MA 105 : Calculus
Division 1, Lecture 13

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Arc length of a smooth curve in the plane
- Special cases, including polar curves. Examples
- Curves in 3-space and their arc-length. Helix as an example
- Surfaces of revolution
- Surface area of a cone and of a frustum of a cone
- Area of surface of revolution obtained by revolving a smooth parametric curve about a line
- Special cases and examples.

- Euclidean spaces
- Notion of addition and scalar multiplication of vectors
- Dot product. (Cauchy-) Schwarz inequality and Triangle inequality
- Notions of orthogonal vectors, angle between nonzero vectors, neighbourhood of a vector

Remaining Tut Problem

In the last tutorial, all problems of Tut Sheet 5, except # 4, 5, 7, were covered. Of those that remain, (7) was worked out in Lecture 11 and (5) in the last lecture. We will now work out:

(4) Find the arc length of the each of the curves described below.

(i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$

(ii) $y = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4.$

Solution: The arc length in (i) is given by

$$\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)}dt = \int_0^{2\pi} 2|\sin(t/2)|dt = 8.$$

and using the FTC, the arc length in (ii) is given by

$$\int_0^{\pi/4} \sqrt{1+y'^2}dx = \int_0^{\pi/4} \sqrt{1+\cos(2x)}dx = \sqrt{2}\int_0^{\pi/4} |\cos(x)|dx = 1.$$

Thus Tutorial Sheet 5 is now completely done!

Euclidean Spaces

Let $m \in \mathbb{N}$, and consider the m dimensional **Euclidean space**

$$\mathbb{R}^m := \{(x_1, \ldots, x_m) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, m\}.$$

If $m = 1$, then an element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar** and if $m > 1$, then an element of $\mathbb{R}^m$ is called a **vector**.
We shall consider sequences in $\mathbb{R}^m$, and functions from subsets of $\mathbb{R}^m$ to $\mathbb{R}$, and we shall study their properties.
We shall set up the basic structure in $\mathbb{R}^m$, and then restrict ourselves to $m := 2, 3$.

For $\boldsymbol{x} := (x_1, \ldots, x_m)$, $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$, let

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &:= (x_1 + y_1, \ldots, x_m + y_m) \in \mathbb{R}^m, \\ a\,\boldsymbol{x} &:= (a\,x_1, \ldots, a\,x_m) \in \mathbb{R}^m, \end{aligned}$$

called the **sum** of $\boldsymbol{x}$ and $\boldsymbol{y}$, and the **scalar multiple** of a and $\boldsymbol{x}$.

Norm of an element of $\mathbb{R}^m$

Let $\boldsymbol{x} := (x_1, \ldots, x_m) \in \mathbb{R}^m$. We define the **norm** of $\boldsymbol{x}$ by

$$\|\boldsymbol{x}\| := (x_1^2 + \cdots + x_m^2)^{1/2}.$$

For $m = 1$, the norm of $x \in \mathbb{R}$ is its absolute value $|x|$. Note:

$$\max\{|x_1|, \ldots, |x_m|\} \leq \|\boldsymbol{x}\| \leq |x_1| + \cdots + |x_m|.$$

Let $\mathbf{0} := (0, \ldots, 0)$. Then $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbf{0}$.

For $\boldsymbol{x} := (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, let

$$\boldsymbol{x} \cdot \boldsymbol{y} \;\; := \;\; x_1 y_1 + \cdots + x_m y_m \in \mathbb{R},$$

called the **dot product** or the **scalar product** of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Theorem (Cauchy-Schwarz Inequality and Triangle Inequality)

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ and $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$. We say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are **orthogonal** or **perpendicular** (to each other) if

$$\boldsymbol{x} \cdot \boldsymbol{y} = 0.$$

If $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$, then the **angle** between them is the unique $\theta \in [0, \pi]$ such that $\cos\theta = \boldsymbol{x} \cdot \boldsymbol{y} / \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$. (Cosine Rule)

Let $m \in \mathbb{N}$ and $\boldsymbol{x}_0 \in \mathbb{R}^m$. For $\boldsymbol{x} \in \mathbb{R}^m$, $\|\boldsymbol{x} - \boldsymbol{x}_0\|$ is called the **distance** of $\boldsymbol{x}$ from $\boldsymbol{x}_0$. For $r > 0$, the subset

$$B(\boldsymbol{x}_0, r) := \{\boldsymbol{x} \in \mathbb{R}^m : \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\}$$

of $\mathbb{R}^m$ is called the **neighbourhood** of $\boldsymbol{x}_0$ of radius r.

