MA 105 : Calculus
Division 1, Lecture 15

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Recap of the previous lecture

- Continuity and Limit
- Limit theorems for functions of two variables
- Equivalent conditions for continuity and limits. Examples
- Partial differentiation
- Examples
- Higher order partial derivatives. Mixed Partials Theorem
- Directional derivatives. Examples
- Differentiation of a function of two real variables
- Differentiability and Total Derivative
- Linear Maps and Derivatives
- Increment Lemma. Application: Differentiability implies Continuity

Differentiability and Total Derivative

Suppose $D \subset \mathbb{R}^2$ and (x_0, y_0) is an interior point of D. Consider a funtion $f : D \to \mathbb{R}$.

Definition

We say that f is differentiable at (x_0, y_0) if there is $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0.$$

In this case, the pair $(\alpha, \beta) \in \mathbb{R}^2$ is called the **total derivative** of f at (x_0, y_0). We have seen that if f is differentiable at (x_0, y_0), then both the partial derivatives at (x_0, y_0) exist and in fact, $(\alpha, \beta) = (\nabla f)(x_0, y_0)$. Moreover, directional derivatives at (x_0, y_0) exists along any unit vector $\boldsymbol{u}$ exist, and

$$(\boldsymbol{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}.$$

Increment Lemma

We have the following two-dimensional analogue of the Carathéodory Lemma.

Proposition (Increment Lemma)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Then $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0) if and only if there exist functions $f_1, f_2 : D \to \mathbb{R}$ such that f_1 and f_2 are continuous at (x_0, y_0), and for all $(x, y) \in D$,

$$f(x, y) - f(x_0, y_0) = (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y).$$

In this case, $\nabla f(x_0, y_0) = \big(f_1(x_0, y_0), f_2(x_0, y_0)\big)$.

A pair (f_1, f_2) of functions stated in the Increment Lemma is called a **pair of increment functions** associated with the function f and the point (x_0, y_0).

We now consider an important consequences of the Increment Lemma.

Proposition (Differentiability $\Longrightarrow$ Continuity)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose $f : D \to \mathbb{R}$ is differentiable at (x_0, y_0). Then f is continuous at (x_0, y_0).

Proof:
If (f_1, f_2) is a pair of increment functions associated with f and (x_0, y_0), then

$$f(x, y) = f(x_0, y_0) + (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y)$$

for all $(x, y) \in D$. Consequently, the continuity of f at (x_0, y_0) follows from the continuity of f_1 and f_2 at (x_0, y_0). $\square$

Differentiability: Necessary Conditions

Let (x_0, y_0) be an interior point of a subset D of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$. Suppose f is differentiable at (x_0, y_0). Then the following conditions necessarily hold, that is, they are necessary conditions.

- Both partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.
- The directional derivative $(\boldsymbol{D_u} f)(x_0, y_0)$ exists for every unit vector $\boldsymbol{u} \in \mathbb{R}^2$.
- $(\boldsymbol{D_u} f)(x_0, y_0) = \big(f_x(x_0, y_0), f_y(x_0, y_0)\big) \cdot \boldsymbol{u} = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u} \in \mathbb{R}^2$.
- f is continuous at (x_0, y_0).

None of the above conditions are sufficient to conclude that a function is differentiable. In fact, we shall give an example of a function for which all the above conditions hold at (x_0, y_0), but the function is not differentiable at (x_0, y_0) (Example (vii)).

Proposition (Sufficient condition for differentiability)

Let $(x_0, y_0) \in \mathbb{R}^2$, $r > 0$, and let $f : S \to \mathbb{R}$, where $S := \{(x, y) \in \mathbb{R}^2 : |x - x_0| < r \text{ and } |y - y_0| < r\}$. Suppose one of the partial derivatives of f exists on S and is continuous at (x_0, y_0), while the other exists at (x_0, y_0). Then f is differentiable at (x_0, y_0).

Proof. Without loss of generality, suppose f_x exists on S and is continuous at (x_0, y_0), while f_y exists at (x_0, y_0).
For $(x, y) \in S$, $f(x, y) - f(x_0, y_0) = A(x, y) + B(y)$, where

$$A(x, y) := f(x, y) - f(x_0, y) \quad \text{and} \quad B(y) := f(x_0, y) - f(x_0, y_0).$$

Define $f_1, f_2 : S \to \mathbb{R}$ as follows.

$$f_1(x, y) := \begin{cases} \dfrac{A(x, y)}{x - x_0} & \text{if } x \neq x_0, \\ f_x(x_0, y) & \text{if } x = x_0, \end{cases} \qquad f_2(x, y) := \begin{cases} \dfrac{B(y)}{y - y_0} & \text{if } y \neq y_0, \\ f_y(x_0, y_0) & \text{if } y = y_0. \end{cases}$$

