MA 105 : Calculus
Division 1, Lecture 18

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Recap of the previous lecture

- Local extremum
- Necessary condition for a local extremum
- Saddle point. Examples
- Bivariate Mean Value Theorem
- Extended Bivariate Mean Value Theorem
- Bivariate Taylor's theorem
- Discriminant Test for functions of two variables
- Examples
- Double integrals on a rectangle
- Riemann condition and domain additivity
- Integrable functions
- Algebraic and order properties of double integrals

Double Integral on a Rectangle

Recall that if $R := [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$ with $a < b$, $c < d$. and $f : R \to \mathbb{R}$ be a bounded function, then for a partition P of R, we defined the **lower double sum** $L(P, f)$ and the **upper double sum** $U(P, f)$ of f with respect to P. Moreover, we defined the **lower double integral** $L(f)$ of f and the **upper double integral** $U(f)$ of f by

$$\begin{aligned} L(f) &:= \sup\{L(P, f) : P \text{ is a partition of } R\}, \\ U(f) &:= \inf\{U(P, f) : P \text{ is a partition of } R\}. \end{aligned}$$

Definition

A bounded function $f : R \to \mathbb{R}$ is said to be **(double) integrable** on R if $L(f) = U(f)$.

We have also seen examples of bounded functions that are integrable and also those that are not integrable.

Riemann condition and Domain additivity

Basic results about integrability over a rectangle are:

Theorem (Riemann condition)

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b] \times [c, d]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

(Domain Additivity)

Let $R := [a, b]\times[c, d]$, and let $f : R \to \mathbb{R}$ be a bounded function. Let $s \in (a, b)$, $t \in (c, d)$. Then f is integrable on R if and only if f is integrable on the four subrectangles $[a, s]\times[c, t]$, $[a, s]\times[t, d]$, $[s, b]\times[c, t]$ and $[s, b]\times[t, d]$.
In this case, the integral of f on R is the sum of the integrals of f on the four subrectangles.

Let $R := [a,b]\times[c,d]$, and let $f : R \to \mathbb{R}$.

(i) If f is monotonic in each of the two variables, then f is integrable on R.

(ii) If f is bounded on R, and has at most a finite number of discontinuities in R, then f is integrable on R.

Let $R := [a,b]\times[c,d]$. If $f, g : R \to \mathbb{R}$ are integrable, then

(i) $f+g$ is integrable, and $\iint_R (f+g) = \iint_R f + \iint_R g$,

(ii) αf is integrable, and $\iint_R \alpha f = \alpha \iint_R f$ for all $\alpha \in \mathbb{R}$,

(iii) $f \cdot g$ is integrable,

(iv) If there is $\delta > 0$ such that $|f(x,y)| \geq \delta$ for all $(x,y) \in R$ (so that $1/f$ is bounded), then $1/f$ is integrable,

(v) If $f \leq g$, then $\iint_R f \leq \iint_R g$,

(vi) $|f|$ is integrable, and $|\iint_R f| \leq \iint_R |f|$.

Evaluation of a Double Integral

Suppose a function is integrable on a rectangle R. How can we find its double integral?

To evaluate the Riemann integral of an integrable function f on an interval $[a, b]$, we used a powerful result known as the Fundamental Theorem of Calculus (Part II). If we find a function F on $[a, b]$ whose derivative is equal to f on $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a).$$

Later in this course, we shall see some versions of the Fundamental Theorem of Calculus for functions of two/three variables, known as the Green/Stokes theorem.

For the present, we consider an easy and widely used method for evaluating double integrals in which the problem is reduced to a repeated evaluation of Riemann integrals.

Theorem (Fubini Theorem on a Rectangle)

Let $R := [a, b] \times [c, d]$, and let $f : R \to \mathbb{R}$ be integrable. Let I denote the double integral of f on R.

(i) If for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)dy$ exists, then the **iterated integral** $\int_a^b \left(\int_c^d f(x, y)dy \right) dx$ exists and is equal to I.

(ii) If for each fixed $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)dx$ exists, then the **iterated integral** $\int_c^d \left(\int_a^b f(x, y)dx \right) dy$ exists and is equal to I.

(iii) If the hypotheses in both (i) and (ii) above hold, and in particular, if f is continuous on R, then

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx = I = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

The Fubini theorem can be proved by using the Riemann condition for a double integral and for a Riemann integral.

