MA 105 : Calculus
Division 1, Lecture 19

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Recap of the previous lecture

- Fubini's theorem on a rectangle. Examples
- Remark about a theorem of Fichtenholtz and Lichtenstein
- Double Riemann sums
- Double integrals over bounded sets
- Volume of a solid bounded by two surfaces defined over a planar domain
- Notion of sets of (two-dimensional) content zero. Examples
- Necessary condition for the integrability of a bounded function over a bounded subset
- Elementary regions
- Fubini Theorem over Elementary Regions. Examples
- Area of a bounded subset of R2
- General version of domain additivity

Change of Variables in a Double Integral

The calculation of the double integral $\iint_D f(x,y)d(x,y)$ can often be simplified by reducing it to another double integral $\iint_E g(u,v)d(u,v)$, where the pair (u,v) of new variables is related to the the pair (x,y) of the given variables by a suitable transformation.

In doing so, we look for a function g which is 'simpler' than the given function f such that the domain E of g is also 'simpler' than the given domain D of f.

Let $x^\circ, y^\circ, a_1, b_1, a_2, b_2 \in \mathbb{R}$, and consider the **affine transformation** $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi := (\psi_1, \psi_2)$, where

$$x = \psi_1(u,v) := x^\circ + a_1 u + b_1 v \quad \text{and} \quad y = \psi_2(u,v) := y^\circ + a_2 u + b_2 v.$$

Note that the above equations have a unique solution, that is, the transformation Ψ is one-one $\iff$ $a_1 b_2 - a_2 b_1 \neq 0$.

In this case, the square region $E := [0,1]\times[0,1]$ is transformed to the parallelogram $D := \Psi(E)$ with vertices $\Psi(0,0) = (x^\circ, y^\circ)$, $\Psi(1,0) = (x^\circ + a_1, y^\circ + a_2)$, $\Psi(0,1) = (x^\circ + b_1, y^\circ + b_2)$ and $\Psi(1,1) = (x^\circ + a_1 + b_1, y^\circ + a_2 + b_2)$. Note that Area $(E) = 1$, and it gets scaled to the area $|a_1 b_2 - a_2 b_1|$ of $D = \Psi(E)$.

In general, let $E \subset \mathbb{R}^2$, and consider $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$. Let $Q_0 := (u_0, v_0)$ be an interior point of E. If $x := \phi_1(u, v)$ and $y := \phi_2(u, v)$ for $(u, v) \in E$, and the functions x and y have partial derivatives at (u_0, v_0), then the **Jacobian** of the transformation Φ at (u_0, v_0) is defined by

$$J(\Phi)(u_0, v_0) := \frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) := \det \begin{bmatrix} \frac{\partial x}{\partial u}(Q_0) & \frac{\partial x}{\partial v}(Q_0) \\ \frac{\partial y}{\partial u}(Q_0) & \frac{\partial y}{\partial v}(Q_0) \end{bmatrix}.$$

In particular, $J(\Psi)(u, v) = a_1 b_2 - a_2 b_1$, $(u, v) \in \mathbb{R}^2$, for the affine transformation Ψ.

Now suppose E is a square region about Q_0, and the transformation $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$ is such that the functions ϕ_1 and ϕ_2 are differentiable at Q_0.

Let $P_0 := \Phi(u_0, v_0)$. For $(u, v) \in \mathbb{R}^2$, let

$$\psi_1(u,v) := \phi_1(u_0,v_0) + \frac{\partial \phi_1}{\partial u}\Big|_{(u_0,v_0)}(u-u_0) + \frac{\partial \phi_1}{\partial v}\Big|_{(u_0,v_0)}(v-v_0),$$

$$\psi_2(u,v) := \phi_2(u_0,v_0) + \frac{\partial \phi_2}{\partial u}\Big|_{(u_0,v_0)}(u-u_0) + \frac{\partial \phi_2}{\partial v}\Big|_{(u_0,v_0)}(v-v_0).$$

By the definition of differentiability of ϕ_1 and ϕ_2 at (u_0, v_0),

$$\phi_1(u,v) = \psi_1(u,v)+\epsilon_1(u,v) \quad \text{and} \quad \phi_2(u,v) = \psi_2(u,v)+\epsilon_2(u,v),$$

