MA 105 : Calculus
Division 1, Lecture 16

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Recap of the previous lecture

- Necessary condition for differentiability
- Sufficient condition for differentiability
- Examples
- Basic properties of differentiable functions
- Chain Rules (in three avatars). Examples
- Geometric Interpretation of the Gradient
- Tangent plane and normal line
- Tangent vector to a parametrized curve

Tangent Plane

Let $D \subset \mathbb{R}^2$, (x_0, y_0) be an interior point of D, and let a function $f : D \to \mathbb{R}$ be differentiable at (x_0, y_0). Then the equation of the **tangent plane** to the surface $z = f(x, y)$ in $\mathbb{R}^3$ at $(x_0, y_0, f(x_0, y_0))$ is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

More generally, suppose $E \subset \mathbb{R}^3$ and $P_0 := (x_0, y_0, z_0)$ is an interior point of E. Let a function $F : E \to \mathbb{R}$ have partial derivatives at P_0, and let $(\nabla F)(P_0) \neq (0, 0, 0)$. Suppose F (implicitly) defines a surface S in $\mathbb{R}^3$ by the equation $F(x, y, z) = 0$ for $(x, y, z) \in E$, and P_0 lies on S. Then the equation of the **tangent plane** to S at P_0 is given by

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0.$$

Note: As before, if we let $F(x, y, z) := z - f(x, y)$, then we recover the earlier equation of the tangent plane.

Normal Line

Now $(\nabla F)(P_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$ for all (x, y, z) on the tangent plane to the surface S, that is, $(\nabla F)(P_0)$ is perpendicular to the tangent plane. The line passing through $P_0 = (x_0, y_0, z_0)$ and parallel to the nonzero vector $(\nabla F)(P_0) = (F_x(P_0), F_y(P_0), F_z(P_0))$ is called the **normal line** to the surface defined by $F(x, y, z) = 0$ at P_0. The parametric equations of this normal line are

$$x = x_0 + F_x(P_0)t, \quad y = y_0 + F_y(P_0)t, \quad z = z_0 + F_z(P_0)t, \quad t \in \mathbb{R}.$$

If all $F_x(P_0), F_y(P_0), F_z(P_0)$ are nonzero, then the equations are

$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}.$$

Also, the parametric equations of the normal line to the surface defined by $z - f(x, y) = 0$ at (x_0, y_0, z_0) are

$$x = x_0 - f_x(x_0, y_0)t, \quad y = y_0 - f_y(x_0, y_0)t, \quad z = f(x_0, y_0) + t, \quad t \in \mathbb{R}.$$

Tangent Vector to a Curve

We now describe a crucial property of the normal line to a surface. Let C be a parametrized smooth curve in $\mathbb{R}^3$ given by

$$t \longmapsto (x(t), y(t), z(t)), \quad t \in [\alpha, \beta].$$

Let $t_0 \in (\alpha, \beta)$ be such that $(x'(t_0), y'(t_0), z'(t_0)) \neq (0, 0, 0)$. Then the vector $\big(x'(t_0), y'(t_0), z'(t_0)\big)$ is called the **tangent vector** to the curve C at the point $\big(x(t_0), y(t_0), z(t_0)\big)$ on it.

Now suppose that the curve C lies on a surface S defined by $F(x, y, z) = 0$, that is, $\phi(t) := F\,(x(t), y(t), z(t)) = 0$ for all $t \in [\alpha, \beta]$. By the Chain Rule,

$$0 = \phi'(t_0) = F_x(P_0)x'(t_0) + F_y(P_0)y'(t_0) + F_z(P_0)z'(t_0).$$

Thus the normal line to the surface S at P_0 is perpendicular to tangent vectors to all curves on S that pass through P_0.

