MA 105 : Calculus
Division 1, Lecture 01

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Generalities about the Course

- Instructor: Prof. Sudhir R. Ghorpade, 106B, Maths Dept.
- Lecture Hours: Mon, Thu 2.00 – 3.25 PM, in LA 001.
- Tutorial: Wed, 2 – 2.55 pm, in LT 001 – 006.
- Attendance: Compulsory! (Also, it will be good for you!)
- Office Hours: Mondays 11.30 am – 12.30 pm.
- Evaluation Plan: Short Quizzes in Tuts (10 %), Common quizzes (10% $\times$ 2), Mid-Sem (30 %), End-Sem (40%).
- More Information:
  - The Booklet
  - Moodle page of the course
  - Instructor's web page, and especially, the course page: http://www.math.iitb.ac.in/~srg/autumn2019.html

Text, References, and Acknowledgements

The treatment of calculus in these lectures will be based on the following two books by S. R. Ghorpade and B. V. Limaye, which are published by Springer, New York.

[GL-1] A Course in Calculus and Real Analysis, 2nd Ed., 2018.

[GL-2] A Course in Multivariable Calculus and Analysis, 2010.

Besides these, the other references listed in the booklet, especially the book of Thomas and Finney, may be consulted. For later parts of the course, it is also useful to see the book Basic Multivariable Calculus by J. E. Marsden, A. J. Tromba and A. Weinstein (Springer, New York, 1993).

Acknowledgement: I shall mainly use the slides of Calculus lectures prepared recently by Prof. B. V. Limaye. These slides acknowledged the use of the lecture notes of similar courses given by myself and by Prof. Prachi Mahajan in the past.

Notation

- $\mathbb{N} := \{1, 2, 3, \ldots\}$ [the set of positive integers]
- $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ [the set of integers]
- $\mathbb{Q} :=$ the set of all rational numbers
  $= \{m/n : m, n \in \mathbb{Z},\ n \neq 0\}$

There is no rational number whose square is 2.

Proof: Suppose not! Then $(p/q)^2 = 2$, that is, $p^2 = 2q^2$ for some $p, q \in \mathbb{Z}$ such that $q \neq 0$, and p and q have no common factor. Now p^2 is even, and so p is even. Hence there is an integer r such that $p = 2r$. Then $2q^2 = p^2 = (2r)^2 = 4r^2$, and so $q^2 = 2r^2$. Thus q^2 is even, and so q is also even. Thus 2 is a common factor of p and q, which is a contradiction.

Optional Exercise: If $d \in \mathbb{N}$ is not the square of an integer, then show that there is no rational number whose square is d.

Let $\mathbb{R}$ denote the set of all real numbers. We will assume the following things about the set $\mathbb{R}$.

The set $\mathbb{Q}$ of all rational numbers is contained in $\mathbb{R}$, and the set $\mathbb{R}$ of all real numbers satisfies

- Algebraic Properties regarding addition and multiplication.
- Order Properties: There is a subset $\mathbb{R}^+$ of $\mathbb{R}$ such that
  (i) Given $a \in \mathbb{R}$, exactly one of the following holds:

  $a \in \mathbb{R}^+$ or $a = 0$ or $-a \in \mathbb{R}^+$

  (ii) $a, b \in \mathbb{R}^+ \Longrightarrow a + b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$.

  Define $a < b$ if $b - a \in \mathbb{R}^+$. Thus $\mathbb{R}^+ = \{x \in \mathbb{R} : 0 < x\}$.
- Completeness Property, which we shall state later.

Elements of the set $\mathbb{R} \setminus \mathbb{Q}$, that is, those real numbers which are not rational numbers, are called **irrational numbers**.

Notation: We write $a \leq b$ if $a < b$ or $a = b$.
Also, we write $a > b$ if $b < a$, and $a \geq b$ if $a > b$ or $a = b$.

Boundedness of a subset of $\mathbb{R}$

Let E be a subset of $\mathbb{R}$, that is, $E \subset \mathbb{R}$.

- E is called **bounded above** if there is $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in E$.
  Any such α is an **upper bound** of E.
- E is called **bounded below** if there is $\beta \in \mathbb{R}$ such that $x \geq \beta$ for all $x \in E$.
  Any such β is a **lower bound** of E.
- E is **bounded** if it is bounded above and bounded below.