Special case: Let $\phi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ be Riemann integrable. Define $f : R \to \mathbb{R}$ by $f(x, y) := \phi(x)\psi(y)$, $(x, y) \in R$. Then f is integrable on R, and its double integral is equal to

$$\int_a^b \left(\int_c^d \phi(x)\psi(y)dy \right) dx = \left(\int_a^b \phi(x)dx \right) \left(\int_c^d \psi(y)dy \right).$$

In particular, if $r, s \in \mathbb{R}$ with $r \geq 0$ and $s \geq 0$, then

$$\iint_{[a,b]\times[c,d]} x^r y^s d(x, y) = \left(\frac{b^{r+1} - a^{r+1}}{r+1} \right) \left(\frac{d^{s+1} - c^{s+1}}{s+1} \right),$$

provided $0 < a < b$ and $0 < c < d$.

Geometrically, the Fubini theorem says that if f is a nonnegative integrable function on $R := [a,b]\times[c,d]$, then the volume of the solid D under the surface $z = f(x,y)$ and above the rectangle R can be found by the slice method. For $x \in [a,b]$, we find the area $A(x) := \int_c^d f(x,y)dy$, of the cross-section of D perpendicular to the x-axis, or alternatively, for $y \in [c,d]$, we find the area $B(y) := \int_a^b f(x,y)dx$, of the cross-section of D perpendicular to the y-axis. (See Figure.)

$$\text{Vol}\,(D) = \int_a^b A(x)dx = \int_c^d B(y)dy.$$

Examples:

(i) Let $R := [0,1]\times[0,1]$, and $f(x,y) := (x+y)^2$, $(x,y) \in R$. Then f is continuous on R. The double integral of f on R is

$$\begin{aligned}\int_0^1\left(\int_0^1 (x+y)^2dx\right)dy &= \frac{1}{3}\int_0^1 (x+y)^3\Big|_0^1 dy \\ &= \frac{1}{3}\int_0^1 \left((1+y)^3 - y^3\right)dy = \frac{7}{6}.\end{aligned}$$

(ii) Let $R := [0,1]\times[0,1]$, $f(0,0) := 0$, and for $(x,y) \neq (0,0)$, let $f(x,y) := xy(x^2-y^2)/(x^2+y^2)^3$. For $x \in [0,1]$, $x \neq 0$,

$$A(x) := \int_0^1 f(x,y)dy = \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}dy = \frac{x}{2(1+x^2)^2}.$$

(Substitute $x^2+y^2 = u$.) Also, $A(0) = \int_0^1 0dy = 0$. Hence

$$\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = \int_0^1 A(x)dx = \int_0^1 \frac{x}{2(1+x^2)^2}dx = \frac{1}{8}.$$

By interchanging x and y, $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = -\frac{1}{8}$.

Thus the two iterated integrals exist, but they are not equal.

Note that since $f(1/n, 1/2n) = 24n^2/125$ for all $n \in \mathbb{N}$, the function f is not bounded on R, and so it is not integrable on R. Thus Fubini's theorem is not applicable.

Remark

Around 1910, Fichtenholz and Lichtenstein proved the following result.

Theorem

Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. If for each fixed $x \in [a,b]$, the Riemann integral $\int_c^d f(x,y)dy$ exists, and if for each fixed $y \in [c,d]$, the Riemann integral $\int_a^b f(x,y)dx$ exists, then the **iterated integrals** $\int_a^b \big(\int_c^d f(x,y)dy \big) dx$ and $\int_c^d \big(\int_a^b f(x,y)dx \big) dy$ exist and are equal.

The above result is stronger than the Fubini theorem since it does not assume that the function $f : [a,b] \times [c,d] \to \mathbb{R}$ is integrable; it only assumes that f is bounded on $[a,b]\times[c,d]$.

Double Riemann Sum

If we are not able to evaluate the double integral exactly, we attempt to find its approximations.

Given a bounded function $f : R \to \mathbb{R}$, and a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$, of $R := [a, b] \times [c, d]$, a double sum of the form

$$S(P, f) := \sum_{i=1}^{n} \sum_{j=1}^{k} f(s_i, t_j)(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $(s_i, t_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, is called a double Riemann sum for f corresponding to P.

