MA 105 : Calculus
Division 1, Lecture 14

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Completion of Tut Sheet 5
- Sequence in $\mathbb{R}^2$ and its limit
- Functions on subsets of $\mathbb{R}^2$. Examples
- Continuity of real-valued functions defined on subsets of $\mathbb{R}^2$
- Basic properties of continuous functions
- Limits of functions of two variables. Examples
- Continuity and Limit
- Limit theorems for functions of two variables

Continuity and Limit

Let us relate the concepts of continuity and limit for functions of two variables.

Let $D \subset \mathbb{R}^2$. We say that $(x_0, y_0) \in \mathbb{R}^2$ is an **interior point** of D if there is $r > 0$ such that $B((x_0, y_0), r) \subset D$.

Suppose (x_0, y_0) be an interior point of D [or more generally, suppose $(x_0, y_0) \in D$ and (x_0, y_0) is a limit point of D]. Then $f : D \to \mathbb{R}$ is continuous at $(x_0, y_0) \iff \lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and it is equal to $f(x_0, y_0)$.

Thus for f to be continuous at an interior point (x_0, y_0) of D,

(i) $f(x_0, y_0)$ has to be defined,

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ must exist, and

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Limit theorems for functions of two real variables

Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell_f$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \ell_g$.
The limit theorems for sequences in $\mathbb{R}$ show the following:

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = \ell_f \pm \ell_g$ resp.

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \cdot g)(x,y) = \ell_f \cdot \ell_g$.

(iii) If $\ell_g \neq 0$, then $\lim\limits_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{\ell_f}{\ell_g}$.

(iv) If $f(x,y) \leq g(x,y)$ for all $(x,y) \neq (x_0,y_0)$ near (x_0,y_0), then $\ell_f \leq \ell_g$.

(v) (Sandwich theorem) If $f(x,y) \leq h(x,y) \leq g(x,y)$ for all (x,y) near (x_0,y_0) and $(x,y) \neq (x_0,y_0)$, and if $\ell_f = \ell = \ell_g$, then $\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = \ell$.

Equivalent condition for a limit to exist

(Recall the ϵ-δ condition for Continuity)

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at (x_0, y_0) $\iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$(x, y) \in D,\ \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \epsilon.$$

Similarly,

(ϵ-δ condition for Limit)

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Suppose $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subset D$ for some $r > 0$ [or more generally, suppose, (x_0, y_0) is a limit point of D]. Let $\ell \in \mathbb{R}$. Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = \ell \iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there is $\delta > 0$ such that $(x, y) \in D$, $0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - \ell| < \epsilon$.

Examples:

(i) Let $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y)| = |x|\frac{|xy|}{x^2+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

Let $\epsilon > 0$. If $\delta := 2\epsilon$, then

$$0 < \|(x,y)\| < \delta \implies |f(x,y)-0| \leq \frac{|x|}{2} \leq \frac{\|(x,y)\|}{2} < \frac{\delta}{2} = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Aliter: If we define $f(0,0) := 0$, then f is continuous at $(0,0)$ as we have seen before, and so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to 0.

(ii) Let $f(x,y) := \dfrac{x^2+y^2}{1-\sqrt{x^2+y^2+1}}$ for $(x,y) \neq (0,0)$.

'Rationalizing' the fraction, we obtain

$f(x,y) = -1 - \sqrt{x^2+y^2+1}$ for $(x,y) \neq (0,0)$, and so
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = -1 - \sqrt{1} = -2$. (Check: $\delta := \epsilon$ works.)

(iii) Let $f(x,y) := y \sin \dfrac{1}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

Since $|f(x,y) - 0| \leq |y| \leq \|(x,y)\|$ for $(x,y) \neq (0,0)$,
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. (Clearly, $\delta := \epsilon$ works.)

Thumb rule: Suppose $f(x,y) = p(x,y)/q(x,y)$ for $(x,y) \neq (0,0)$, where $p(x,y)$ and $q(x,y)$ are homogeneous polynomials of degrees m and n respectively, and $p(0,0) = 0 = q(0,0)$. If $m > n$, then $f(x,y) \to 0$ as $(x,y) \to (0,0)$, and if $m \leq n$, then this limit does not exist.