We say that M is the **maximum** of E if M is an upper bound of E and $M \in E$, and we say that m is the **minimum** of E if m is a lower bound of E and $m \in E$.

Supremum (sup or lub) and Infimum (inf or glb)

Let $E \subset \mathbb{R}$.

- A real number α is called a **supremum** or a **least upper bound** of E if α is an upper bound of E (that is, $x \leq \alpha$ for all $x \in E$), and $\alpha \leq u$ for every upper bound u of E.

- A real number β is called an **infimum** or a **greatest lower bound** of E if β is a lower bound of E (that is, $\beta \leq x$ for all $x \in E$), and $v \leq \beta$ for every lower bound v of E.

- If E has a supremum, then it is unique, and it is denoted by sup E or lub E. Similarily, if E has an infimum, then it is unique and is denoted by inf E or glb E.

Example: Let $E := \{x \in \mathbb{R} : 0 < x \leq 1\}$. Then $\sup E = 1$ and $\inf E = 0$. Also, $\max E = 1$, but E has no minimum.

Completeness Property of $\mathbb{R}$:

A nonempty subset of $\mathbb{R}$ that is bounded above has a supremum, that is, a least upper bound.

Consequences of the Completeness Property:

- A nonempty subset E of $\mathbb{R}$ that is bounded below has an infimum, that is, a greatest lower bound.
  In fact, the set $F := \{-x : x \in E\}$ is bounded above, and if α is the lub of F, then $\beta := -\alpha$ is the glb of E.
- Archimedean Property: Given $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$.
  Proof: Suppose not! Then $n \leq x$ for all $n \in \mathbb{N}$, that is, x is an upper bound of the set $\mathbb{N}$. Let $\alpha := \sup \mathbb{N}$. Then $\alpha - 1$ is not an upper bound of $\mathbb{N}$, that is, there is $n_0 \in \mathbb{N}$ such that $\alpha - 1 < n_0$. But then $\alpha < n_0 + 1 \leq \alpha$, since $(n_0 + 1) \in \mathbb{N}$ and α is an upper bound of $\mathbb{N}$. Thus we obtain $\alpha < \alpha$, which is a contradiction.

- Let $x \in \mathbb{R}$. Applying the Archimedean property to x and $-x$, we see that there are $\ell, n \in \mathbb{N}$ such that $-\ell < x < n$. The largest among finitely many integers k satisfying $-\ell \leq k \leq n$ and also $k \leq x$ is called the **integer part** of x, and is denoted by $[x]$ or by $\lfloor x \rfloor$. Note that $[x]$ is the largest integer $\leq x$ and it is characterized by the following two properties: (i) $[x] \in \mathbb{Z}$ and (ii) $x - 1 \leq [x] \leq x$.

- Let $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then there is a unique $b \in \mathbb{R}^+$ such that $b^n = a$. This real number b is called the **positive nth root of a**, and we denote it by $a^{1/n}$.

  Example (the positive square root of 2):
  Let $S := \{x \in \mathbb{R} : x^2 \leq 2\}$. Then S is nonempty since $1 \in S$ and S is bounded above by 2. By the completeness property of $\mathbb{R}$, let $b := \sup S$. Then $b \geq 1 > 0$. Also, we obtain $b^2 = 2$ by showing that both $b^2 < 2$ and $b^2 > 2$ lead to contradictions. (Verify!) Thus $b := \sqrt{2}$. Since $b \in S$, we see that $b = \max S$.

- Let $a < b$ in $\mathbb{R}$. Then there is a rational number r such that $a < r < b$. In fact, we can consider $r := m/n$, where $n > 1/(b-a)$ and $m := [na] + 1$.
- Let $a < b$ in $\mathbb{R}$. Then there is an irrational number s such that $a < s < b$. In fact, since $a + \sqrt{2} < b + \sqrt{2}$, let $r \in \mathbb{Q}$ be such that $a + \sqrt{2} < r < b + \sqrt{2}$. Then $a < r - \sqrt{2} < b$, where $s := r - \sqrt{2}$ is an irrational number.

Thus we obtain the following important result.

Between any two real numbers, there is a rational number as well as an irrational number.

Optional Exercise: Write down more detailed versions of the "proofs" sketched above. Consult [GL-1], if desired.