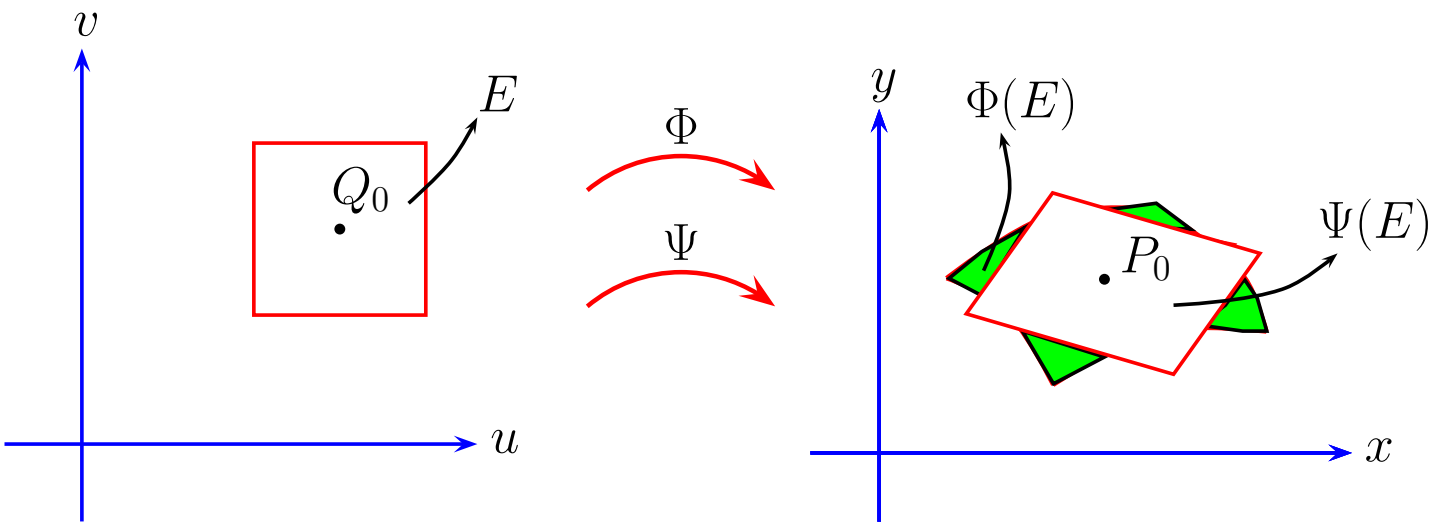
for all $(u, v) \in E$, where $\epsilon_1(u, v)/\|(u - u_0, v - v_0)\| \to 0$ and $\epsilon_2(u, v)/\|(u - u_0, v - v_0)\| \to 0$ as $(u, v) \to (u_0, v_0)$.

Thus the affine transformation $\Psi := (\psi_1, \psi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ maps Q_0 to P_0, and approximates the transformation Φ around Q_0.

Also, by the definition of $\Psi = (\psi_1, \psi_2)$, we obtain

$$J(\Psi) = \left(\frac{\partial \psi_1}{\partial u}\frac{\partial \psi_2}{\partial v} - \frac{\partial \psi_1}{\partial v}\frac{\partial \psi_2}{\partial u}\right) = \left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}\right)(Q_0) = J(\Phi)(Q_0).$$

Hence $\text{Area}\,(\Psi(E)) = |J(\Psi)|\text{Area}\,(E) = |J(\Phi)(Q_0)|\text{Area}\,(E)$.



To state our next result, we need the following definition.
A subset Ω of $\mathbb{R}^2$ is called **open** if every point of Ω is an interior point of Ω.

Theorem (Change of Variables Formula)

Let D be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose Ω is an open subset of $\mathbb{R}^2$ and $\Phi : \Omega \to \mathbb{R}^2$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2)$, where ϕ_1 and ϕ_2 have continuous partial derivatives in Ω and $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D$. Then E is a closed and bounded subset of Ω, and E has an area. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iint_D f(x, y)d(x, y) = \iint_E (f \circ \Phi)(u, v)|J(\Phi)(u, v)|d(u, v).$$

Note: The hypothesis $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$ can be weakened by assuming only that $J(\Phi)(u, v) \geq 0$ for all $(u, v) \in \Omega$ or $J(\Phi)(u, v) \leq 0$ for all $(u, v) \in \Omega$, and $J(\Phi)(u, v) = 0$ only for (u, v) in a subset of Ω of (two-dimensional) content zero.