Partial Differentiation

Recall: Let $D \subset \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to have a **derivative** at c if $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists, and then it is denoted by $f'(c)$.

Definition

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D.
A function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** x at (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, and then it is denoted by $f_x(x_0, y_0)$ or by $\dfrac{\partial f}{\partial x}(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the horizontal line segment $\{(x, y_0) : x \in (x_0 - r, x_0 + r)\} \subset D$.

Geometric interpretation

Let C denote the curve obtained by intersecting the graph of f by the horizontal plane $y = y_0$. The partial derivative of f with respect to x at (x_0, y_0) equals the slope of the tangent to the curve C at (x_0, y_0). It can be interpreted as the rate of change in f along the x-axis at (x_0, y_0).

Computationally, $f_x(x_0, y_0)$ is obtained by differentiating f with respect to x at x_0, treating y as the constant y_0.

Similarly, a function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** y at (x_0, y_0) if

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

exists, and then it is denoted by $f_y(x_0, y_0)$ or by $\dfrac{\partial f}{\partial y}(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the vertical line segment $\{(x_0, y) : y \in (y_0 - r, y_0 + r)\} \subset D$.

Let $a < b$ and $c < d$. Consider

$$D := \{(x, c) \in \mathbb{R}^2 : a \leq x \leq b\} \cup \{(a, y) \in \mathbb{R}^2 : c \leq y \leq d\},$$

and let $f : D \to \mathbb{R}$. Although there is no interior point of D, we can still define $f_x^+(a, c)$, $f_x^-(b, c)$, $f_y^+(a, c)$ and $f_y^-(a, d)$ as the appropriate left hand and the right hand partial derivatives.

Partial derivatives of sums, products, quotients and compositions of functions of two variables can be found in exactly the same manner as the derivatives of a function of one variable.

Definition

If the partial derivatives of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f exist at (x_0, y_0), then

$$(\nabla f)(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0)) \in \mathbb{R}^2$$

is called the **gradient** of f at (x_0, y_0).

Partial Differentiation: Examples

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$. Partial derivatives of f exist at all points of $\mathbb{R}^2$. In fact, for $(x_0, y_0) \in \mathbb{R}^2$,

$$f_x(x_0, y_0) = 2x_0 \quad \text{and} \quad f_y(x_0, y_0) = 2y_0.$$

(ii) Let $f(x,y) := \sqrt{x^2 + y^2}$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0, y_0) \neq (0,0)$. Then

$$f_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \quad \text{and} \quad f_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}.$$

But f does not have either partial derivative at $(0,0)$ since $\lim_{h \to 0} |h|/h$ does not exist. Note: f is continuous at $(0,0)$.

(iii) Let $f(x,y) := xy/(x^2 + y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) = 0$. It is easy to see that $f_x(0,0) = 0 = f_y(0,0)$. We have already seen that f is not continuous at $(0,0)$.

Higher Order Partial Derivatives

Let $D \subset \mathbb{R}^2$, and $f : D \to \mathbb{R}$. Suppose $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$.

Also, if $f_x : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{xy}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

Similarly, we define $f_{yy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$, or $\dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$.

In general, the **mixed partial derivatives** $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ may not be equal.

Mixed Partials

Example: Let $f(x,y) := x\,y\dfrac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Then $f_x(0,y_0) = -y_0$ for $y_0 \in \mathbb{R}$, and $f_y(x_0,0) = x_0$ for $x_0 \in \mathbb{R}$. Hence $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

Theorem (Mixed Partials Theorem)

Let $D \subset \mathbb{R}^2$, and let (x_0,y_0) be an interior point of D. Then there is $r > 0$ such that
$S := \{(x,y) \in \mathbb{R}^2 : |x-x_0| < r \text{ and } |y-y_0| < r\} \subset D$.
Consider $f : S \to \mathbb{R}$, and suppose f_x and f_y exist on S. If one of the mixed partials f_{xy} or f_{yx} exists on S, and it is continuous at (x_0,y_0), then the other mixed partial exists at (x_0,y_0), and $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

We omit a proof of this result since it is rather involved.