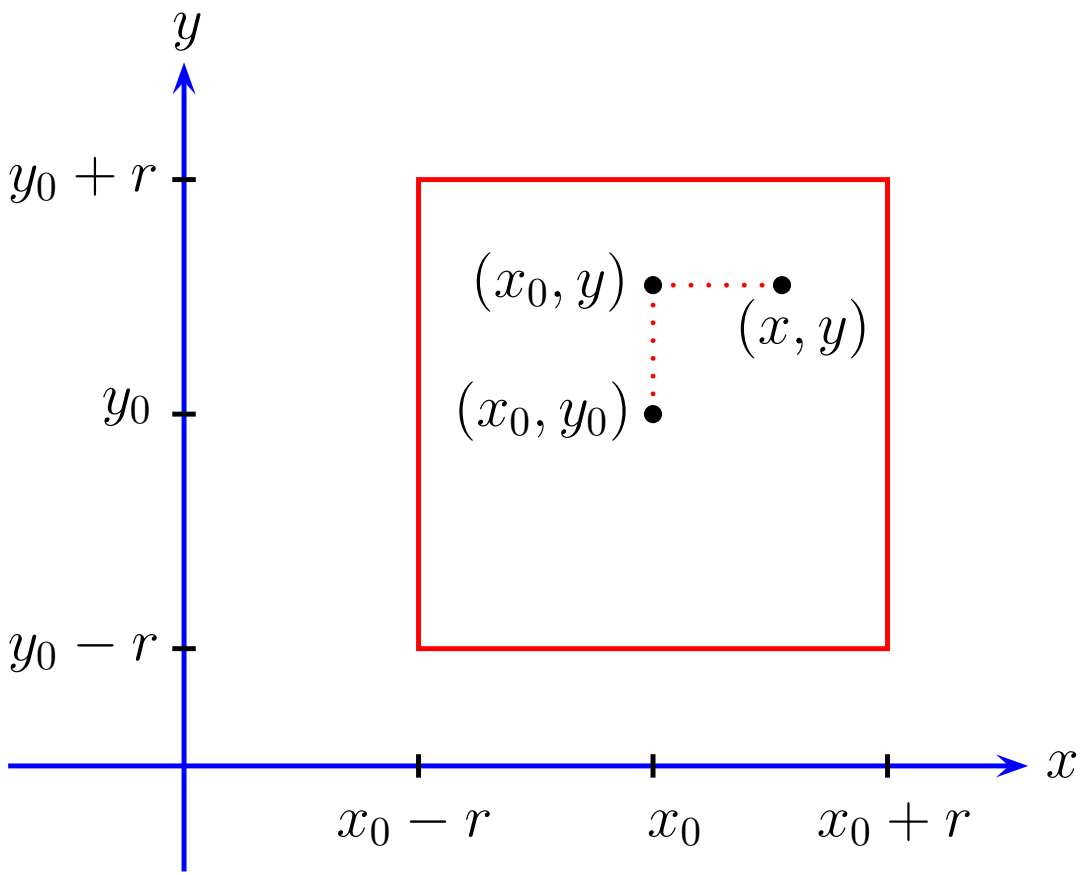

Figure: The dotted lines linking (x, y) to (x_0, y), and (x_0, y) to (x_0, y_0).

By considering the cases $x \neq x_0,\ x = x_0,\ y \neq y_0$ and $y = y_0$, it follows that for all $(x, y) \in S$,

$$f(x, y) - f(x_0, y_0) = (x - x_0)f_1(x, y) + (y - y_0)f_2(x, y).$$

To show f_1 is continuous at (x_0, y_0), consider $x \in (x_0 - r, x_0) \cup (x_0, x_0 + r)$ and $y \in (y_0 - r, y_0 + r)$. Since f_x exists on S, there is $c_y \in \mathbb{R}$ between x_0 and x such that

$$A(x, y) = f(x, y) - f(x_0, y) = (x - x_0)f_x(c_y, y).$$

by the MVT. Hence $f_1(x, y) = f_x(c_y, y)$. Also, by definition, $f_1(x_0, y) = f_x(x_0, y)$ for all $y \in (y_0 - r, y_0 + r)$. Since f_x is continuous at (x_0, y_0), we see that f_1 is continuous at (x_0, y_0).

Also, f_2 is continuous at (x_0, y_0) since $f_y(x_0, y_0)$ exists.

By the Increment Lemma, f is differentiable at (x_0, y_0). □

How to check differentiability?

To check the differentiability of a function of two variables, we may see whether f violates one of the necessary conditions for differentiability. Also, we may see whether f satisfies a sufficient condition for differentiability. Alternatively, we may turn to the definition of differentiability itself.

Examples:

(i) Let $f(x,y) := \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.
Then f is continuous at $(0,0)$. But $(\boldsymbol{D_u} f)(0,0)$ does not exist for any unit vector $\boldsymbol{u}$. Hence f is not differentiable at $(0,0)$.