Compare the above result on Change of Variables for double integrals with the result on Integration by Substitution for Riemann integrals proved in Lecture 12:

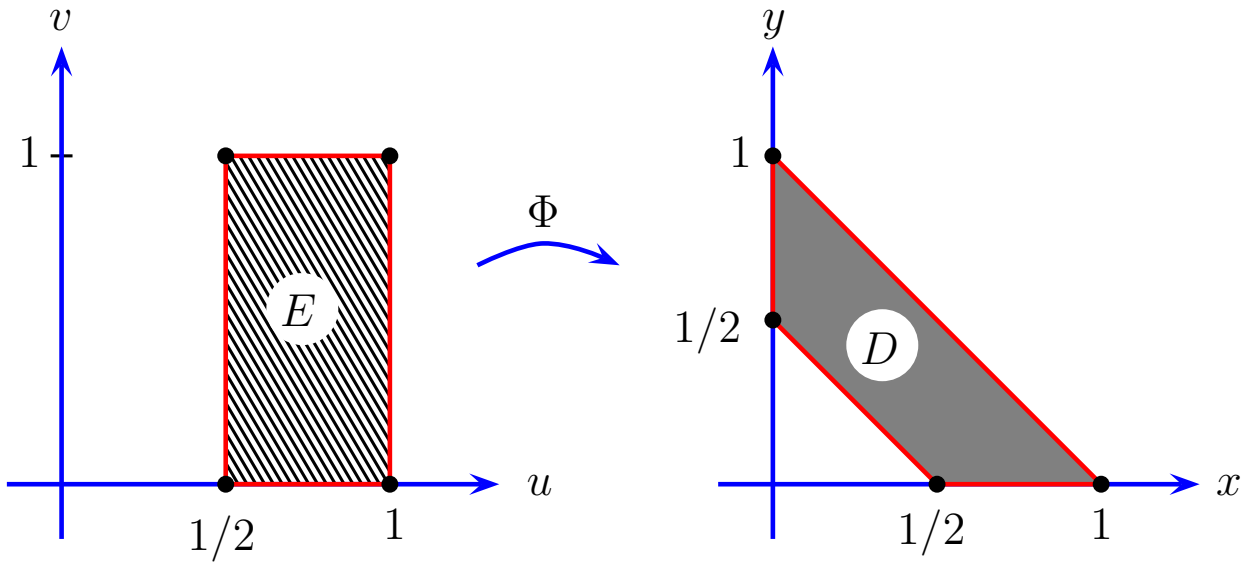
(Integration by substitution)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function such that $\phi([\alpha, \beta]) = [a, b]$. If $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))|\phi'(t)|dt.$$

Example

Let $D := \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0 \text{ and } 1 \leq 2(x + y) \leq 2\}$, and define $f : D \to \mathbb{R}$ by $f(x, y) := y/(x + y)$ for $(x, y) \in D$.



Let $u := x+y,\ v := y/(x+y)$, that is, $x := u(1-v),\ y := uv$.

We let $\Omega := \{(u, v) \in \mathbb{R}^2 : u > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^2$ by $\Phi(u, v) = \big(u(1 - v), uv\big)$. Then the map Φ is one-one from Ω onto $\{(x, y) \in \mathbb{R}^2 : x + y > 0\}$. If $\Phi = (\phi_1, \phi_2)$, then the partial derivatives of ϕ_1 and ϕ_2 exist and are continuous.

Also,

$$J(\Phi)(u,v) = \det \begin{bmatrix} 1-v & -u \\ v & u \end{bmatrix} = u \neq 0 \quad \text{for all } (u,v) \in \Omega.$$

Further, if $E := [1/2, 1] \times [0, 1]$, then $\Phi(E) = D$ and $(f \circ \Phi)(u, v) = v$ for all $(u, v) \in E$. Since f is continuous on D, the Change of Variables formula gives

$$\begin{aligned} \iint_D f(x,y)\,d(x,y) &= \iint_E f(u(1-v), uv)|u|\,d(u,v) \\ &= \iint_E uv\,d(u,v) \\ &= \left(\int_{1/2}^{1} u\,du\right)\left(\int_0^1 v\,dv\right) \\ &= \frac{3}{16}. \end{aligned}$$