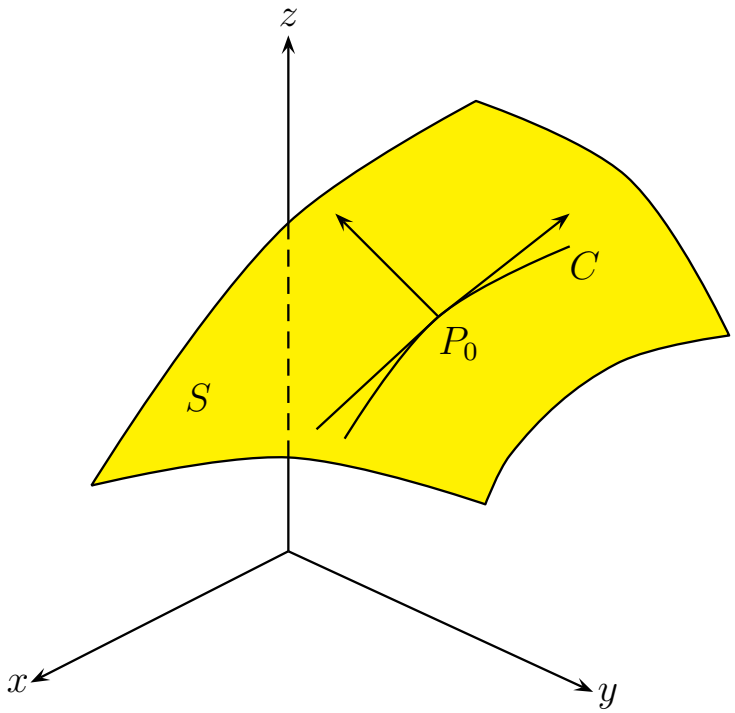

Figure: Normal line to the surface S at P_0 and a curve C on S

Special case: Suppose a surface S is given by $z = f(x, y)$ for $(x, y) \in D$. Let $Q_0 := (x_0, y_0) \in D$, and let a parametrized curve given by $(x(t), y(t))$, $t \in [\alpha, \beta]$, pass through Q_0. Let $z(t) := f(x(t), y(t))$, $t \in [\alpha, \beta]$. Then the curve C given by $(x(t), y(t), z(t))$, $t \in [\alpha, \beta]$, lies on S, and passes through $P_0 := f(Q_0)$.

Hence the normal line

$$x = x_0 - f_x(Q_0)t,\ y = y_0 - f_y(Q_0)t,\ z = f(Q_0) + t,\ t \in \mathbb{R},$$

to the surface S at P_0 is perpendicular to the tangent vector $(x'(t_0), y'(t_0), z'(t_0))$ to the curve C at P_0, that is, $(-f_x(Q_0), -f_y(Q_0), 1) \cdot (x'(t_0), y'(t_0), z'(t_0)) = 0$.
(This can also be independently verified as $z'(t_0) = f_x(Q_0)x'(t_0) + f_y(Q_0)y'(t_0)$ by the chain rule.)

In particular, let $c_0 := f(x_0, y_0)$, and suppose the level curve $\{(x, y) \in D : f(x, y) = c_0\}$ is parametrized by $(x(t), y(t))$ with $t \in [\alpha, \beta]$. Then the contour line C is parametrized by $(x(t), y(t), c_0)$ with $t \in [\alpha, \beta]$, and it lies on the surface S. Hence the normal line to S at $P_0 := (x_0, y_0, c_0)$ is perpendicular to the corresponding contour line C.

Closed Sets and Boundary Points

Let $D \subset \mathbb{R}^2$. We say that D is **closed** if every sequence in D that converges in $\mathbb{R}^2$ converges to a point of D itself.

A point $(x_0, y_0) \in \mathbb{R}^2$ is called a **boundary point** of D if there is a sequence in D that converges to (x_0, y_0) and also a sequence in the complement of D that converges to (x_0, y_0).

The set of all boundary points of D in $\mathbb{R}^2$ is called the **boundary** of D (in $\mathbb{R}^2$). We shall denote it by ∂D.

One can show that D is closed $\iff$ $\partial D \subset D$ (Tutorial 9).

Examples: Let $D_1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $D_2 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then D_1 is closed, but D_2 is not. In fact, $\partial D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = \partial D_2$.

Let $D := [a, b] \times [c, d]$. Then D is a closed subset of $\mathbb{R}^2$ and $\partial D = \big\{(x, y) \in D : x \in \{a, b\}$ or $y \in \{c, d\}\big\}$, which is the union of the 4 sides of D.