(ii) Let $f(x,y) := 1$ if $0 < y < x^2$ and $f(x,y) := 0$ otherwise.
If $\boldsymbol{u} := (u_1, u_2)$, then $f(tu_1, tu_2) = 0$ for all t in an interval about 0, and so $(\boldsymbol{D_u} f)(0,0) = 0 = (\nabla f)(0,0) \cdot \boldsymbol{u}$. But f is not continuous at $(0,0)$ since $f(0,0) = 0$ and $f(1/n, 1/2n^2) = 1$ for all $n \in \mathbb{N}$. Hence f is not differentiable at $(0,0)$.

(iii) Let $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Then f is continuous at $(0,0)$, and $(\boldsymbol{D_u} f)(0,0) = u_1^2 u_2$ for every unit vector $\boldsymbol{u} := (u_1, u_2)$. But $(\boldsymbol{D_u} f)(0,0) \neq (\nabla f)(0,0) \cdot \boldsymbol{u}$ unless $u_1 = 0$ or $u_2 = 0$. Hence f is not differentiable at $(0,0)$.

(iv) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$. Let $(x_0, y_0) \in \mathbb{R}^2$. Then f is continuous at (x_0, y_0) and for every unit vector $\boldsymbol{u}$, $(\boldsymbol{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u}$. Since f_x and f_y exist on $\mathbb{R}^2$ and f_x is continuous at (x_0, y_0), f is differentiable at (x_0, y_0).

(v) Let $f(x,y) := x^2y^2/(x^4+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) := 0$. We check that f is continuous at $(0,0)$, and $(\boldsymbol{D_u} f)(0,0) = 0 = (\nabla f)(0,0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$.

Let us find $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ for $(x_0, y_0) \neq (0,0)$.

For $(x_0, y_0) \neq (0,0)$, it can be checked that

$$f_x(x_0, y_0) = \frac{2x_0y_0^2(y_0^2 - x_0^4)}{(x_0^4 + y_0^2)^2} \quad \text{and} \quad f_y(x_0, y_0) = \frac{2x_0^6y_0}{(x_0^4 + y_0^2)^2}.$$

Now f_x is continuous at $(0,0)$ as $y_0^4, 2x_0^4y_0^2 \leq (x_0^4 + y_0^2)^2$. Hence f is differentiable at $(0,0)$. (We note that f_y is not continuous at $(0,0)$ since $f_y(1/n, 1/n^2) = 1/2$ for $n \in \mathbb{N}$.)
(Aliter: Here is a direct proof. For $(h,k) \neq (0,0)$,

$$0 \leq \frac{h^2k^2 - 0 - 0\cdot h - 0\cdot k}{(h^4 + k^2)\sqrt{h^2 + k^2}} = h\,\frac{k^2}{(h^4 + k^2)}\,\frac{h}{\sqrt{h^2 + k^2}} \leq |h| \to 0$$

as $(h,k) \to (0,0)$, and so f is differentiable at $(0,0)$.)

(vi) Let $f(x,y) := |xy|$ for $(x,y) \in \mathbb{R}^2$. Then f is continuous at $(0,0)$, and $(\boldsymbol{D_u} f)(0,0) = 0 = (\nabla f)(0,0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$. On the other hand, $f_x(0,y_0)$ does not exist if $y_0 \neq 0$ and $f_y(x_0,0)$ does not exist if $x_0 \neq 0$. For $(h,k) \neq (0,0)$,

$$Q(h,k) := \frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{|hk|}{\sqrt{h^2+k^2}} \leq |h| \to 0$$

as $(h,k) \to (0,0)$. Hence f is differentiable at $(0,0)$.

(vii) Let $f(x,y) := x^3y/(x^4+y^2)$ for $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Then $(\boldsymbol{D_u} f)(0,0) = 0 = (\nabla f)(0,0) \cdot \boldsymbol{u}$ for every unit vector $\boldsymbol{u}$. Also, f is continuous at $(0,0)$. Further, f_x and f_y are not continuous at $(0,0)$. (Check!) For $(h,k) \neq (0,0)$,

$$Q(h,k) := \frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{h^3k}{(h^4+k^2)\sqrt{h^2+k^2}} \not\to 0$$

as $(h,k) \to (0,0)$ since $Q(1/n, 1/n^2) \to 1/2$. Hence f is not differentiable at $(0,0)$.

We obtain basic properties of differentiable functions, and also the gradients of a sum, of a difference, of a product and of a quotient of functions from the Increment Lemma as follows.