Polar Transformation Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\Phi(r, \theta) := (r\cos\theta,\ r\sin\theta)$. If $\Phi := (\phi_1, \phi_2)$, then ϕ_1 and ϕ_2 have continuous partial derivatives in $\mathbb{R}^2$ and

$$J(\Phi)(r,\theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r \quad \text{for all } (r,\theta) \in \mathbb{R}^2.$$

Thus the Jacobian of Φ is nonzero on $\{(r,\theta) \in \mathbb{R}^2 : r \neq 0\}$.
Also, the function $\Phi : \{(r,\theta) \in \mathbb{R}^2 : r > 0 \text{ and } -\pi < \theta \leq \pi\}$ $\to \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ is one-one and onto.
In this case, the following result holds.

Proposition

Let D be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose the set $E := \{(r,\theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi, (r\cos\theta, r\sin\theta) \in D\}$, has an area. Then the function $(r,\theta) \longmapsto f(r\cos\theta, r\sin\theta)$ is continuous on E and
$\iint_D f(x,y)d(x,y) = \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta)$.

Examples

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Define

$E := \{(r, \theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi \text{ and } (r\cos\theta, r\sin\theta) \in D\}$.

Then $E := [0, 1] \times [-\pi, \pi]$.

(i) Let $f(x, y) := \sqrt{1 - x^2 - y^2}$ for $(x, y) \in D$. Then

$$\begin{aligned}\iint_D f(x, y)d(x, y) &= \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r, \theta)\\ &= \int_{-\pi}^{\pi}\left(\int_0^1 \sqrt{1 - r^2}\, r\, dr\right) d\theta\\ &= \int_{-\pi}^{\pi}\frac{1}{2}\left(\int_0^1 \sqrt{s}\, ds\right) d\theta = \frac{2\pi}{3}.\end{aligned}$$

(ii) Let $f(x, y) := e^{x^2+y^2}$ for $(x, y) \in D$. Then

$$\iint_D f(x, y)d(x, y) = \int_{-\pi}^{\pi}\left(\int_0^1 e^{r^2} r\ dr\right)d\theta = \int_{-\pi}^{\pi}\left(\frac{e-1}{2}\right)d\theta = \pi(e-1).$$

Triple Integrals

The theory of double integrals of functions of two variables readily extends to triple integrals of functions of three variables. No new ideas are needed. We shall, therefore, mention only a few important points.

Consider a **cuboid** $K := [a, b] \times [c, d] \times [p, q]$, where $a < b$, $c < d$, $p < q$. We consider a partition P of K into several subcuboids.

For a bounded function $f : K \to \mathbb{R}$, we define the corresponding lower triple sum $L(P, f)$, the upper triple sum $U(P, f)$, the lower triple integral $L(f)$, and the upper triple integral $U(f)$. As before, f is **(triple) integrable** on K, if $L(f) = U(f)$. In this case, its **triple integral** is denoted by

$$\iiint_K f(x, y, z) d(x, y, z) \quad \text{or} \quad \iiint_K f.$$

If $f := 1$ on K, then f is integrable on K, and its triple integral is equal to the volume of K, whereas if f is the trivariate Dirichlet function on K, then f is not integrable on K.

Further, f is integrable on K if and only if it satisfies the Riemann condition. If f is monotonic in each of the three variables, or if f is continuous on K, then f is integrable.

Fubini's theorem on cuboids

Let f be integrable on K, and let I denote its triple integral. Suppose for each fixed $x \in [a,b]$, the double integral $\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)$ exists. Then the **iterated integral** $\int_a^b \big(\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)\big)dx$ exists and equals I. Further, if for each fixed $(x,y) \in [a,b]\times[c,d]$, the Riemann integral $\int_p^q f(x,y,z)dz$ exists, then the **iterated integral** $\int_a^b \big[\int_c^d \big(\int_p^q f(x,y,z)dz\big)dy\big]dx$ exists and equals I.

There are other versions of this result with the roles of x, y, z interchanged. They are used in evaluating triple integrals.

Let D be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a cuboid $K := [a, b]\times[c, d]\times[p, q]$ such that $D \subset K$, and define $f^* : K \to \mathbb{R}$ by

$$f^*(x, y, z) := \begin{cases} f(x, y, z) & \text{if } (x, y, z) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is **integrable** over D if f^* is integrable on K, and in this case, the **triple integral** of f (over D) is defined to be the triple integral of f^* (on K), that is,

$$\iiint_D f(x, y, z)d(x, y, z) := \iiint_K f^*(x, y, z)d(x, y, z).$$

We may also denote the triple integral by $\iiint_D f$.