If $m := 1$, then $B(\boldsymbol{x}_0, r)$ is the interval $(x_0 - r, x_0 + r)$ in $\mathbb{R}$.

If $m := 2$ and $\boldsymbol{x}_0 := (x_0, y_0)$, then $B(\boldsymbol{x}_0, r)$ is the disk $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$ in $\mathbb{R}^2$.

If $m := 3$ and $\boldsymbol{x}_0 := (x_0, y_0, z_0)$, then $B(\boldsymbol{x}_0, r)$ is the ball $\{(x, y, z) \in \mathbb{R}^3 : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$ in $\mathbb{R}^3$.

Sequence in $\mathbb{R}^2$ and its Limit

A **sequence** in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. We denote the nth term of such a sequence by (x_n, y_n), and the sequence itself by $\big((x_n, y_n)\big)$.

Definition

A sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ is **convergent** in $\mathbb{R}^2$ if both (x_n) and (y_n) are convergent sequences in $\mathbb{R}$. In this case, we write

$$(x_n, y_n) \to (x_0, y_0), \text{ where } x_n \to x_0 \text{ and } y_n \to y_0.$$

For $n \in \mathbb{N}$,

$$\max\{|x_n - x_0|, |y_n - y_0|\} \leq \|(x_n, y_n) - (x_0, y_0)\| \leq |x_n - x_0| + |y_n - y_0|.$$

Hence $(x_n, y_n) \to (x_0, y_0) \iff \|(x_n, y_n) - (x_0, y_0)\| \to 0.$

Functions on subsets of $\mathbb{R}^2$

Let D be a subset of $\mathbb{R}^2$, and let f be a real-valued function defined on D. The subset

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in D\}$$

is called the **graph** of f. It is the surface $z = f(x, y)$ in $\mathbb{R}^3$.

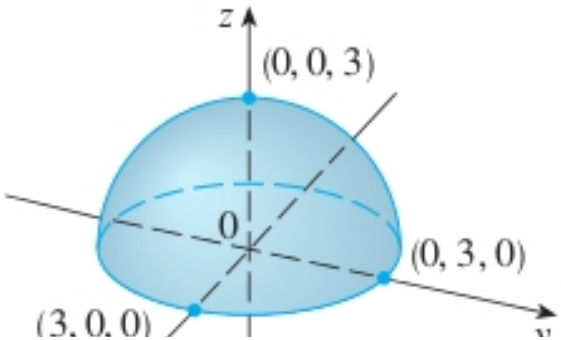
Let $c \in \mathbb{R}$. Then the set
$\{(x, y, c) \in \mathbb{R}^3 : (x, y) \in D$ and $f(x, y) = c\}$ is called a **contour line** of f. It is the intersection of the graph of f by the horizontal plane $z = c$ in $\mathbb{R}^3$.

Also, $\{(x, y) \in D : f(x, y) = c\} \subset \mathbb{R}^2$ is called a **level curve** of f. It is the projection of the corresponding contour line on the plane $z = 0$ in $\mathbb{R}^3$.

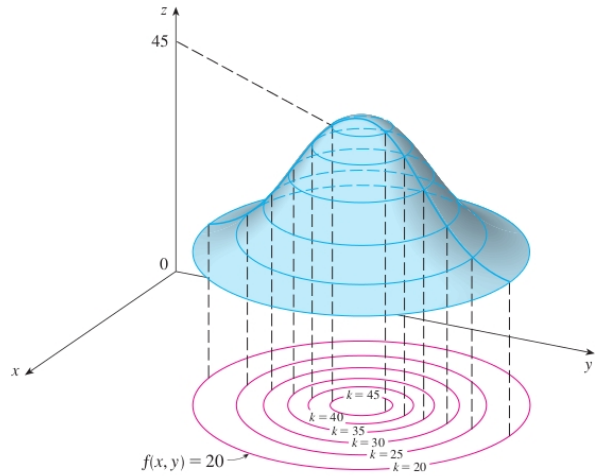
Contour lines and level curves help us visualize the behaviour of a function f on its domain in $\mathbb{R}^2$. Such a visualization is not possible for functions defined on subsets of $\mathbb{R}^3$.

Examples

(i) Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$, and let $f(x, y) := \sqrt{9 - x^2 - y^2}$ for $(x, y) \in D$. The graph of f is the upper hemisphere of radius 3 centered at the origin.



(ii) Contour lines



and level curves.

(iii) Let $D := \mathbb{R}^2$, and let $f(x, y) := x^2 + y^2$ for $(x, y) \in D$. The graph of f is a paraboloid.