Directional Derivatives

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D.
Let $\boldsymbol{u} := (u_1, u_2) \in \mathbb{R}^2$ be a **unit vector**, that is, $\|\boldsymbol{u}\| = 1$.

Definition

A function $f : D \to \mathbb{R}$ is said to have a **directional derivative along $\boldsymbol{u}$** at (x_0, y_0) if

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t}$$

exists, and then it is denoted by $(\boldsymbol{D_u} f)(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the (slanted) line segment $\{(x_0 + tu_1, y_0 + tu_2) : t \in (-r, r)\} \subset D$.

$(\boldsymbol{D_u} f)(x_0, y_0)$ is the rate of change in f along $\boldsymbol{u}$ at (x_0, y_0).

Clearly, if $\boldsymbol{u} := (1, 0)$, then $\boldsymbol{D_u} f = f_x$, and if $\boldsymbol{u} := (0, 1)$, then $\boldsymbol{D_u} f = f_y$. Also, $\boldsymbol{D_{-u}} f = -\boldsymbol{D_u} f$ for every unit vector $\boldsymbol{u}$.

(Carathéodory's lemma for a directional derivative)

Let (x_0, y_0) be an interior point of $D \subset \mathbb{R}^2$. Let E denote the subset $\{(x_0 + tu_1, y_0 + tu_2) \in D : t \in \mathbb{R}\}$ of D. Then $(\boldsymbol{D_u} f)(x_0, y_0)$ exists $\iff$ there is a function $f_1 : E \to \mathbb{R}$ which is continuous at (x_0, y_0) and satisfies $f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0) = t\, f_1(x_0 + tu_1, y_0 + tu_2)$ for all $(x_0 + tu_1, y_0 + tu_2) \in E$.

In this case, $(\boldsymbol{D_u} f)(x_0, y_0) = f_1(x_0, y_0)$.

Proof. Suppose $(\boldsymbol{D_u} f)(x_0, y_0)$ exists. Define $f_1 : E \to \mathbb{R}$ by

$$f_1(x_0+tu_1, y_0+tu_2) := \begin{cases} \dfrac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} & \text{if } t \neq 0, \\ (\boldsymbol{D_u} f)(x_0, y_0) & \text{if } t = 0. \end{cases}$$

Then f_1 is continuous at (x_0, y_0) since
$f_1(x_0 + tu_1, y_0 + tu_2) \to (\boldsymbol{D_u} f)(x_0, y_0) = f_1(x_0, y_0)$ as $t \to 0$.

Conversely, suppose there is $f_1 : E \to \mathbb{R}$ as stated. Then

$$\begin{aligned} \lim_{t\to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} &= \lim_{t\to 0} f_1(x_0 + tu_1, y_0 + tu_2) \\ &= f_1(x_0, y_0), \end{aligned}$$

since f_1 is continuous at (x_0, y_0). Hence $(\boldsymbol{D_u} f)(x_0, y_0)$ exists.

In this case, it follows that $(\boldsymbol{D_u} f)(x_0, y_0) = f_1(x_0, y_0)$. □

Examples of directional derivatives

Examples:

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0, y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1, u_2)$ be a unit vector. For $t \neq 0$,

$$\begin{aligned} & \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0)}{t} \\ = \; & \frac{(x_0 + tu_1)^2 + (y_0 + tu_2)^2 - x_0^2 - y_0^2}{t} \\ = \; & 2x_0u_1 + 2y_0u_2 + t. \end{aligned}$$

Letting $t \to 0$, we obtain

$(\boldsymbol{D_u} f)(x_0, y_0) = 2x_0u_1 + 2y_0u_2.$

Note: $(\boldsymbol{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$.