Intervals

Given any $a, b \in \mathbb{R}$, we define

- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ [open interval]
- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ [closed interval]
- Semi-open intervals $(a, b]$ and $[a, b)$ are defined similarly.
- It is also useful to consider symbols ∞ and $-\infty$ and define the infinite intervals

$$(a, \infty) := \{x \in \mathbb{R} : x > a\}, \quad [a, \infty) := \{x \in \mathbb{R} : x \geq a\},$$
$$(-\infty, a) := \{x \in \mathbb{R} : x < a\}, \quad (-\infty, a] := \{x \in \mathbb{R} : x \leq a\}.$$

  Also, one writes $\mathbb{R} = (-\infty, \infty)$ and refers to this as an infinite interval, or sometimes, a doubly infinite interval.
- In general, a subset I of $\mathbb{R}$ is an interval if

$$x, y \in I, \ x < y \Longrightarrow [x, y] \subseteq I.$$

- One can show that every interval in $\mathbb{R}$ is open, closed, semi-open, or infinite interval.

Absolute Value

- For $x \in \mathbb{R}$, the absolute value or the modulus of x is

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- Basic Properties: For any $x, y \in \mathbb{R}$,
  - $|x + y| \leq |x| + |y|$ [Triangle Inequality]
  - $||x| - |y|| \leq |x - y|$.
- Optional Exercises: (i) Show that for any $a, b \in \mathbb{R}$ with $a \geq 0$, $b \geq 0$, and $n \in \mathbb{N}$,

$$\left| \sqrt[n]{a} - \sqrt[n]{b} \right| \leq \sqrt[n]{|a - b|}.$$

(ii) For any $n \in \mathbb{N}$ and any nonnegative real numbers $a_1, \ldots, a_n$, prove the AM-GM inequality:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Functions

- Given sets D, E, a function $f : D \to E$ assigns to each $x \in D$, a unique element of E, denoted $f(x)$. We refer to D as the domain of f and E as the co-domain of f. The set $\{f(x) : x \in D\}$ of all values taken by the function is called the range of f. [For a formal definition, see [GL-1].]
- A function $f : D \to E$ is said to be:
  - one-one (or injective) if for any $x_1, x_2 \in D$,
$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$
  - onto (or surjective) if its range is E.
- If $f : D \to E$ is bijective, then it has an inverse $g : E \to D$ with the property that the composites $g \circ f$ and $f \circ g$ are identity functions, i.e.,

$$g(f(x)) = x \text{ for all } x \in D \quad \text{and} \quad f(g(y)) = y \text{ for all } y \in E.$$

Examples and Types of Functions

- Example/Exercise: Consider $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

  $$f_1(x) = 2x+1, \quad f_2(x) = x^2, \quad f_3(x) = x^3, \quad \text{and} \quad f_4(x) = \frac{1}{x}.$$

  Are these one-one/onto/bijective?
- Functions that mainly arise in Calculus are usually of following types.
  - Polynomial functions
  - Rational functions
  - Algebraic functions
  - Transcendental functions (this includes logarithmic, exponential and trigonometric functions).
- Besides these, we can construct functions by pieceing together known functions (such as those belonging to the above classes). A good example is the absolute value function $x \longmapsto |x|$.

Sequences

Definition

Let X be any set. A sequence in X is a function from the set $\mathbb{N}$ of natural numbers to the set X.

The value of this function at $n \in \mathbb{N}$ is denoted by $a_n \in X$, and a_n is called the nth **term** of the sequence.

We shall use the notation (a_n) to denote a sequence.

Note: $\{a_n : n \in \mathbb{N}\}$ is the set of all terms of the sequence (a_n). Thus if $X := \mathbb{R}$ and $a_n := (-1)^n$ for $n \in \mathbb{N}$, then the sequence (a_n) is given by $-1, 1, -1, 1, \ldots$, but $\{a_n : n \in \mathbb{N}\} = \{-1, 1\}$.

Initially, we let $X := \mathbb{R}$, that is, we consider sequences in $\mathbb{R}$. Later, we shall consider sequences in $\mathbb{R}^2$ and in $\mathbb{R}^3$.