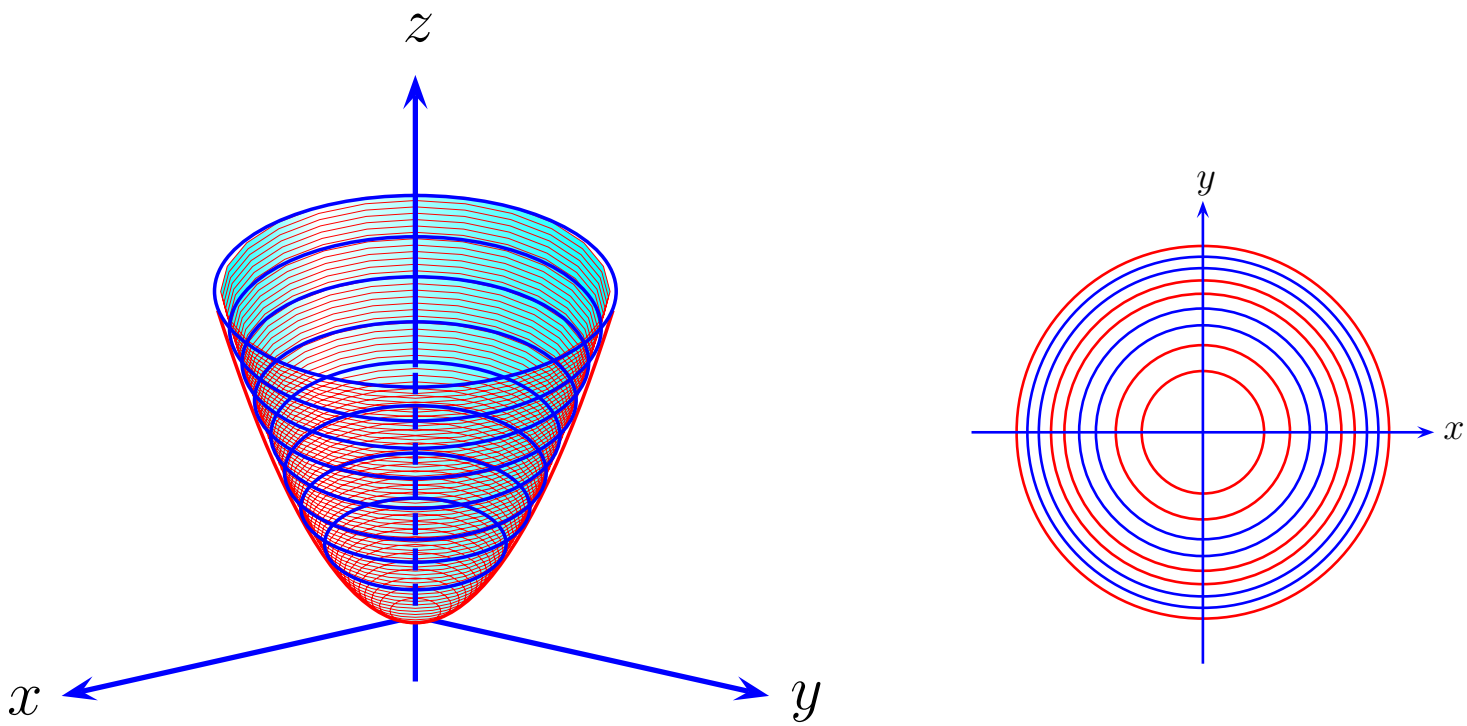


Figure: Contour lines and the corresponging level curves

(iv) Let $D := \{(x, y) \in \mathbb{R}^2 : x \neq y\}$, and let $f(x, y) := (x + y)/(x - y)$ for $(x, y) \in D$. Then the contour line corresponding to $c = 2$ is $\{(x, x/3, 2) : x \in \mathbb{R} \setminus \{0\}\}$.

Continuity of a function defined on a subset of $\mathbb{R}^2$

Definition

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. We say that f is **continuous** at (x_0, y_0) if

$$(x_n, y_n) \to (x_0, y_0) \implies f(x_n, y_n) \to f(x_0, y_0)$$

for every sequence $\big((x_n, y_n)\big)$ in D.

We say that f is **continuous on** D if f is continuous at each point of D.

We say that f is **discontinuous** on D if it is not continuous at some point of D.