Finding Global Extrema

Let $D \subset \mathbb{R}^2$. We say that D is **bounded** if there is $\alpha > 0$ such that $\|(x, y)\| \leq \alpha$ for all $(x, y) \in D$.

Theorem (Extreme Value Theorem)

Let D be a nonempty closed and bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous. Then f is bounded and attains its bounds on D, that is, there are $(x_1, y_1), (x_2, y_2) \in D$ such that

$$\begin{aligned} f(x_1, y_1) &= \min\{f(x, y) : (x, y) \in D\}, \\ f(x_2, y_2) &= \max\{f(x, y) : (x, y) \in D\}. \end{aligned}$$

As in the case of the extreme value theorem for functions of one variable, the proof is quite involved, and we shall skip it.

Critical Points and Global Exrema

Let $D \subset \mathbb{R}^2$, and let $f : D \to \mathbb{R}$. An interior point (x_0, y_0) of D is called a **critical point** of f if either $(\nabla f)(x_0, y_0)$ does not exist, or $(\nabla f)(x_0, y_0)$ exists and is equal to $(0, 0)$.

Proposition

Let D be a nonempty, closed and bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be continuous. Then the global minimum m as well as the global maximum M of f is attained either at a critical point of f, or at a boundary point of D.

Proof. Let $(x_1, y_1) \in D$ be such that $f(x_1, y_1) = m$. If $(x_1, y_1) \in \partial D$, then we are done. Otherwise (x_1, y_1) is an interior point of D, and f has a local minimum at (x_1, y_1). If $(\nabla f)(x_1, y_1)$ does not exist, then (x_1, y_1) is a critical point of f. If $(\nabla f)(x_1, y_1)$ exists, then necessarily $(\nabla f)(x_1, y_1) = (0, 0)$, and so (x_1, y_1) is a critical point of f. Similarly, for M. □

Finding Global Extrema

Let D be a nonempty closed and bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a continuous function. Here is a procedure for finding the global extrema of f on D.

(i) Determine the critical points of f in D. (Do not wonder whether f has a local extremum or a saddle point.)

(ii) Determine the boundary ∂D of D. Often ∂D consists of closed and bounded 'one dimensional pieces'.

(iii) Restrict the function f to each 'one dimensional piece' of the boundary of D. Use methods of one variable calculus to find the global extrema of f on that piece.

(iv) Compare the values of f at the critical points of f and at the extreme values of f on the one dimensional piece(s) of ∂D. The largest (resp., smallest) among these values is the global maximum (resp., minimum) of f on D.

We shall later give an alternative to (iii) above, called the Orthogonal Gradient Theorem.

Finding Global Extrema: Example

Example:
Let $D := \{(x, y) \in \mathbb{R}^2 : x,\ y \geq 0 \text{ and } x + y \leq 9\}$, and let $f(x, y) := 2 + 2x + 2y - x^2 - y^2$ for $(x, y) \in D$.

(i) (1,1) is the only critical point of f, and $f(1, 1) = 4$.

(ii) The boundary of D has three closed one dimensional pieces, one each on the lines $x = 0$, $y = 0$ and $y = 9 - x$.

(iii) $f(0, y) = 2 + 2y - y^2$ for $y \in [0, 9]$. At the critical point $y = 1$, $f(0, 1) = 3$. Also, $f(0, 0) = 2$ and $f(0, 9) = -61$.

$f(x, 0) = 2 + 2x - x^2$ for $x \in [0, 9]$. At the critical point $x = 1$, $f(1, 0) = 3$. Also, $f(0, 0) = 2$ and $f(9, 0) = -61$.

$f(x, 9 - x) = -2x^2 + 18x - 61$ for $x \in [0, 9]$. At the critical point $x = 9/2$, $f(9/2, 9/2) = -41/2$. Also, $f(0, 9) = -61 = f(9, 0)$.

