorial Sheet No. 11

(1) (i) $\displaystyle\int\int_R (g_x - f_y)dxdy \; = \; \int_{\partial R}(fdx+gdy) \; = \; \frac{1}{3}.$

(ii) $\displaystyle\int\int_R (g_x - f_y)dxdy \; = \; \int_{\partial R}(fdx+gdy) \; = \; 1.$

(2) (i) -4, (ii) 4, (iii) 4π.

(3) (i) $\dfrac{3a^2\pi}{2}$, (ii) $\dfrac{a^2}{2}$.

(4) (i) $\dfrac{a^2}{8}(3\pi-8)$, (ii) $2\pi a^2$, (iii) $\dfrac{3\pi-8}{4}$.

(5) 0.

(7) $\displaystyle\int\int_D \mathbf{u}\cdot\mathbf{v}dxdy \; = \; 0$ and $\displaystyle\int\int_D \mathbf{u}\cdot\mathbf{w}dxdy \; = \; -\pi\,.$

(8) 0.

(9) (a) 0.

(10) (i) When the curve does not enclose the origin, the intergral is 0 otherwise -2π, (ii) $\dfrac{-\pi}{4}$, (iii) -2π.

Tutorial Sheet No. 12

(1) (i) $\mathbf{r}(u,v) = u\mathbf{i}+v\mathbf{j}+\frac{1}{2}(4+v-u)\mathbf{k}\,, u,v \in \mathbb{R}\,, \mathbf{r}_u\times\mathbf{r}_v = \frac{1}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+\mathbf{k}$.

(ii) $\mathbf{r}(u,v) = u\mathbf{i} + a\sin v\mathbf{j} + a\cos v\mathbf{k}$ for $u \in \mathbb{R},\ 0 \le v \le 2\pi$ and $\mathbf{r}_u \times \mathbf{r}_v = a\sin v\,\mathbf{j} + a\cos v\,\mathbf{k}$.

(iii) $\mathbf{r}(u,v) = \cos u\,\mathbf{x} + \sin u\,\mathbf{y} + v\mathbf{e}, 0 \le u \le 2\pi,\ v \in \mathbb{R}$, where $\mathbf{x}$ is a unit vector through the origin on the planar cross section of the cylinder through the origin, $\mathbf{e} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ and $\mathbf{y} = \mathbf{e}\times\mathbf{x}$.

$$\mathbf{r}_u \times \mathbf{r}_v = \cos u\,\mathbf{x} + \sin u\,\mathbf{y}\,.$$

(3) $2a^2(\pi - 2)$.

(4) (i) S is a portion of a paraboloid of revolution.

(ii) $\mathbf{r}_u \times \mathbf{r}_v = -2u^2(\cos v\,\mathbf{i} + \sin v\,\mathbf{j}) + u\mathbf{k}$

(iii) $n = 6$.

(5) $\dfrac{2\pi}{3}(3\sqrt{3} - 1)a^2$.

(7) $\frac{1}{2}$.

(8) 0.

(9) $\dfrac{2\pi}{3}$.

(10) $\dfrac{2\pi}{3}$.

Tutorial Sheet No. 13

(1) (a) $\displaystyle\int\int_S \mathrm{curl}\mathbf{F}\cdot\mathbf{n}dS = \int_C \mathbf{F}\cdot d\mathbf{r} = \frac{\pi a^2}{2}\,.$

(b) $\displaystyle\int\int_S \mathrm{curl}\mathbf{F}\cdot\mathbf{n}dS = \int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi a^2\,.$

(2) 0.

(3) -108π.

(4) -2π.

(5) $-\dfrac{9a^3}{2}$.

(6) $-\pi a^2$.

(7) $\pi\|\mathbf{u}\times\mathbf{v}\|$.

Tutorial Sheet No. 14

(1) $\displaystyle\int\int_{\partial R} \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_R \mathrm{div}\ \mathbf{F}dV = \frac{4^4 6\pi}{5}\,.$

(2) $\displaystyle\int\int_{\partial R} \mathbf{F}\cdot\mathbf{n}dS \;=\; \int\int\int_{R} \text{div }\mathbf{F}dV \;=\; \frac{abc}{24}(a+b+c)\,.$

(4) 8π.

(6) 3.

(7) 0.

(8) 0.