Proposition

Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D. Suppose $f, g : D \to \mathbb{R}$ are functions that are differentiable at (x_0, y_0). Then $f \pm g$ and fg are differentiable at (x_0, y_0), and moreover,

$$\nabla(f \pm g)(x_0, y_0) = (\nabla f)(x_0, y_0) \pm (\nabla g)(x_0, y_0) \text{ resp.,}$$

$$\nabla(fg)(x_0, y_0) = g(x_0, y_0)(\nabla f)(x_0, y_0) + f(x_0, y_0)(\nabla g)(x_0, y_0).$$

If $g(x_0, y_0) \neq 0$, then f/g is differentiable at (x_0, y_0) and

$$\nabla\left(\frac{f}{g}\right)(x_0, y_0) = \frac{g(x_0, y_0)(\nabla f)(x_0, y_0) - f(x_0, y_0)(\nabla g)(x_0, y_0)}{g(x_0, y_0)^2}.$$

Proposition (Chain Rules)

Let $D \subset \mathbb{R}^2$ and let (x_0, y_0) be an interior point of D, and let $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0).

(i) Let $E \subset \mathbb{R}$ be such that $f(D) \subset E$ and $z_0 := f(x_0, y_0)$ is an interior point of E. If $g : E \to \mathbb{R}$ is differentiable at $f(x_0, y_0)$, then the function $h := g \circ f : D \to \mathbb{R}$ is differentiable at (x_0, y_0),

$$h_x(x_0, y_0) = g'(z_0)\, f_x(x_0, y_0) \quad \text{and} \quad h_y(x_0, y_0) = g'(z_0)\, f_y(x_0, y_0).$$

(ii) Let $E \subset \mathbb{R}$, and let t_0 be an interior point of E. If $x, y : E \to \mathbb{R}$ are differentiable at t_0, and if $(x(t), y(t)) \in D$ for all $t \in E$ and $(x(t_0), y(t_0)) := (x_0, y_0)$, then the function $\phi : E \to \mathbb{R}$ defined by $\phi(t) := f(x(t), y(t))$ for $t \in E$ is differentiable at t_0, and

$$\phi'(t_0) = f_x(x_0, y_0)\, x'(t_0) + f_y(x_0, y_0)\, y'(t_0).$$

Proposition (Chain Rules: continued)

(iii) Let $E \subset \mathbb{R}^2$, and let (u_0, v_0) be an interior point of E. If $x, y : E \to \mathbb{R}$ are differentiable at (u_0, v_0), and if $(x(u, v), y(u, v)) \in D$ for all $(u, v) \in E$ and $(x(u_0, v_0), y(u_0, v_0)) := (x_0, y_0)$, then the function $F : E \to \mathbb{R}$ defined by $F(u, v) := f(x(u, v), y(u, v))$ for $(u, v) \in E$ is differentiable at (u_0, v_0),

$$F_u(u_0, v_0) = f_x(x_0, y_0)\, x_u(u_0, v_0) + f_y(x_0, y_0)\, y_u(u_0, v_0),$$

and

$$F_v(u_0, v_0) = f_x(x_0, y_0)\, x_v(u_0, v_0) + f_y(x_0, y_0)\, y_v(u_0, v_0).$$

It is often helpful to write the identities given by the Chain Rules in an informal but suggestive notation as follows.

(i) If $z = f(x, y)$ and $w = g(z)$, then w is a function of (x, y), and $\dfrac{\partial w}{\partial x} = \dfrac{dw}{dz}\dfrac{\partial z}{\partial x}, \quad \dfrac{\partial w}{\partial y} = \dfrac{dw}{dz}\dfrac{\partial z}{\partial y}.$

(ii) If $z = f(x, y)$ and if $x = x(t)$, $y = y(t)$, then z is a function of t, and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

(iii) If $z = f(x, y)$ and if $x = x(u, v)$, $y = y(u, v)$, then z is a function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

It should be noted that the identities in (i), (ii), and (iii) above are valid when the concerned (partial) derivatives are evaluated at appropriate points and when, for instance, each of the (partial) derivatives exists in a neighbourhood of the relevant point and is continuous at that point.

The conclusions of Chain Rules (i), (ii) and (iii) can be written in terms of the gradient $(\nabla f)(x_0, y_0)$ of f at (x_0, y_0) as follows:

$$\text{(i)} \quad (\nabla h)(x_0, y_0) = g'(z_0)(\nabla f)(x_0, y_0)$$

$$\text{(ii)} \quad \phi'(t_0) = (\nabla f)(x_0, y_0) \cdot \big(x'(t_0), y'(t_0)\big)$$

$$\text{(iii)} \quad (\nabla F)(u_0, v_0) = \Big((\nabla f)(x_0, y_0) \cdot \big(x_u(u_0, v_0), y_u(u_0, v_0)\big), \; (\nabla f)(x_0, y_0) \cdot \big(x_v(u_0, v_0), y_v(u_0, v_0)\big)\Big)$$

The Chain Rules given above can be proved by using the Increment Lemma.