Finding Global Extrema

(iv) The above values of f are tabulated as follows:

(x, y)	(1,1)	(0,1)	(0,0)	(0,9)	(1,0)	(9,0)	(9/2,9/2)
$f(x, y)$	4	3	2	-61	3	-61	-41/2

Thus, the global maximum of f on D is 4, and it is attained at $(1, 1)$, while the global minimum of f on D is -61, and it is attained at $(0, 9)$ as well as at $(9, 0)$.

Aliter: Clearly, $f(x, y) = 4 - (x - 1)^2 - (y - 1)^2 \leq 4$ for all $(x, y) \in \mathbb{R}^2$. Also, since $(9, 0)$ and $(0, 9)$ are the farthest points of D from the point $(1, 1)$, we see that $f(x, y) \geq 4 - (9 - 1)^2 - (0 - 1)^2 = -61$ for all $(x, y) \in D$.

In fact, $f(1, 1) = 4$ and $f(9, 0) = f(0, 9) = -61$.

If D is a closed and bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ is continuous, then the values of f on the boundary ∂D of D play a special role when we find the global extrema of f on D.

Since the boundary of D often consists of one or more curves, we need to find the extreme values of f, when it is restricted to a curve in D. In this connection the following result is useful.

Proposition (Orthogonal Gradient Theorem)

Let $E \subset \mathbb{R}^2$ and (x_0, y_0) be an interior point of E. Let C be a parametrized curve in E passing through (x_0, y_0) given by $(x(t), y(t))$, where $x, y : [\alpha, \beta] \to E$, and let $t_0 \in (\alpha, \beta)$ with $x(t_0) = x_0$, $y(t_0) = y_0$. Suppose a function $f : E \to \mathbb{R}$, when restricted to C, has a local extremum at (x_0, y_0). If x and y are differentiable at t_0, and f is differentiable at (x_0, y_0), then $(\nabla f)(x_0, y_0) \cdot (x'(t_0), y'(t_0)) = 0$, that is, the gradient of f at (x_0, y_0) is perpendicular to the tangent vector of C at (x_0, y_0).

Proof. Define $\phi : [\alpha, \beta] \to \mathbb{R}$ by $\phi(t) := f(x(t), y(t))$. Then ϕ has a local extremum at t_0, and so $\phi'(t_0) = 0$. But by the chain rule (ii), $\phi'(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0)$. □

Example: Let $E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 2\}$, and $f(x, y) := x\,y$ for $(x, y) \in E$. Let us find the global extrema of f on $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

- $(\nabla f)(x, y) = (y, x)$ if $x^2 + y^2 < 1$. Hence $(0, 0)$ is the only critical point of f, and $f(0, 0) = 0$.
- The boundary of D is given by $(x(t), y(t))$, where $x(t) := \cos t$, $y(t) := \sin t$ for $t \in [-\pi, \pi]$. Each point of D is an interior point of E.
- $(\nabla f)(x(t), y(t)) \cdot (x'(t), y'(t)) = -\sin^2 t + \cos^2 t = 0$ $\iff$ $t = \pm\pi/4, \pm 3\pi/4$, that is, $(x(t), y(t))$ $= \big(\pm 1/\sqrt{2},\ \pm 1/\sqrt{2}\big)$.
- $f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/2$ and $f(-1/\sqrt{2}, 1/\sqrt{2}) = f(1/\sqrt{2}, -1/\sqrt{2}) = -1/2$.

So the maximum of f is $1/2$ and the minimum of f is $-1/2$.

Constrained Extrema

Let $D \subset \mathbb{R}^2$, and let $f, g : D \to \mathbb{R}$. Let us address the question of finding the maximum and the minimum of f on D subject to the constraint $g = 0$. The equation $g(x, y) = 0$, $(x, y) \in D$ often defines a curve C in D implicitly.

For example, let $D := \{(x, y) \in \mathbb{R}^2 : 1/4 \leq x^2 + y^2 \leq 4\}$, and let $g(x, y) := x^2 + y^2 - 1$ for $(x, y) \in D$. Then the curve C defined by $g(x, y) = 0$ is the unit circle in $\mathbb{R}^2$, and it lies in D.