Basic properties of continuous functions

- Let $D \subset \mathbb{R}^2$, and $f, g : D \to \mathbb{R}$. Let $(x_0, y_0) \in D$, and let f and g be continuous at (x_0, y_0). Then $f + g$ and $f \cdot g$ are continuous at (x_0, y_0). Further, f/g is continuous at (x_0, y_0) if $g((x_0, y_0)) \neq 0$. (Limit theorems for sequences.)
- Let $D \subset \mathbb{R}^2, E \subset \mathbb{R}$. Suppose $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions such that $f(D) \subset E$, f is continuous at $(x_0, y_0) \in D$, and g is continuous at $f(x_0, y_0) \in E$. Then the **composite function** $g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0). (The proof is easy.)

Examples

(i) Let p be a **polynomial function in two variables**, that is, let $k = 0$ or $k \in \mathbb{N}$ and for $(x, y) \in \mathbb{R}^2$, let

$$p(x, y) := \sum_{i \geq 0, j \geq 0, i+j \leq k} a_{i,j} x^i y^j, \quad \text{where } a_{i,j} \in \mathbb{R}.$$

Then p is a continuous function on $\mathbb{R}^2$.

(ii) A **rational function in two variables**, that is, a function given by $r(x, y) := p(x, y)/q(x, y)$, where p and q are polynomial functions in two variables, is continuous at each $(x_0, y_0) \in \mathbb{R}^2$ at which $q((x_0, y_0)) \neq 0$.

(iii) Let $p(x, y)$ be a polynomial function in two variables, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := e^{p(x,y)}$ for $(x, y) \in \mathbb{R}^2$. Then f is continuous on $\mathbb{R}^2$ since the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) := e^z$ for $z \in \mathbb{R}$, is continuous on $\mathbb{R}$.

Equivalent condition for continuity of a function

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at $(x_0, y_0) \iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there exists $\delta > 0$ such that $(x, y) \in D$ and

$$\|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \epsilon.$$

Examples:
Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f(0, 0) := 0$.

(i) Suppose $f(x, y) := xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$.
Let $(x_n, y_n) := (1/n, 1/n)$, $n \in \mathbb{N}$. Then $(x_n, y_n) \to (0, 0)$, but $f(x_n, y_n) \to 1/2 \neq f(0, 0)$. Hence f is not continuous at $(0, 0)$.

(ii) Suppose $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y) - f(0,0)| = |x| \frac{|xy|}{x^2+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

If $(x_n, y_n) \to (0,0)$, then $f(x_n, y_n) \to 0 = f(0,0)$. Hence f is continuous at 0.

(iii) Suppose $f(x,y) := x^2y/(x^4+y^2)$ for $(x,y) \neq (0,0)$.
If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = n/(1+n^2) \to 0$.
Can we say f is continuous at $(0,0)$?
Let $(x_n, y_n) := (1/n, 1/n^2)$, then $(x_n, y_n) \to (0,0)$, but
$f(x_n, y_n) \to 1/2 \neq f(0,0)$. Hence f is not continuous at $(0,0)$.

(iv) Suppose $f(x,y) := x^3y/(x^4+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y) - f(0,0)| = |x| \frac{|x^2y|}{x^4+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

As in (ii) above, f is continuous at $(0,0)$.

Limits of Functions of Two Variables

Definition

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$. Suppose $(x_0, y_0) \in \mathbb{R}^2$ is such that $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subset D$ for some $r > 0$ [or suppose, more generally, that (x_0, y_0) is a limit point of D, which means that every neighborhood of (x_0, y_0) contains a point of D other than (x_0, y_0)]. We say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists if there is $\ell \in \mathbb{R}$ such that $f(x_n, y_n) \to \ell$ for every sequence $\big((x_n, y_n)\big)$ in D with $(x_n, y_n) \neq (x_0, y_0)$ for all $n \in \mathbb{N}$ and $(x_n, y_n) \to (x_0, y_0)$.

Examples: (i) Let

$$f(x, y) := \begin{cases} x + y & \text{if } (x, y) \in \mathbb{R}^2,\ x \neq y, \\ 1 & \text{if } (x, y) \in \mathbb{R}^2,\ x = y. \end{cases}$$

If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = 1$, and
if $(x_n, y_n) := (1/n, -1/n)$, then $f(x_n, y_n) = 0$ for all $n \in \mathbb{N}$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(ii) Suppose $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$.
If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = 1/2$, and
if $(x_n, y_n) := (1/n, -1/n)$, then $f(x_n, y_n) = -1/2$ for $n \in \mathbb{N}$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Aliter: Let $(x_n, y_n) := (1/n, (-1)^n/n)$, then for $n \in \mathbb{N}$,
$f(x_n, y_n) = (-1)^n/2$, and so the sequence $(f(x_n, y_n))$ is
divergent. Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(iii) Suppose $f(x,y) := x^2y/(x^4+y^2)$ for $(x,y) \neq (0,0)$.
Suppose $(x_n, y_n) \to (0,0)$ and $(x_n, y_n) \neq (0,0)$. If $x_n := 0$ or
$y_n := 0$, then $f(x_n, y_n) = 0$. Also, if $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and
$y_n := \alpha x_n$ for $n \in \mathbb{N}$, then $f(x_n, y_n) = \alpha x_n/(x_n^2 + \alpha^2) \to 0$.