Examples illustrating the Chain Rules:

(i) Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x, y) := xy$ for $(x, y) \in \mathbb{R}^2$, and $g(z) := \sin z$ for $z \in \mathbb{R}$. By the Chain Rule, the composite function $(g \circ f)(x, y) := \sin(xy)$ is differentiable at every point of $\mathbb{R}^2$, and

$$\frac{\partial(g\circ f)}{\partial x} = \frac{dg}{dz}\frac{\partial z}{\partial x} = (\cos xy)y \quad \text{and} \quad \frac{\partial(g\circ f)}{\partial y} = \frac{dg}{dz}\frac{\partial z}{\partial y} = (\cos xy)x.$$

(ii) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$, and $x, y : \mathbb{R} \to \mathbb{R}$ by $x(t) := e^t$ and $y(t) := \sin t$ for $t \in \mathbb{R}$. By the Chain Rule, the composite function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(t) := f(x(t), y(t)) = e^{2t} + \sin^2 t$ is differentiable at every point of $\mathbb{R}$, and

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (2x(t))e^t + (2y(t))\cos t \\ &= 2e^{2t} + 2\sin t \cos t. \end{aligned}$$

(iii) As in (ii) above, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$, and $x, y : \mathbb{R}^2 \to \mathbb{R}$ by $x(u, v) := u^2 - v^2$ and $y(u, v) := 2uv$ for $(u, v) \in \mathbb{R}^2$. By the Chain Rule, the composite function $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\begin{aligned} F(u, v) &:= f(x(u, v), y(u, v)) = (u^2 - v^2)^2 + (2uv)^2 \\ &= u^4 + 2u^2v^2 + v^4 \end{aligned}$$

is differentiable at every point of $\mathbb{R}^2$, and

$$\begin{aligned} \frac{\partial F}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2x(2u) + 2y(2v) \\ &= 2(u^2 - v^2)(2u) + 2(2uv)(2v) = 4u(u^2 + v^2), \\ \frac{\partial F}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2x(-2v) + 2y(2u) \\ &= 2(u^2 - v^2)(-2v) + 2(2uv)(2u) = 4v(u^2 + v^2). \end{aligned}$$

Geometric Interpretation of the Gradient

Suppose $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is differentiable at an interior point (x_0, y_0) of D, and suppose $(\nabla f)(x_0, y_0) \neq (0, 0)$. Then for every unit vector $\boldsymbol{u}$,

$$(\boldsymbol{D_u} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \boldsymbol{u} = \|(\nabla f)(x_0, y_0)\| \cos\theta,$$

where $\theta \in [0, \pi]$ is the angle between $(\nabla f)(x_0, y_0)$ and $\boldsymbol{u}$. Hence

- $(\boldsymbol{D_u} f)(x_0, y_0)$ is maximum when $\cos\theta = 1$, that is, $\theta = 0$, so that $\boldsymbol{u} = \dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$.
- $(\boldsymbol{D_u} f)(x_0, y_0)$ is minimum when $\cos\theta = -1$, that is, $\theta = \pi$, so that $\boldsymbol{u} = -\dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$.
- $(\boldsymbol{D_u} f)(x_0, y_0) = 0$ when $\cos\theta = 0$, that is, $\theta = \pi/2$, so that $\boldsymbol{u}$ is perpendicular to $(\nabla f)(x_0, y_0)$.

Example:

Let $f(x,y) := 4 - x^2 - y^2$ for $(x,y) \in \mathbb{R}^2$. Then f is differentiable on $\mathbb{R}^2$, and $(\nabla f)(x_0, y_0) = (-2x_0, -2y_0)$ for all $(x_0, y_0) \in \mathbb{R}^2$.

Let $(x_0, y_0) := (1,1)$. Then $(\nabla f)(1,1) = (-2,-2)$.

Consider the surface $z = f(x,y)$. At $(1,1)$,

- the direction of steepest ascent is $(-2,-2)/\|(-2,-2)\| = (-1/\sqrt{2}, -1/\sqrt{2})$,
- the direction of steepest descent is $-(-2,-2)/\|(-2,-2)\| = (1/\sqrt{2}, 1/\sqrt{2})$,
- the directions (u, u_2) of no change (in height) satisfy $(-2,-2) \cdot (u_1, u_2) = 0$, and so, they are $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$.

See the picture on the next slide.