Examples

Examples:

1. $a_n := n$ for $n \in \mathbb{N}$: $1, 2, 3, 4, \ldots$
2. $a_n := 1/n$ for $n \in \mathbb{N}$: $1,\ 1/2,\ 1/3,\ 1/4, \ldots$
3. $a_n := n^2$ for $n \in \mathbb{N}$: $1, 4, 9, 16, \ldots$
4. $a_n := \sqrt{2}$ for $n \in \mathbb{N}$: $\sqrt{2}, \sqrt{2}, \ldots$ This is an example of a constant sequence.
5. $a_n := 2^n$ for $n \in \mathbb{N}$: $2, 4, 8, 16, \ldots$
6. $a_n := (-1)^n$ for $n \in \mathbb{N}$: $-1, 1, -1, 1, \ldots$
7. $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3$: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ This sequence is known as the Fibonacci sequence.
8. $a_n := 1/2 + \cdots + 1/2^n$ for $n \in \mathbb{N}$. Check: $a_n = 1 - (1/2^n)$.

Bounded sequences

A sequence (a_n) of real numbers is said to be **bounded above** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded above, that is, if there is a real number α such that $a_n \leq \alpha$ for every $n \in \mathbb{N}$.

A sequence (a_n) of real numbers is said to be **bounded below** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded below, that is, if there is a real number β such that $\beta \leq a_n$ for every $n \in \mathbb{N}$.

A sequence (a_n) of real numbers is said to be **bounded** if it is bounded above as well as bounded below, that is, if there are real numbers α, β such that $\beta \leq a_n \leq \alpha$ for every $n \in \mathbb{N}$.

If a sequence is not bounded, it is said to be **unbounded**.

Let us check which of the sequences mentioned earlier are bounded above and/or bounded below.

Examples of bounded and unbounded sequences:

- $a_n := n$ for $n \in \mathbb{N}$: $\beta = 1$.
- $a_n := 1/n$ for $n \in \mathbb{N}$: $\beta = 0$, $\alpha = 1$.
- $a_n := n^2$ for $n \in \mathbb{N}$: $\beta = 1$.
- $a_n := \sqrt{2}$ for $n \in \mathbb{N}$: $\beta = \sqrt{2} = \alpha$.
- $a_n := 2^n$ for $n \in \mathbb{N}$: $\beta = 2$.
- $a_n := (-1)^n$ for $n \in \mathbb{N}$: $\beta = -1$, $\alpha = 1$.
- $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3$: $\beta = 1$.
- $a_n := 1/2 + \cdots + 1/2^n$ for $n \in \mathbb{N}$: $\beta = 1/2$, $\alpha = 1$.

Note: The sequence given by $a_n := n$ for $n \in \mathbb{N}$, is not bounded above by the Archimedean property of $\mathbb{R}$.

Toward convergence of a sequence

Let (a_n) be a sequence in $\mathbb{R}$, and let a be a real number.

- Convergence of a sequence (a_n) to a real number a should mean that the term a_n is as close to a as we like for all sufficiently large n.
- Fix any positive number. Construct the sequence

$$|a_1 - a|, |a_2 - a|, \ldots, |a_n - a|, \ldots$$

  After a certain stage, all the entries from this sequence should be smaller than the fixed positive number.
- The fixed positive number is often denoted by ϵ (epsilon).
- Let n_0 indicate how far one needs to go in the sequence to ensure that the entries from $|a_{n_0} - a|$ onward are smaller than ϵ, that is, $a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots$ all belong to $(a - \epsilon, a + \epsilon)$.

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $a \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.

This is known as the ϵ-n_0 definition of convergence of a sequence.

In this case, we say that (a_n) **converges** to a, or that a is a **limit** of (a_n), and we write

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a \ \ (\text{as } n \to \infty).$$

If a sequence does not converge, we say that the sequence **diverges** or it is **divergent**.

The convergence of a sequence is unaltered if a finite number of its terms are replaced by some other terms.

Examples:

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a$. We can let $n_0 := 1$.

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

Let $\epsilon > 0$ be given. We want to find $n_0 \in \mathbb{N}$ such that $|(1/n) - 0| < \epsilon$ for all $n \geq n_0$.

Choose any $n_0 \in \mathbb{N}$ which is greater than $1/\epsilon$. This is possible because of the Archimedean property of $\mathbb{R}$. For example, we can let $n_0 := [1/\epsilon] + 1$.