Note: $L(P, f) \leq S(P, f) \leq U(P, f)$ for any $s_1, \ldots, s_n \in [a, b]$ and $t_1, \ldots, t_k \in [c, d]$.

We define the **mesh** of a partition
$P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ by
$\mu(P) := \max\{x_1 - x_0, \ldots, x_n - x_{n-1},\ y_1 - y_0, \ldots, y_k - y_{k-1}\}$.

Theorem: Let f be integrable on R, and let $\epsilon > 0$. Then there is $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition P satisfying $\mu(P) < \delta$.

Corollary: If f is integrable on R and (P_n) is a sequence of partitions of R with $\mu(P_n) \to 0$, then $U(P_n, f) - L(P_n, f) \to 0$. Further, if $S(P_n, f)$ is a double Riemann sum corresponding to P_n and f, then $S(P_n, f) \to \iint_R f$.

Proof: Given $\epsilon > 0$, find $\delta > 0$ as in the theorem. Then there is $n_0 \in \mathbb{N}$ such that $\mu(P_n) < \delta$ for all $n \geq n_0$, and so $U(P_n, f) - L(P_n, f) < \epsilon$. Thus $U(P_n, f) - L(P_n, f) \to 0$.
Since $L(P_n, f) \leq S(P_n, f) \leq U(P_n, f)$, and
$L(P_n, f) \leq L(f) = \iint_R f = U(f) \leq U(P_n, f)$, we see that
$\left|S(P_n, f) - \iint_R f\right| \leq U(P_n, f) - L(P_n, f) \to 0$. $\square$

Caution:

If we define

$$\mu(P) := \max\{(x_i - x_{i-1})(y_j - y_{j-1}) : 1 \le i \le n;\ 1 \le j \le k\},$$

then we may not obtain $S(P_n, f) \to \int\int_R f$ as $\mu(P_n) \to 0$ for every integrable function f on R. An example of a function of this kind is a bit involved.

The preceding corollary allows us to find limits of certain sequences. For example, let

$$s_n := \frac{1}{n^4} \sum_{i=1}^{n} \sum_{j=1}^{n} (i + j)^2 \quad \text{for } n \in \mathbb{N}.$$

Then $s_n = \dfrac{1}{n^2} \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{i}{n} + \frac{j}{n}\right)^2 = S(\mathrm{P}_n, f)$, where $f(x, y) := (x + y)^2$ for $(x, y) \in [0, 1] \times [0, 1]$, and $\mu(\mathrm{P}_n) = 1/n$ for $n \in \mathbb{N}$. Hence $s_n \to \int_0^1 \int_0^1 (x + y)^2 d(x, y) = 7/6$.

Double Integral over a Bounded Set

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a rectangle $R := [a, b] \times [c, d]$ such that $D \subset R$, and define $f^* : R \to \mathbb{R}$ by

$$f^*(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is **integrable over** D if f^* is integrable on R, and in this case, the **double integral of** f **over** D is defined to be the double integral of f^* on R, that is,

$$\iint_D f(x, y) d(x, y) := \iint_R f^*(x, y) d(x, y).$$

By the domain additivity of double integrals on rectangles, the integrability of f over D and the value of its double integral are independent of the choice of a rectangle R containing D and the corresponding extension f^* of f to R.

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be integrable over D. We may also denote the double integral of f over D by $\iint_D f$.

If f is nonnegative, then the **volume** of the solid under the surface given by $z = f(x, y)$ and above the region D is defined to be the double integral of f over D. Thus

$$\text{Vol}\,(E_f) := \iint_D f(x, y)d(x, y),$$

where $E_f := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}$.

Next, if $g : D \to \mathbb{R}$ is integrable, and $f \leq g$ on D, then

$$\iint_D \big(g(x, y) - f(x, y)\big)d(x, y)$$

is defined to be the **volume between the surfaces** given by $z = f(x, y)$ and $z = g(x, y)$.

The double integral over a bounded subset of $\mathbb{R}^2$ has algebraic and order properties analogous to those of the double integral on a rectangle.

In order to seek conditions under which a bounded function f defined on a bounded subset D of $\mathbb{R}^2$ is integrable over D, we introduce a new concept.