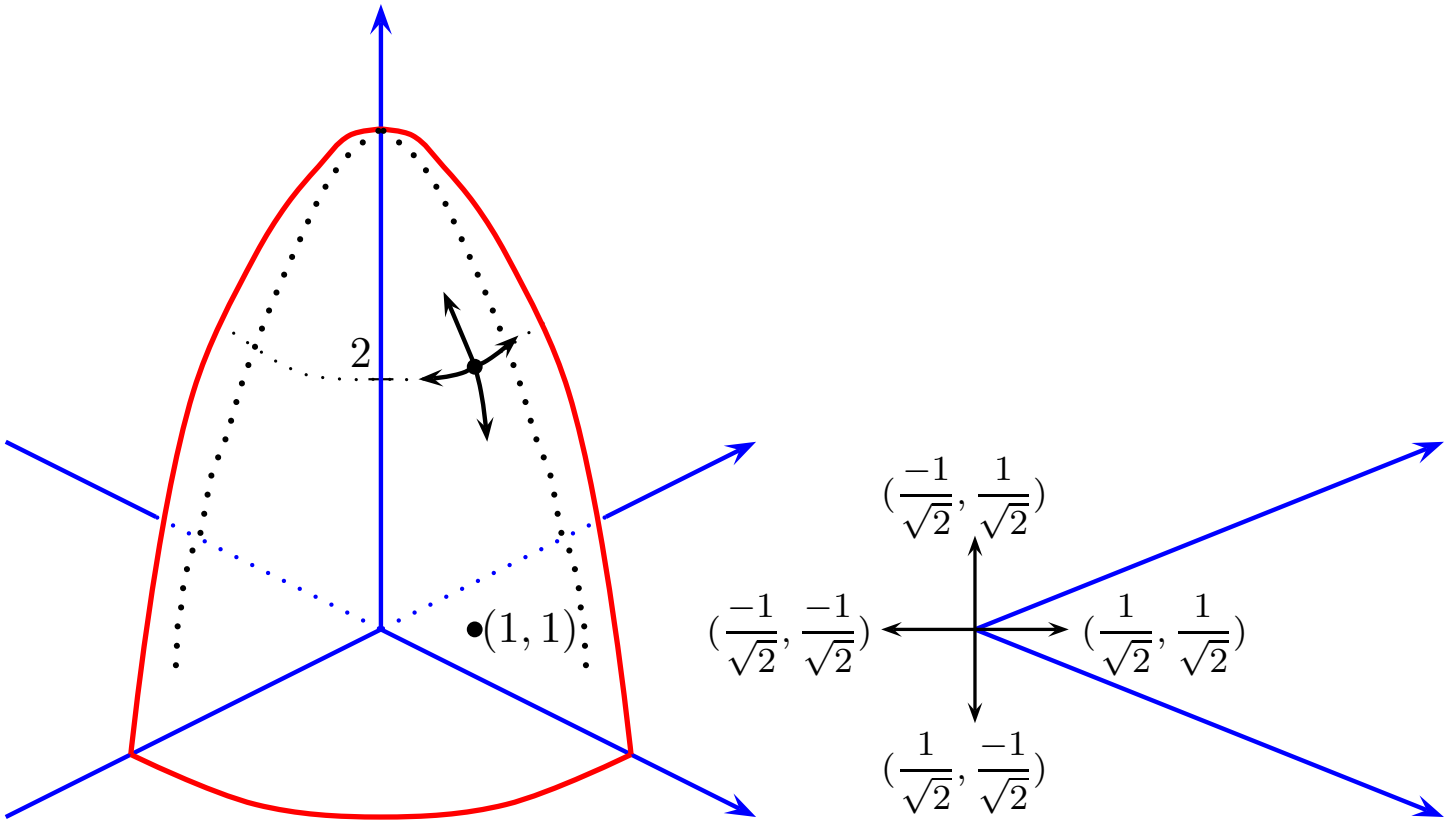

Figure: Directions of steepest ascent, of steepest decent and of no change in height

Tangent Line

Recall the one variable situation:
Let $a < b$ and $x_0 \in (a, b)$. Let a function $f : (a, b) \to \mathbb{R}$ be differentiable at x_0. Then the equation of the **tangent line** to the curve $y = f(x)$ in $\mathbb{R}^2$ at $(x_0, f(x_0))$ is given by

$$y - f(x_0) = f'(x_0)(x - x_0).$$

More generally, suppose $D \subset \mathbb{R}^2$ and $P_0 := (x_0, y_0)$ is an interior point of D. Let a function $F : D \to \mathbb{R}$ have partial derivatives at P_0, and let $(\nabla F)(P_0) \neq (0, 0)$. Suppose F defines a curve C in $\mathbb{R}^2$ (implicitly) by the equation $F(x, y) = 0$ for $(x, y) \in D$, and P_0 lies on C. Then the equation of the **tangent line** to C at P_0 is given by

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) = 0.$$

Note: If $D := (a, b) \times \mathbb{R}$, $f : (a, b) \to \mathbb{R}$, and for $(x, y) \in D$, $F(x, y) := y - f(x)$, then $y_0 = f(x_0)$, $F_x(P_0) = -f'(x_0)$ and $F_y(P_0) = 1$. We recover the earlier equation of the tangent line.