The triple integral over a bounded subset of $\mathbb{R}^3$ has algebraic and order properties analogous to those of a double integral over a bounded subset of $\mathbb{R}^2$.

In order to seek conditions under which a bounded function f defined on a bounded subset D of $\mathbb{R}^3$ is integrable over D, we introduce the following concept.

A bounded subset E of $\mathbb{R}^3$ is of (three-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many cuboids whose union contains E and the sum of whose volumes is less than ϵ.

Examples:
(i) Let D_0 be a bounded subset of $\mathbb{R}^2$, and let $\varphi : D_0 \to \mathbb{R}$ be integrable over D_0. Then its graph $E := \{(x, y, \varphi(x, y)) : (x, y) \in D_0\}$ is of (three-dimensional) content zero. To see this, let R be a rectangle containing D_0, and let $\varphi^* : R \to \mathbb{R}$ be the usual extension of φ. Let $\epsilon > 0$. By the Riemann condition for the integrable function φ^*, there is a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ of R such that $U(P, \varphi^*) - L(P, \varphi^*) < \epsilon$. Then the nk cuboids $K_{i,j} := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [m_{i,j}(\varphi^*), M_{i,j}(\varphi^*)], i = 1, \ldots n; j = 1, \ldots, k$, work.

(ii) Let functions $x,\ y,\ z : [a,b]\times[c,d] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)\times(c,d)$, and let their partial derivatives be bounded on $(a,b)\times(c,d)$. Then the set $E := \{(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3 : (u,v) \in [a,b]\times[c,d]\}$ is of three-dimensional content zero. We prove this as follows.

For $n \in \mathbb{N}$, consider the partition of $[a,b]\times[c,d]$ into n^2 equal parts, and let (u_i, v_j) be the centroid of the (i,j)th subrectangle for $i,j = 1,\ldots,n$.

Let $(u,v) \in [a,b]\times[c,d]$. Then there are $i,j \in \{1,\ldots,n\}$ such that $|u-u_i| \le (b-a)/2n$ and $|v-v_j| \le (d-c)/2n$. By the Bivariate Mean Value Theorem, there are $(p_x,q_x), (p_y,q_y)$ and (p_z,q_z) on the open line segment joining (u,v) and (u_i,v_j) (and hence belonging to $(a,b)\times(c,d)$) such that

$$
\begin{aligned}
x(u,v) - x(u_i,v_j) &= x_u(p_x,q_x)(u-u_i) + x_v(p_x,q_x)(v-v_j),\\
y(u,v) - y(u_i,v_j) &= y_u(p_y,q_y)(u-u_i) + y_v(p_y,q_y)(v-v_j),\\
z(u,v) - z(u_i,v_j) &= z_u(p_z,q_z)(u-u_i) + z_v(p_z,q_z)(v-v_j).
\end{aligned}
$$

Let $\alpha \in \mathbb{R}$ with $|x_u|, |x_v|, |y_u|, |y_v| \leq \alpha$ on $(a,b)\times(c,d)$, and $\beta := b - a + d - c$. Then

$$
|x(u,v)-x(u_i,v_j)|, |y(u,v)-y(u_i,v_j)|, |z(u,v)-z(u_i,v_j)| \leq \frac{\alpha\beta}{2n},
$$

and so $(x(u,v), y(u,v), z(u,v))$ lies in a cube $K_{i,j}$ centered at $(x(u_i,v_j), y(u_i,v_j), z(u_i,v_j))$ and of volume $\alpha^3\beta^3/n^3$.

Thus the set S is contained in the union of n^2 cubes $K_{i,j}$, $i,j = \ldots n$, the sum of whose volumes is equal to $\alpha^3\beta^3/n$.

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $n > \alpha^3\beta^3/\epsilon$. Then S is contained in the union of these n^2, the sum of whose volumes is less that ϵ. Hence S is of three-dimensional content zero.

The above result can be improved by replacing the rectangle $[a, b] \times [c, d]$ by a bounded convex subset E of $\mathbb{R}^2$ whose interior E° is nonempty.