(ii) Let $f(x,y) := \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$. Let $(x_0,y_0) \in \mathbb{R}^2$ and $\boldsymbol{u} := (u_1,u_2)$ be a unit vector. For $t \neq 0$,

$$\begin{aligned} & \frac{f(x_0+tu_1, y_0+tu_2) - f(x_0,y_0)}{t} \\ = & \frac{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} - \sqrt{x_0^2+y_0^2}}{t} \\ = & \frac{2x_0u_1 + 2y_0u_2 + t}{\sqrt{(x_0+tu_1)^2+(y_0+tu_2)^2} + \sqrt{x_0^2+y_0^2}}. \end{aligned}$$

Hence if $(x_0,y_0) \neq (0,0)$, then

$$(\boldsymbol{D_u} f)(x_0,y_0) = \frac{x_0u_1+y_0u_2}{\sqrt{x_0^2+y_0^2}} = (\nabla f)(x_0,y_0) \cdot \boldsymbol{u}.$$

But $(\boldsymbol{D_u} f)(0,0)$ does not exist since $\lim\limits_{t\to 0} t/|t|$ does not exist.

(iii) Let $f(x,y) := xy/(x^2+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$. For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{u_1 u_2}{t}.$$

Hence $(\boldsymbol{D_u} f)(0,0)$ exists $\iff$ $u_1 = 0$ or $u_2 = 0$.

(iv) Let $f(x,y) := x^2 y/(x^4+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$. For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2}.$$

Hence $(\boldsymbol{D_u} f)(0,0) = 0$ if $u_2 = 0$, and $(\boldsymbol{D_u} f)(0,0) = u_1^2/u_2$ if $u_2 \neq 0$. In particular, $(\nabla f)(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$.

Thus $(\boldsymbol{D_u} f)(0,0) \neq (\nabla f)(0,0) \cdot \boldsymbol{u}$ unless $u_1 = 0$ or $u_2 = 0$.

(v) Let $f(x,y) := x^3y/(x^4+y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$. For a unit vector $\boldsymbol{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0+tu_1, 0+tu_2) - f(0,0)}{t} = \frac{tu_1^3u_2}{t^2u_1^4 + u_2^2}.$$

Hence $(\boldsymbol{D_u} f)(0,0) = 0$ if $u_2 = 0$ and also if $u_2 \neq 0$. In particular, $(\nabla f)(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$

Thus $(\boldsymbol{D_u} f)(0,0) = (\nabla f)(0,0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$.

Higher order directional derivatives can also be considered. But we shall not do so here.

Differentiation of a function of two real variables

Let $D \subset \mathbb{R}^2$, and $f : D \to \mathbb{R}$. Suppose (x_0, y_0) is an interior point of D. We have considered partial derivatives, and more generally, directional derivatives of f at (x_0, y_0). Each of them gives us an information about the rate of change in f at (x_0, y_0) in a given direction. Let us see how we can obtain complete information about the rate of change in f at (x_0, y_0).

Let us recall the one variable situation. If $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, and c is an interior point of D, then the derivative of f at c is

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

Analogously, we would have liked to define the 'derivative' of f at (x_0, y_0) by

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{(h, k)}.$$

But this does not make sense since division by a vector $(h, k) \in \mathbb{R}^2$ is not defined even when $(h, k) \neq (0, 0)$.

We note that if $f : D \to \mathbb{R}$, c is an interior point of D and $\alpha \in \mathbb{R}$, then $f'(c) = \alpha \in \mathbb{R}$ if and only if

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{h} = 0,$$

that is,

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - \alpha h}{|h|} = 0.$$

Hence if $D \subset \mathbb{R}^2$, (x_0, y_0) is an interior point of D and $f : D \to \mathbb{R}$, then it is natural to see if there is $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{\|(h, k)\|} = 0.$$

Differentiability and Total Derivative

Suppose $D \subset \mathbb{R}^2$ and (x_0, y_0) is an interior point of D. Consider a funtion $f : D \to \mathbb{R}$.

Definition

We say that f is differentiable at (x_0, y_0) if there is $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0.$$

In this case, the pair $(\alpha, \beta) \in \mathbb{R}^2$ is called the **total derivative** of f at (x_0, y_0). Does it give us the full picture?