Tangent Plane

Let us turn to the two variable situation:
Let $D \subset \mathbb{R}^2$, (x_0, y_0) be an interior point of D, and let a function $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0). Then the equation of the **tangent plane** to the surface $z = f(x, y)$ in $\mathbb{R}^3$ at $(x_0, y_0, f(x_0, y_0))$ is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

More generally, suppose $E \subset \mathbb{R}^3$ and $P_0 := (x_0, y_0, z_0)$ is an interior point of E. Let a function $F : E \to \mathbb{R}$ have partial derivatives at P_0, and let $(\nabla F)(P_0) \neq (0, 0, 0)$. Suppose F (implicitly) defines a surface S in $\mathbb{R}^3$ by the equation $F(x, y, z) = 0$ for $(x, y, z) \in E$, and P_0 lies on S. Then the equation of the **tangent plane** to S at P_0 is given by

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0.$$

Note: As before, if we let $F(x, y, z) := z - f(x, y)$, then we recover the earlier equation of the tangent plane.

Normal Line

Now $(\nabla F)(P_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$ for all (x, y, z) on the tangent plane to the surface S, that is, $(\nabla F)(P_0)$ is perpendicular to the tangent plane. The line passing through $P_0 = (x_0, y_0, z_0)$ and parallel to the nonzero vector $(\nabla F)(P_0) = (F_x(P_0), F_y(P_0), F_z(P_0))$ is called the **normal line** to the surface defined by $F(x, y, z) = 0$ at P_0. The parametric equations of this normal line are

$$x = x_0 + F_x(P_0)t, \quad y = y_0 + F_y(P_0)t, \quad z = z_0 + F_z(P_0)t, \quad t \in \mathbb{R}.$$

If all $F_x(P_0), F_y(P_0), F_z(P_0)$ are nonzero, then the equations are

$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}.$$

Also, the parametric equations of the normal line to the surface defined by $z - f(x, y) = 0$ at (x_0, y_0, z_0) are

$$x = x_0 - f_x(x_0, y_0)t, \quad y = y_0 - f_y(x_0, y_0)t, \quad z = f(x_0, y_0) + t, \quad t \in \mathbb{R}.$$

Tangent Vector to a Curve

We now describe a crucial property of the normal line to a surface. Let C be a parametrized smooth curve in $\mathbb{R}^3$ given by

$$t \longmapsto (x(t), y(t), z(t)), \quad t \in [\alpha, \beta].$$

Let $t_0 \in (\alpha, \beta)$ be such that $(x'(t_0), y'(t_0), z'(t_0)) \neq (0, 0, 0)$. Then the vector $\big(x'(t_0), y'(t_0), z'(t_0)\big)$ is called the **tangent vector** to the curve C at the point $\big(x(t_0), y(t_0), z(t_0)\big)$ on it.

Now suppose that the curve C lies on a surface S defined by $F(x, y, z) = 0$, that is, $\phi(t) := F\,(x(t), y(t), z(t)) = 0$ for all $t \in [\alpha, \beta]$. By the Chain Rule,

$$0 = \phi'(t_0) = F_x(P_0)x'(t_0) + F_y(P_0)y'(t_0) + F_z(P_0)z'(t_0).$$

Thus the normal line to the surface S at P_0 is perpendicular to tangent vectors to all curves on S that pass through P_0.

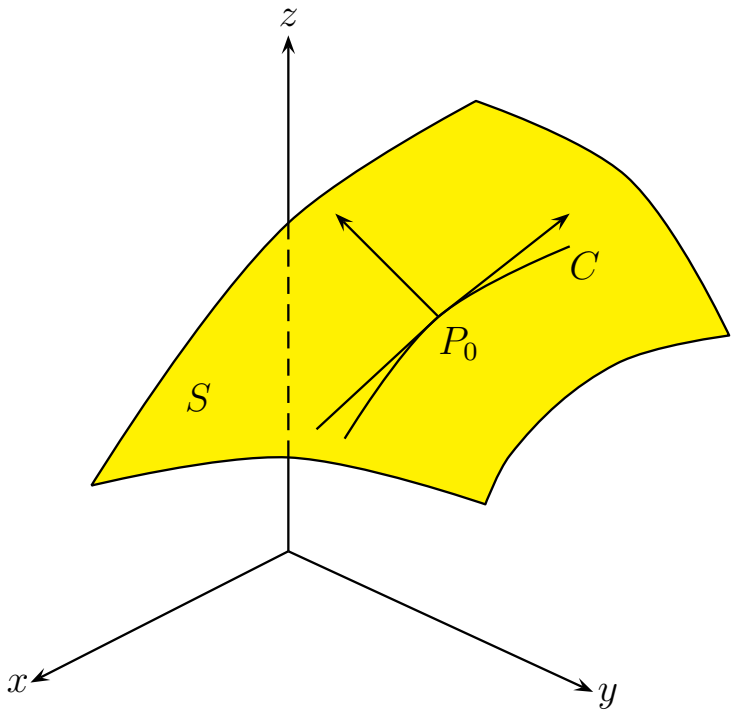

Figure: Normal line to the surface S at P_0 and a curve C on S

Special case: Suppose a surface S is given by $z = f(x, y)$ for $(x, y) \in D$. Let $Q_0 := (x_0, y_0) \in D$, and let a parametrized curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, pass through Q_0. Let $z(t) := f(x(t), y(t))$, $t \in [\alpha, \beta]$. Then the curve C given by $(x(t), y(t), z(t))$, $t \in [\alpha, \beta]$, lies on S, and passes through $P_0 := f(Q_0)$.