Further, if E is any bounded convex subset of $\mathbb{R}^2$, and x, y and z are differentiable functions on an open subset of $\mathbb{R}^2$ containing E, and if their first partial derivatives are bounded on E, then we can prove that the set

$$S := \{(x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^3 : (u, v) \in E\}$$

is of three-dimensional content zero.

Remark: The subset $E := [a, b] \times [c, d] \times \{0\}$ of $\mathbb{R}^3$ is of three-dimensional content zero, but the subset $[a, b] \times [c, d]$ of $\mathbb{R}^2$ is not of two-dimensional content zero.

Theorem

Let D be a bounded subset of $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be a bounded function. Suppose (i) the set of discontinuities of f in D is of (three-dimensional) content zero and (ii) the boundary ∂D of D is of (three-dimensional) content zero. Then f is integrable over D.

We omit the proof.

Suppose D_0 is a subset of $\mathbb{R}^2$ having an area, that is, ∂D_0 is of two-dimensional content zero. Let $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ be continuous, and let $\psi_1 \leq \psi_2$. Consider an **elementary region**

$$D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0 \text{ and } \psi_1(x, y) \leq z \leq \psi_2(x, y)\}.$$

Then ∂D is of three-dimensional content zero. Hence if a function is continuous on D, then it is integrable over D.

Proposition (Cavalieri Principle)

Let D be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be continuous. Suppose $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D_0$ and $\psi_1(x, y) \leq z \leq \psi_2(x, y)\}$, where D_0 is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \leq \psi_2$. Then f is integrable on D, the **iterated integral** exists, and

$$\iiint_D f = \iint_{D_0} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z) dz \right) d(x, y).$$

In particular, if D_0 is an elementary region in $\mathbb{R}^2$ given by $D_0 := \{(x, y) : \mathbb{R}^2 : a \leq x \leq b$ and $\phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \leq \phi_2$, then

$$\iiint_D f = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)\, dz \right) dy \right) dx.$$

The Cavalieri Principle stated above gives a version of Fubini's theorem for an 'elementary domain' D in $\mathbb{R}^3$. Other versions can be obtained similarly.

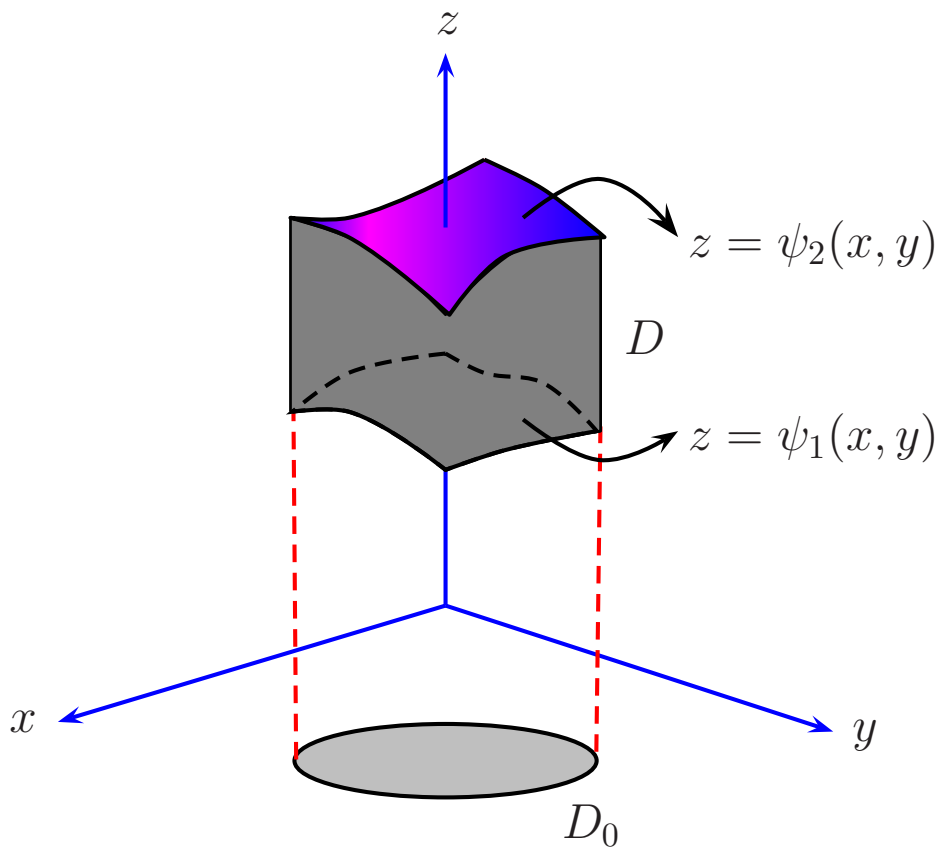


Figure: Illustration of the Cavalieri Principle: a solid between two surfaces defined over D_0.