A bounded subset E of $\mathbb{R}^2$ is said to be of (two-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

A set of (two-dimensional) content zero is a 'thin' subset of $\mathbb{R}^2$.

Examples:
(i) Every finite subset of $\mathbb{R}^2$ is of content zero.

(ii) The infinite subset $\{(1/n, 1/k) : n, k \in \mathbb{N}\}$ of $\mathbb{R}^2$ is of content zero.

(iii) The subset $\{(x, y) \in [0,1]\times[0,1] : x, y \in \mathbb{Q}\}$ of $\mathbb{R}^2$ is not of content zero. (Consider any ϵ less than 1.)

(iv) Let $\varphi : [a, b] \to \mathbb{R}$ be an integrable function. Then its graph $E := \{(x, \varphi(x)) : x \in [a, b]\}$ is of content zero. To see this, let $\epsilon > 0$. By the Riemann condition, there is a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(P, \varphi) - L(P, \varphi) < \epsilon$. Then $E \subset \bigcup_{i=1}^{n} R_i$, where $R_i := [x_{i-1}, x_i]\times[m_i(\varphi), M_i(\varphi)]$ and

$$\text{Area}\,(R_1) + \cdots + \text{Area}\,(R_n) = U(P, \varphi) - L(P, \varphi) < \epsilon.$$

Similarly, if $\psi : [c, d] \to \mathbb{R}$ is an integrable function, then the set $E := \{(\psi(y), y) : y \in [c, d]\}$ is of content zero.

(v) Let functions x, $y : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let their derivatives be bounded on (a, b). Then the set $E := \{(x(t), y(t)) \in \mathbb{R}^2 : t \in [a, b]\}$ is of two-dimensional content zero. We prove this as follows.

For $n \in \mathbb{N}$, consider the partition of $[a, b]$ into n equal parts, and let s_i be the mid-point of the ith subinterval for $i = 1, \ldots, n$. Let $t \in [a, b]$. Then there is $i \in \{1, \ldots, n\}$ such that $|t - s_i| \leq (b - a)/2n$. By the Mean Value Theorem, there are c_x and c_y between t and s_i such that

$$x(t) - x(s_i) = x'(c_x)(t - s_i) \quad \text{and} \quad y(t) - y(s_i) = x'(c_y)(t - s_i).$$

Let $\alpha \in \mathbb{R}$ with $|x'|, |y'| \leq \alpha$ on (a, b), and $\beta := b - a$. Then $|x(t) - x(s_i)|, |y(t) - y(s_i)| \leq \alpha\beta/2n$, and so $(x(t), y(t))$ lies in a square S_i centered at $(x(s_i), y(s_i))$ and of area $\alpha^2\beta^2/n^2$.

Thus E is contained in the union of the n squares $S_1, \ldots, S_n$, the sum of whose areas is equal to $\alpha^2\beta^2/n$.

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $n > \alpha^2(b - a)^2/\epsilon$. Then the set E is contained in the union of these n squares, the sum of whose areas is less that ϵ. Hence E is of two-dimensional content zero.

Theorem

Let D be a bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be a bounded function. If the boundary ∂D of D is of (two-dimensional) content zero and if the set of discontinuities of f in D is also of (two-dimensional) content zero, then f is integrable over D.

We omit the proof.

The above theorem says that the two conditions
(i) 'the boundary ∂D of D is of content zero' and
(ii) 'the set of discontinuities of f in D is of content zero'
together are sufficient to guarantee the integrability of f on D.
While neither condition can be dropped from the above theorem, the integrability of f does not necessarily imply either condition.

(Find examples to illustrate the above remarks.)

Elementary Regions

Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ such that $\phi_1 \leq \phi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}.$$

Alternatively, let $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ be continuous functions on $[c, d]$ such that $\psi_1 \leq \psi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}.$$

In both cases, D is called an **elementary region** in $\mathbb{R}^2$. In the former case, it is said to be of **type** I, and in the latter case, it is said to be of **type** II.

We note that the boundary of an elementary domain in $\mathbb{R}^2$ is of (two-dimensional) content zero. Hence a continuous function defined on such a domain is integrable over it.