Let (x_0, y_0) be an interior point of D such that $g(x_0, y_0) = 0$. Suppose f has a local extremum at (x_0, y_0) subject to the constraint $g(x, y) = 0$. We would like to show that the gradient vectors $(\nabla f)(x_0, y_0)$ and $(\nabla g)(x_0, y_0)$ are parallel.

First we give a general idea, and then we shall quote a precise result. Suppose f and g are differentiable at (x_0, y_0).

Assume that we are able to solve the equation $g(x, y) = 0$ for y in terms of x near x_0, that is, there a function η defined near x_0 such that $g(x, \eta(x)) = 0$ and $\eta(x_0) = y_0$. If $\eta'(x_0)$ exists, then $g_x(x_0, y_0) + g_y(x_0, y_0)\eta'(x_0) = 0$ by the chain rule (ii).

Consider the function $\phi(x) := f(x, \eta(x))$ for x near x_0. Now ϕ has a local extremum at x_0, and so $\phi'(x_0) = 0$, that is, $f_x(x_0, y_0) + f_y(x_0, y_0)\,\eta'(x_0) = 0$ again by the chain rule (ii).

It follows that $f_x(x_0, y_0)g_y(x_0, y_0) = f_y(x_0, y_0)g_x(x_0, y_0)$, that is, the gradient vectors $(\nabla f)(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0))$ and $(\nabla g)(x_0, y_0) = (g_x(x_0, y_0), g_y(x_0, y_0))$ are parallel.

In fact, if $g_y(x_0, y_0) \neq 0$, then $(\nabla f)(x_0, y_0) = \lambda_0(\nabla g)(x_0, y_0)$, where $\lambda_0 := f_y(x_0, y_0)/g_y(x_0, y_0)$.

Similarly, if $g(x, y) = 0$ can be solved for x in terms of y near y_0 and if $g_x(x_0, y_0) \neq 0$, then $(\nabla f)(x_0, y_0) = \lambda_0(\nabla g)(x_0, y_0)$, where $\lambda_0 := f_x(x_0, y_0)/g_x(x_0, y_0)$.

Proposition (Lagrange Multiplier Theorem)

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D. Suppose f, $g : D \to \mathbb{R}$ have continuous partial derivatives in a neighbourhood of (x_0, y_0). Let $C := \{(x, y) \in D : g(x, y) = 0\}$. Suppose (i) $g(x_0, y_0) = 0$, (ii) $(\nabla g)(x_0, y_0) \neq (0, 0)$, and (iii) the function f, when restricted to C, has a local extremum at (x_0, y_0). Then there is $\lambda_0 \in \mathbb{R}$ such that

$$(\nabla f)(x_0, y_0) = \lambda_0 (\nabla g)(x_0, y_0).$$

The real number λ_0 is called a **Lagrange multiplier**.

We now have the following procedure to determine the global extremum of a function $f : D \to \mathbb{R}$, subject to the constraint $g = 0$. Consider a new variable λ, called an **undetermined multiplier**, and seek simultaneous solutions of the following 3 equations in the 3 unknowns x, y, λ:
$(\nabla f)(x, y) = \lambda (\nabla g)(x, y)$ and $g(x, y) = 0$, $(x, y) \in D$.

Suppose f has a global extremum on the set $\{(x, y) \in D : g(x, y) = 0\}$. (For example, when this set is nonempty, closed and bounded, and f is continuous on it). Then it is also a local extremum of f, and so it is attained either at a simultaneous solution (x_0, y_0) of the above two equations where $\nabla g(x_0, y_0) \neq (0, 0)$, or at a point (x_1, y_1) where $g(x_1, y_1) = 0$ and $(\nabla g)(x_1, y_1) = (0, 0)$.

Examples: (i) Consider the problem of maximizing/minimizing $f(x, y) := x\,y$ subject to $x + y = 2$. For this purpose, we find the global extrema of $\phi(x) := f(x, 2 - x) = x(2 - x)$, $x \in \mathbb{R}$. Since $\phi'(x) = 2(1 - x)$, $x \in \mathbb{R}$, we see that $\phi'(x) > 0$ if $x < 1$ and $\phi'(x) < 0$ if $x > 1$. Also, ϕ is continuous at 1. As in the first derivative test, $\phi(1) = 1$ is the global maximum of ϕ, that is, 1 is the constrained maximum of f. Also, $\phi(x) \to -\infty$ as $x \to \pm\infty$, and so f has no constrained minimum.