First, letting $(h, k) \to (0, 0)$ along the x-axis, we obtain

$$\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0) - \alpha h}{|h|} = 0,$$

that is, $f_x(x_0, y_0) = \alpha$.

Next, letting $(h, k) \to (0, 0)$ along the y-axis, we obtain

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0) - \beta k}{|k|} = 0,$$

that is, $f_y(x_0, y_0) = \beta$.

Thus if f is differentiable at (x_0, y_0), and (α, β) is its total derivative, then both partials of f exist at (x_0, y_0), and $f_x(x_0, y_0) = \alpha$, $f_y(x_0, y_0) = \beta$, that is, $(\alpha, \beta) = (\nabla f)(x_0, y_0)$, the gradient of f at (x_0, y_0).

More generally, let $\boldsymbol{u} := (u_1, u_2)$ be a unit vector. Letting $(h, k) \to (0, 0)$ along the parametrized line $x = tu_1$, $y = tu_2$,

$$\lim_{t \to 0} \frac{f(x_0 + tu_1, y_0 + tu_2) - f(x_0, y_0) - \alpha t u_1 - \beta t u_2}{|t|} = 0,$$

that is, $(\boldsymbol{D_u} f)(x_0, y_0) = \alpha u_1 + \beta u_2 = (\alpha, \beta) \cdot \boldsymbol{u} = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}$.

Linear Maps and Derivatives

Let $m \in \mathbb{N}$. A function $L : \mathbb{R}^m \to \mathbb{R}$ is called a linear map if $L(r\boldsymbol{x} + s\boldsymbol{y}) = r\,L(\boldsymbol{x}) + s\,L(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ and $r, s \in \mathbb{R}$.

(i) A function $L : \mathbb{R} \to \mathbb{R}$ is a linear map if and only if there is $\alpha \in \mathbb{R}$ such that $L(x) = \alpha x$ for all $x \in \mathbb{R}$.

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Then f is differentiable at c if and only if there is a linear map $L : \mathbb{R} \to \mathbb{R}$ such that $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c) - L(h)}{h} = 0$, that is,

$$f(c+h) = f(c) + L(h) + R(h), \text{ where } \frac{R(h)}{h} \to 0 \text{ as } h \to 0.$$

Thus the 'remainder term' $R(h)$ is even smaller than h.

The linear map L is known as the **derivative** of f at c.

(ii) A function $L : \mathbb{R}^2 \to \mathbb{R}$ is a linear map if and only if there are $\alpha, \beta \in \mathbb{R}$ such that $L(x, y) = \alpha x + \beta y$ for all $(x, y) \in \mathbb{R}^2$.

Let $D \subset \mathbb{R}^2$, (x_0, y_0) an interior point of D, and $f : D \to \mathbb{R}$. Then f is differentiable at (x_0, y_0) if and only if there is a linear map $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - L(h, k)}{\|(h, k)\|} = 0, \text{ that is,}$$

$$f(x_0+h, y_0+k) = f(x_0, y_0)+L(h, k)+R(h, k), \text{ where } \frac{R(h, k)}{\|(h, k)\|} \to 0$$

as $\|(h, k)\| \to 0$.
Thus the 'remainder term' $R(h, k)$ is even smaller than $\|(h, k)\|$.

The linear map L is known as the **total derivative** of f at (x_0, y_0).

We now consider a two-dimensional analogue of the Carathéodory Lemma. As before, it relates the concept of differentiability to the concept of continuity.

Proposition (Increment Lemma)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Then $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) if and only if there exist functions $f_1, f_2 : D \to \mathbb{R}$ such that f_1 and f_2 are continuous at (x_0, y_0), and for all $(x, y) \in D$,

$$f(x, y) - f(x_0, y_0) = (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y).$$

In this case, $\nabla f(x_0, y_0) = \big(f_1(x_0, y_0), f_2(x_0, y_0)\big)$.

Proof.