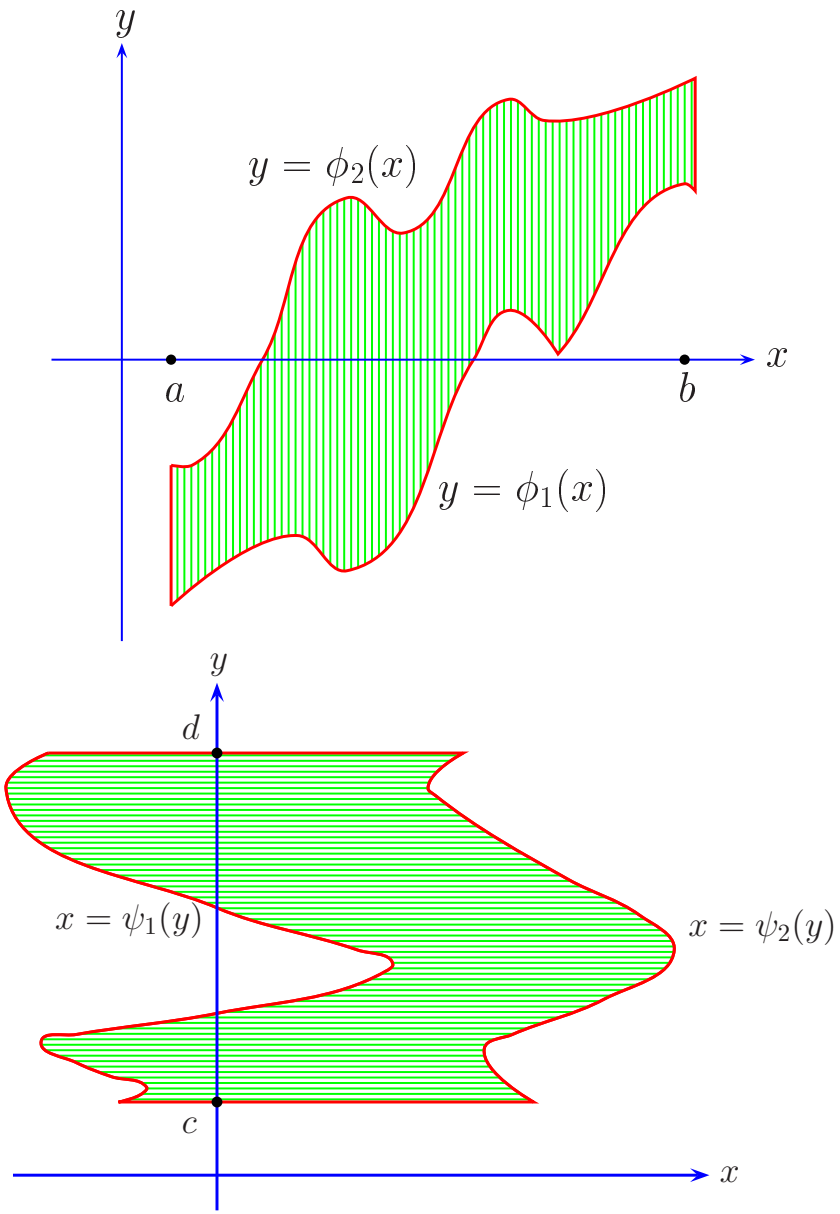

Figure: Elementary regions with boundaries in red colour

Clearly, a rectangle is an elementary region in $\mathbb{R}^2$ of type I as well as of type II.
Also, if $a > 0$, then the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$ is an elementary region in $\mathbb{R}^2$, since

$$D = \left\{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a \text{ and } -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}\right\}.$$

Essential feature of an elementary region D in $\mathbb{R}^2$ of type I:
For every $x \in \mathbb{R}$, the vertical cross-section of D at x is a closed and bounded interval.
Essential feature of an elementary region D in $\mathbb{R}^2$ of type II:
For every $y \in \mathbb{R}$, the horizontal cross-section of D at y is a closed and bounded interval.

There do exist bounded subsets of $\mathbb{R}^2$ that are not elementary regions. For example, a star-shaped subset of $\mathbb{R}^2$ or an annulus is not an elementary region, since for some $x \in \mathbb{R}$, the vertical cross-section of D at x is not an interval, and for some $y \in \mathbb{R}$, the horizontal cross-section of D at y is not an interval.