Hence the normal line

$$x = x_0 - f_x(Q_0)t,\ y = y_0 - f_y(Q_0)t,\ z = f(Q_0) + t,\ t \in \mathbb{R},$$

to the surface S at P_0 is perpendicular to the tangent vector $(x'(t_0), y'(t_0), z'(t_0))$ to the curve C at P_0, that is,
$(-f_x(Q_0), -f_y(Q_0), 1) \cdot (x'(t_0), y'(t_0), z'(t_0)) = 0.$
(This can also be independently verified as
$z'(t_0) = f_x(Q_0)x'(t_0) + f_y(Q_0)y'(t_0)$ by the chain rule.)

In particular, let $c_0 := f(x_0, y_0)$, and suppose the level curve $\{(x, y) \in D : f(x, y) = c_0\}$ is parametrized by $(x(t), y(t))$ with $t \in [\alpha, \beta]$. Then the contour line C is parametrized by $(x(t), y(t), c_0)$ with $t \in [\alpha, \beta]$, and it lies on the surface S. Hence the normal line to S at $P_0 := (x_0, y_0, c_0)$ is perpendicular to the corresponding contour line C.

Examples:

(i) Let $F(x, y, z) := x^2 + y^2 + z^2 - 1$ for $(x, y, z) \in \mathbb{R}^3$. Then $(\nabla F)(x, y, z) = (2x, 2y, 2z)$ and $(\nabla F)(0, 0, 1) = (0, 0, 2)$. The tangent plane to the surface defined by $F(x, y, z) = 0$ at the point $(0, 0, 1)$ is $0(x-0)+0(y-0)+2(z-1) = 0$, that is, $z = 1$. The normal line at $(0, 0, 1)$ is $x = 0, y = 0, z = 1 + 2t$, where $t \in \mathbb{R}$, that is, the z-axis. Note that the surface is the unit sphere in $\mathbb{R}^3$, and the point $(0, 0, 1)$ is its north pole.

(ii) Let $F(x, y, z) := e^x + \sin y - \cos z$ for $(x, y, z) \in \mathbb{R}^3$. Then $(\nabla F)(x, y, z) = (e^x, \cos y, \sin z)$ for $(x, y, z) \in \mathbb{R}^3$ and $(\nabla F)(0, 0, 0) = (1, 1, 0)$. The tangent plane to the surface defined by $F(x, y, z) = 0$ at $(0, 0, 0)$ is $x + y + 0z = 0$, that is, $x + y = 0$. The normal line at $(0, 0, 0)$ is $x = t, y = t, z = 0$, where $t \in \mathbb{R}$, that is, the intersection of the planes given by $x = y$ and $z = 0$.

(iii) Let $F(x, y, z) := x^2 + y^2 - 1$ for $(x, y, z) \in \mathbb{R}^3$. Then $(\nabla F)(x, y, z) = (2x, 2y, 0)$ for $(x, y, z) \in \mathbb{R}^3$. The tangent plane to the surface defined by $F(x, y, z) = 0$ at the point $P_0 := (\cos\theta_0, \sin\theta_0, z_0)$ on the surface is $2\cos\theta_0(x - \cos\theta_0) + 2\sin\theta_0(y - \sin\theta_0) + 0(z - z_0) = 0$, that is, $\cos\theta_0\ x + \sin\theta_0\ y = 1$. The normal line at P_0 is $x = \cos\theta_0(1 + 2t)$, $y = \sin\theta_0(1 + 2t)$, $z = z_0$, where $t \in \mathbb{R}$. Note that the surface is a cylinder in $\mathbb{R}^3$.

A surface in $\mathbb{R}^3$ can be parametrically given by $\big(x(u, v), y(u, v), z(u, v)\big)$ for $(u, v) \in E$, where $E \subset \mathbb{R}^2$, and $x,\ y,\ z : E \to \mathbb{R}$ are continuous functions. A point $P_0 := (x_0, y_0, z_0)$ on this surface is given by $\big(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0)\big)$ for some $(u_0, v_0) \in E$. In a subsequent lecture, we shall define a normal vector to this parametrically given surface at P_0, and also obtain the equation of the tangent plane to the surface at P_0.