(ii) Consider the problem of maximizing/minimizing $f(x, y) := x\,y$, subject to $x^2 + y^2 - 1 = 0$.

Let $g(x, y) := x^2 + y^2 - 1$ for $(x, y) \in \mathbb{R}^2$. Note that the set $\{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$, that is, the unit circle, is nonempty, closed and bounded, and f is continuous on it. Now $(\nabla f)(x, y) = \lambda\,(\nabla g)(x, y)$ and $g(x, y) = 0$ means

$$y = 2\lambda x, \quad x = 2\lambda y, \quad \text{and} \quad x^2 + y^2 - 1 = 0.$$

Then $y\,x = 4\lambda^2 xy$, that is, $4\lambda^2 = 1$, since $x\,y \neq 0$. Thus $\lambda = \pm 1/2$, and the simultaneous solutions of the above equations are given by $(x, y) = \left(\pm 1/\sqrt{2},\, \pm 1/\sqrt{2}\right)$. Also, $(\nabla g)(x, y) \neq (0, 0)$ whenever $g(x, y) = 0$.

Thus the hypotheses of the Lagrange Multiplier Theorem are satisfied. Hence the maximum of f on the unit circle is $f\left(1/\sqrt{2},\, 1/\sqrt{2}\right) = f(-1/\sqrt{2},\, -1/\sqrt{2}) = 1/2$, while the minimum of f on the unit circle is $f\left(1/\sqrt{2},\, -1/\sqrt{2}\right) = f\left(-1/\sqrt{2},\, 1/\sqrt{2}\right) = -1/2$.

Sequences in $\mathbb{R}^3$ and Functions of 3 Variables

A sequence in $\mathbb{R}^3$ is denoted by $\big((x_n, y_n, z_n)\big)$, where $x_n, y_n, z_n \in \mathbb{R}$ for $n \in \mathbb{N}$. We say that a sequence $\big((x_n, y_n, z_n)\big)$ in $\mathbb{R}^3$ is **convergent** in $\mathbb{R}^3$ if the sequences $(x_n), (y_n)$ and (z_n) are convergent in $\mathbb{R}$.

Let $D \subset \mathbb{R}^3$ and $f : D \to \mathbb{R}$. We say that f is continuous at $(x_0, y_0, z_0) \in D$ if $f(x_n, y_n, z_n) \to f(x_0, y_0, z_0)$ whenever $(x_n, y_n, z_n) \to (x_0, y_0, z_0)$ in D.

We have already considered the partial derivatives of f, and denoted the gradient of f at an interior point $P_0 := (x_0, y_0, z_0)$ of D by

$$(\nabla f)(P_0) := (f_x(P_0), f_y(P_0), f_z(P_0)).$$

Also, the differentiability of f, and the local and global extrema of f can be defined in a similar manner.

Here is an analogue of the Mean Value Theorem for real-valued functions of three variables.

Let $D \subset \mathbb{R}^3$. Suppose every point of D is an interior point and the line segment joining any two points in D lies in D. Let $(x_0, y_0, z_0), (x_1, y_1, z_1) \in D$, and let $f : D \to \mathbb{R}$.

(Trivariate Mean Value Theorem)

Suppose f has continuous partial derivatives on D. Then there is $(c, d, e) \in D$ lying on the open line segment joining (x_0, y_0, z_0) and (x_1, y_1, z_1) such that $f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$ is equal to

$$(x_1 - x_0)f_x(c, d, e) + (y_1 - y_0)f_y(c, d, e) + (z_1 - z_0)f_z(c, d, e).$$

Proof: Define $\phi : [0, 1] \to \mathbb{R}$ by $\phi(t) := f(x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0))$, and use the one variable Lagrange mean value theorem for ϕ. □

In particular, if $\nabla f = 0$ on D, then f is constant on D.