Proposition (Fubini Theorem over Elementary Regions)

Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous, then the iterated integral $\int_a^b \big(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \big) dx$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

(ii) If $D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$, where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are continuous, then the iterated integral $\int_c^d \big(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \big) dy$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

Proof: (i) Let $c := \inf\{\phi_1(x) : x \in [a, b]\}$ and $d := \sup\{\phi_2(x) : x \in [a, b]\}$. Then $D \subset R := [a, b] \times [c, d]$, and the extended function $f^* : R \to \mathbb{R}$ is integrable. Now use the Fubini theorem for f^*. (ii) A similar argument works here.

Examples:

(i) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1/2 \text{ and } 0 \le y \le x^2\}$ and $f(x, y) := x + y$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$I := \int\int_D (x + y) d(x, y) = \int_0^{1/2} \left(\int_0^{x^2} (x + y) dy \right) dx,$$

which is equal to

$$\int_0^{1/2} \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x^2} dx = \int_0^{1/2} \left(x^3 + \frac{x^4}{2} \right) dx = \frac{3}{160}.$$

Alternatively, since
$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1/4 \text{ and } \sqrt{y} \le x \le 1/2\}$,

$$I = \int_0^{1/4} \left(\int_{\sqrt{y}}^{1/2} (x+y) dx \right) dy = \int_0^{1/4} \left[\frac{x^2}{2} + xy \right]_{x=\sqrt{y}}^{x=1/2} dy = \frac{3}{160}.$$

(ii) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } 0 \le y \le 2x\}$ and $f(x, y) := e^{x^2}$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$\iint_D f = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = \int_0^1 2x\, e^{x^2} dx = e - 1.$$

Also, since $D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 2 \text{ and } y/2 \le x \le 1\}$,

$$\iint_D f = \int_0^2 \left(\int_{y/2}^1 e^{x^2} dx \right) dy.$$

However, the integral $\int_{y/2}^1 e^{x^2} dx$ cannot be evaluated in terms of known functions.

This example shows that an iterated integral may not always be useful in evaluating a double integral.

Constant Function on a Bounded Subset of $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x, y) := 1 \text{ for all } (x, y) \in D.$$

If the boundary ∂D of D is of (two-dimensional) content zero, then the continuous function 1_D is integrable on D as we saw. The converse also holds, that is, if the function 1_D is integrable on D, then ∂D is of (two-dimensional) content zero.

Examples: Let $R := [a, b] \times [c, d]$. If $D := R$, then we have seen that 1_D is integrable and its double integral is equal to $(b - a)(d - c)$. But if $D := \{(x, y) \in R : x, y \in \mathbb{Q}\}$, then the function $1_D^* : R \to \mathbb{R}$, obtained by extending the function 1_D as usual, is the bivariate Dirichlet function on R. We have seen that 1_D^* is not integrable on R, that is, 1_D is not integrable over D. Note: In this case $\partial D = [a, b] \times [c, d]$, and so ∂D is not of (two-dimensional) content zero.

Area of a Bounded Subset of $\mathbb{R}^2$

Let $D \subset \mathbb{R}^2$ be bounded. We say that D has an **area** if the function 1_D is integrable over D, and then we define

$$\text{Area}\,(D) := \iint_D 1_D(x,y)\,d(x,y).$$

Thus a bounded subset D of $\mathbb{R}^2$ has an area $\iff$ ∂D is of content zero.

Important special case: Consider an elementary domain $D := \{(x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$, where $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ are continuous. Then ∂D is of content zero, and by the Fubini theorem,

$$\text{Area}\,(D) = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 1\,dy \right) dx = \int_a^b \big(\phi_2(x) - \phi_1(x)\big)\,dx,$$

which was our definition of the **area between the curves** $y = \phi_1(x)$ and $y = \phi_2(x)$, $x \in [a,b]$.

The following basic inequality is important.

Let D be a bounded subset of $\mathbb{R}^2$ which has an area. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \leq \alpha$ on D, then

$$\left| \iint_D f(x,y)d(x,y) \right| \leq \iint_D |f(x,y)|d(x,y) \leq \alpha \,\text{Area}\,(D).$$

The following result is useful in evaluating a double integral.

Theorem (Domain Additivity)

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (two-dimensional) content zero. If f is integrable over D_1 and over D_2, then f is integrable over D and

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$