But if $(x_n, y_n) := (1/n, 1/n^2)$ for $n \in \mathbb{N}$, then $f(x_n, y_n) = 1/2$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity and Limit

Let us relate the concepts of continuity and limit for functions of two variables.

Let $D \subset \mathbb{R}^2$. We say that $(x_0, y_0) \in \mathbb{R}^2$ is an **interior point** of D if there is $r > 0$ such that $B((x_0, y_0), r) \subset D$.

Suppose (x_0, y_0) be an interior point of D [or more generally, suppose $(x_0, y_0) \in D$ and (x_0, y_0) is a limit point of D]. Then $f : D \to \mathbb{R}$ is continuous at $(x_0, y_0) \iff \lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and it is equal to $f(x_0, y_0)$.

Thus for f to be continuous at an interior point (x_0, y_0) of D,

(i) $f(x_0, y_0)$ has to be defined,

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ must exist, and

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Limit theorems for functions of two real variables

Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell_f$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \ell_g$.
The limit theorems for sequences in $\mathbb{R}$ show the following:

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = \ell_f \pm \ell_g$ resp.

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \cdot g)(x,y) = \ell_f \cdot \ell_g$.

(iii) If $\ell_g \neq 0$, then $\lim\limits_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{\ell_f}{\ell_g}$.

(iv) If $f(x,y) \leq g(x,y)$ for all $(x,y) \neq (x_0,y_0)$ near (x_0,y_0), then $\ell_f \leq \ell_g$.

(v) (Sandwich theorem) If $f(x,y) \leq h(x,y) \leq g(x,y)$ for all (x,y) near (x_0,y_0) and $(x,y) \neq (x_0,y_0)$, and if $\ell_f = \ell = \ell_g$, then $\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = \ell$.

Equivalent condition for a limit to exist

(Recall the ϵ-δ condition for Continuity)

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at (x_0, y_0) $\iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$(x, y) \in D,\ \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \epsilon.$$

Similarly,

(ϵ-δ condition for Limit)

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Suppose $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subset D$ for some $r > 0$ [or more generally, suppose, (x_0, y_0) is a limit point of D]. Let $\ell \in \mathbb{R}$. Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = \ell \iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there is $\delta > 0$ such that $(x, y) \in D$, $0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - \ell| < \epsilon$.

Examples:

(i) Let $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y)| = |x|\frac{|xy|}{x^2+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

Let $\epsilon > 0$. If $\delta := 2\epsilon$, then

$$0 < \|(x,y)\| < \delta \implies |f(x,y)-0| \leq \frac{|x|}{2} \leq \frac{\|(x,y)\|}{2} < \frac{\delta}{2} = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Aliter: If we define $f(0,0) := 0$, then f is continuous at $(0,0)$ as we have seen before, and so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to 0.

(ii) Let $f(x,y) := \dfrac{x^2+y^2}{1-\sqrt{x^2+y^2+1}}$ for $(x,y) \neq (0,0)$.

'Rationalizing' the fraction, we obtain

$f(x,y) = -1 - \sqrt{x^2+y^2+1}$ for $(x,y) \neq (0,0)$, and so
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = -1 - \sqrt{1} = -2$. (Check: $\delta := \epsilon$ works.)

(iii) Let $f(x,y) := y \sin \dfrac{1}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

Since $|f(x,y) - 0| \leq |y| \leq \|(x,y)\|$ for $(x,y) \neq (0,0)$,
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. (Clearly, $\delta := \epsilon$ works.)

Thumb rule: Suppose $f(x,y) = p(x,y)/q(x,y)$ for $(x,y) \neq (0,0)$, where $p(x,y)$ and $q(x,y)$ are homogeneous polynomials of degrees m and n respectively, and $p(0,0) = 0 = q(0,0)$. If $m > n$, then $f(x,y) \to 0$ as $(x,y) \to (0,0)$, and if $m \leq n$, then this limit does not exist.