The following analogue of the Orthogonal Gradient Theorem would be useful.

Let $D \subset \mathbb{R}^3$, and let (x_0, y_0, z_0) be an interior point of D. Let C be a parametrized curve in D given by $(x(t), y(t), z(t))$, where $x, y, z : [\alpha, \beta] \to D$. Suppose there is $t_0 \in (\alpha, \beta)$ such that $x(t_0) = x_0$, $y(t_0) = y_0$, $z(t_0) = z_0$, and let a function $f : D \to \mathbb{R}$, when restricted to C, have a local extremum at (x_0, y_0, z_0). If x, y and z are differentiable at t_0, and if f is differentiable at (x_0, y_0, z_0), then

$$(\nabla f)(x_0, y_0, z_0) \cdot (x'(t_0), y'(t_0), z'(t_0)) = 0,$$

that is, the gradient of f at (x_0, y_0, z_0) is perpendicular to the tangent vector of C at the point (x_0, y_0, z_0).

As before, the result follows by the Chain Rule (part (ii)) applied to the function $\phi : [\alpha, \beta] \to \mathbb{R}^3$ given by

$$\phi(t) := f(x(t), y(t), z(t)).$$

Lagrange Multipliers for Functions of 3 Variables

The Lagrange Multiplier Theorem can be adapted to a subset D of $\mathbb{R}^3$ and functions $f, g : D \to \mathbb{R}$. If f and g have continuous partial derivatives, then a global extremum of f on the set $\{(x, y, z) \in D : g(x, y, z) = 0\}$ can be found by comparing the values of f at the simultaneous solutions of the following 4 equations in the 4 unknowns x, y, z, λ:
$\nabla f = \lambda \nabla g$ and $g = 0$ at which $\nabla g \neq \mathbf{0}$. We must also consider the values of f at points where $g = 0$ and $\nabla g = \mathbf{0}$.

Further, the result can be adapted to a situation where there are two constraints, namely $g = 0$ and $h = 0$. In this case, we compare the values of f at the simultaneous solutions of the following 5 equations in the 5 unknowns x, y, z, λ, μ:
$\nabla f = \lambda \nabla g + \mu \nabla h$ and $g = 0 = h$ at which $\nabla g \neq \mathbf{0}, \nabla h \neq \mathbf{0}$ and $\nabla g \nparallel \nabla h$. We must also consider the values of f at points where $g = 0 = h$, and $\nabla g = \mathbf{0}$ or $\nabla h = \mathbf{0}$ or $\nabla g \parallel \nabla h$.

Multiple constraints

Explanation: Let $D \subset \mathbb{R}^3$ and $f : D \to \mathbb{R}$. Let C denote the curve in D obtained by intersecting the surfaces S_g and S_h defined by $g = 0$ and $h = 0$ respectively. Suppose f, when restricted to C, has a local extremum at a point P_0 of C, and the tangent vector to the curve C at P_0 is nonzero. By the three-dimensional analogue of the orthogonal gradient theorem, this tangent vector is perpendicular to $(\nabla f)(P_0)$

Suppose $(\nabla g)(P_0) \neq \mathbf{0}$, $(\nabla h)(P_0) \neq \mathbf{0}$ and $\nabla g \not\parallel \nabla h$. Now the normal lines to the surfaces S_g and S_h at P_0 are perpendicular to the tangent vector to the curve C at P_0, since C lies in S_g as well as in S_h. But the plane in $\mathbb{R}^3$ perpendicular to the tangent vector to the curve C at P_0 is two-dimensional, and so there are $\lambda_0, \mu_0 \in \mathbb{R}$ such that

$$(\nabla f)(P_0) = \lambda_0(\nabla g)(P_0) + \mu_0(\nabla h)(P_0).$$

Examples:

(i) Let us find the maximum and the minimum values of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

The continuous function f attains its bounds on the nonempty, closed and bounded subset $\{(x, y, z) : x - y + z = 1 = x^2 + y^2\}$.