Let $\alpha, \beta \in \mathbb{R}$. For $(x, y) \in D$, consider the 'remainder function'

$$F(x, y) := f(x, y) - f(x_0, y_0) - \alpha(x - x_0) - \beta(y - y_0).$$

Suppose $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). Then $\alpha := f_x(x_0, y_0)$ and $\beta := f_y(x_0, y_0)$ exist.
Define $f_1, f_2 : D \to \mathbb{R}$ as follows. Let $f_1(x_0, y_0) := \alpha$, $f_2(x_0, y_0) := \beta$, and for $(x, y) \neq (x_0, y_0)$, let

$$f_1(x, y) := \alpha + \frac{(x - x_0)F(x, y)}{(x - x_0)^2 + (y - y_0)^2},$$

$$f_2(x, y) := \beta + \frac{(y - y_0)F(x, y)}{(x - x_0)^2 + (y - y_0)^2}.$$

Let $(x, y) \in D$ with $(x, y) \neq (x_0, y_0)$. Then

$$\begin{aligned}
&(x - x_0)f_1(x, y) + (y - y_0)f_2(x, y) \\
&= \alpha(x - x_0) + \frac{(x - x_0)^2 F(x, y)}{(x - x_0)^2 + (y - y_0)^2} \\
&\quad + \beta(y - y_0) + \frac{(y - y_0)^2 F(x, y)}{(x - x_0)^2 + (y - y_0)^2} \\
&= \alpha(x - x_0) + \beta(y - y_0) + F(x, y) = f(x, y) - f(x_0, y_0),
\end{aligned}$$

and this equation also holds for $(x, y) := (x_0, y_0)$, as desired.

Also, since

$$\frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \le 1 \quad \text{and} \quad \frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \le 1,$$

and since f is differentiable at (x_0, y_0), we obtain

$$|f_1(x,y)-\alpha|,\ |f_2(x,y)-\beta| \le \left|\frac{F(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right| \to 0$$

as $(x,y) \to (x_0,y_0)$. Hence f_1 and f_2 are continuous at (x_0,y_0).

Conversely, suppose there are $f_1, f_2 : D \to \mathbb{R}$ as stated. Let $\alpha := f_1(x_0,y_0)$ and $\beta := f_2(x_0,y_0)$. Then for $(x,y) \in D$,

$$F(x,y) = (x-x_0)(f_1(x,y)-\alpha) + (y-y_0)(f_2(x,y)-\beta),$$

and so $|F(x,y)|/\|(x-x_0, y-y_0)\| \le |f_1(x,y)-\alpha| + |f_2(x,y)-\beta|$.

Now $|f_1(x,y)-\alpha|+|f_2(x,y)-\beta| \to 0$, since f_1 and f_2 are continuous at (x_0,y_0). Hence

$$\left|\frac{F(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right| \to 0 \quad \text{as } (x,y)\to(x_0,y_0).$$

Thus f is differentiable at (x_0,y_0), and
$\nabla f(x_0,y_0)=(\alpha,\beta)=(f_1(x_0,y_0),f_2(x_0,y_0))$. □

A pair (f_1,f_2) of functions stated in the Increment Lemma is called a **pair of increment functions** associated with the function f and the point (x_0,y_0). Unlike the increment function f_1 stated in the Carathéodory Lemma, such a pair (f_1,f_2) of functions is not unique. Indeed, if $g: D\to\mathbb{R}$ is any function that is continuous at (x_0,y_0), and for $(x,y)\in D$, we let

$$\begin{aligned} g_1(x,y) &:= f_1(x,y)+(y-y_0)g(x,y),\\ g_2(x,y) &:= f_2(x,y)-(x-x_0)g(x,y), \end{aligned}$$

then (g_1,g_2) is also a corresponding pair of increment functions.

We now consider an important consequences of the Increment Lemma.

Proposition (Differentiability $\Longrightarrow$ Continuity)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). Then f is continuous at (x_0, y_0).

Proof:
If (f_1, f_2) is a pair of increment functions associated with f and (x_0, y_0), then

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y)$$

for all $(x, y) \in D$. Consequently, the continuity of f at (x_0, y_0) follows from the continuity of f_1 and f_2 at (x_0, y_0). $\square$