The undetermined multipliers $\lambda, \mu \in \mathbb{R}$ satisfy

$$(1, 2, 3) = \lambda\,(1, -1, 1) + \mu\,(2x, 2y, 0),$$

that is, $\lambda + 2\mu x = 1, \quad -\lambda + 2\mu y = 2$ and $\lambda = 3$. This gives

$$\mu x = -1, \quad \mu y = 5/2, \quad \text{and so} \quad \mu^2 = (-1)^2 + (5/2)^2.$$

Hence $\mu = \pm\sqrt{29}/2$, $x = \mp 2/\sqrt{29}$ and $y = \pm 5/\sqrt{29}$, and so $z = 1 - x + y = 1 \pm 7/\sqrt{29}$. Also, $(1, -1, 1) \not\parallel (2x, 2y, 0)$.

It follows that the constrained maximum of f is $3 + \sqrt{29}$ and the constrained minimum of f is $3 - \sqrt{29}$.

(ii) Let us determine a point on the intersection of the planes $x+y+z=1$ and $3x+2y+z=6$ that is closest to the origin.

Define $f(x,y,z) := x^2+y^2+z^2$ for $(x,y,z) \in \mathbb{R}^3$. We wish to find a point $(x,y,z) \in \mathbb{R}^3$ at which f attains its minimum subject to constraints $g(x,y,z) = x+y+z-1 = 0$ and $h(x,y,z) = 3x+2y+z-6 = 0$.

The set $E := \{(x,y,z) \in \mathbb{R}^3 : g(x,y,z) = 0 = h(x,y,z)\}$ is nonempty since $(3,-1,-1) \in E$. It is closed since g and h are continuous, but it is not bounded. Let

$$E_1 := \{(x,y,z) \in E : \|(x,y,z)\| \leq \|(3,-1,-1)\|\}.$$

Then the set E_1 is nonempty, closed and bounded, and so the continuous function attains its minimum on E_1. Further, the minimum of f on E_1 is equal to the minimum of f on E. Thus f attains its minimum on E.

The equations $\nabla f = \lambda \nabla g + \mu \nabla h$ yield

$$x = \frac{\lambda + 3\mu}{2}, \quad y = \frac{\lambda + 2\mu}{2}, \quad z = \frac{\lambda + \mu}{2}.$$

Since $g(x, y, z) = 0 = h(x, y, z)$, we obtain $3\lambda + 6\mu = 2$ and $3\lambda + 7\mu = 6$. This gives $\lambda = -22/3$ and $\mu = 4$. Hence $P_0 := (x_0, y_0, z_0) = (7/3, 1/3, -5/3)$.

We note that $(\nabla g)(P_0) = (1, 1, 1)$ and $(\nabla h)(P_0) = (3, 2, 1)$ are nonzero and nonparallel. Hence f attains its constrained minimum on E at P_0.

We remark that f is not bounded above on the straight line E, and so it cannot have a constrained maximum on E.

ALITER:

$x + y + z = 1$ and $3x + 2y + z = 6 \implies 1 - x - y = z = 6 - 3x - 2y \implies y = 5 - 2x$ and $z = -4 + x$. Hence $x^2 + y^2 + z^2 = x^2 + (5 - 2x)^2 + (-4 + x)^2 = 6x^2 - 28x + 41$.

Therefore our problem reduces to minimizing the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := 6x^2 - 28x + 41$, $x \in \mathbb{R}$.

Now $f'(x) = 4(3x - 7)$ for $x \in \mathbb{R}$. Clearly,

$x < 7/3 \implies f'(x) < 0 \implies f$ is decreasing on $(-\infty, 7/3)$,

$x > 7/3 \implies f'(x) > 0 \implies f$ is increasing on $(7/3, \infty)$.

Also, f is continuous at $7/3$. As in the First Derivative Test, f has a global minimum at $x = 7/3$.

Further, $x = 7/3 \implies y = 1/3$, $z = -5/3$.

Thus the point on the intersection of the planes that is closest to the origin is $(7/3, 1/3, -5/3)$.

‘Lagrange’ on the Eiffel Tower

‘Lagrange’ is among the 72 names inscribed on Eiffel Tower.