Constant Function on a Bounded Subset of $\mathbb{R}^3$

Let D be a bounded subset of $\mathbb{R}^3$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x, y, z) := 1 \text{ for all } (x, y, z) \in D.$$

If the boundary ∂D of D is of (three-dimensional) content zero, then the continuous function 1_D is integrable on D. The converse also holds, that is, if the function 1_D is integrable on D, then ∂D is of (three-dimensional) content zero.

Examples:
Let $K := [a, b] \times [c, d] \times [p, q]$. If $D := K$, then 1_D is integrable and its triple integral is equal to $(b - a)(d - c)(q - p)$. But if $D := \{(x, y, z) \in K : x, y, z \in \mathbb{Q}\}$, then the function $1_D^* : K \to \mathbb{R}$, obtained by extending the function 1_D as usual, is the trivariate Dirichlet function on K. Now 1_D^* is not integrable on K, that is, 1_D is not integrable over D. Note: In this case, $\partial D = K$ is not of three-dimensional content zero.

Volume of a Bounded Subset of $\mathbb{R}^3$

Let $D \subset \mathbb{R}^3$ be bounded. We say that D has a **volume** if the function 1_D is integrable over D, and then we define

$$\mathrm{Vol}\,(D) := \iiint_D 1_D(x,y,z)\,d(x,y,z).$$

Important special case: Consider an elementary domain $D := \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D_0$ and $\psi_1(x,y) \le z \le \psi_2(x,y)\}$, where D_0 is a subset of $\mathbb{R}^2$ having an area, $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$.

By the Cavalier Principle, the volume of D is equal to

$$\iint_{D_0} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} dz \right) d(x,y) = \iint_{D_0} \big(\psi_2(x,y) - \psi_1(x,y)\big)\, d(x,y),$$

which was our definition (Lecture 18) of the **volume between the surfaces** $z = \psi_1(x,y)$ and $z = \psi_2(x,y)$, $(x,y) \in D_0$.

For instance, let

$$D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 4z^2 \leq 6\}.$$

Then $D = \left\{(x, y, z) \in \mathbb{R}^3 : |x| \leq \sqrt{6},\ |y| \leq \sqrt{6 - x^2}\right.$ and $\left.|z| \leq \sqrt{6 - x^2 - y^2}/2\right\}$. Hence the volume of the ellipsoidal solid D is equal to

$$8 \int_0^{\sqrt{6}} \left(\int_0^{\sqrt{6-x^2}} \left(\int_0^{\sqrt{6-x^2-y^2}/2} dz \right) dy \right) dx = 4\sqrt{6}\,\pi.$$

Note: It can be shown that the definitions of the volume of a solid used in the slice method, the washer method and the shell method agree with the general definition of the volume of a bounded subset of $\mathbb{R}^3$ given above.

The following basic inequality is important. Let D be a bounded subset of $\mathbb{R}^3$ which has a volume. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \leq \alpha$ on D, then

$$\left| \iiint_D f(x,y,z)d(x,y,z) \right| \leq \iiint_D |f(x,y,z)|d(x,y,z) \leq \alpha \,\mathrm{Vol}\,(D).$$

The following result is useful in evaluating a triple integral.

Theorem (Domain Additivity)

Let D be a bounded subset of $\mathbb{R}^3$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (three-dimensional) content zero. If f is integrable over D_1 and over D_2, then f is integrable over D and

$$\iiint_D f = \iiint_{D_1} f + \iiint_{D_2} f.$$

Change of Variables in a Triple Integral

To simplify the calculation of $\iiint_D f(x,y,z)d(x,y,z)$, we may reduce it to another triple integral $\iiint_E g(u,v,w)d(u,v,w)$.

Let $E \subset \mathbb{R}^3$ and $\Phi := (\phi_1, \phi_2, \phi_3) : E \to \mathbb{R}^3$. Suppose $Q_0 := (u_0, v_0, w_0)$ is an interior point of E.
If $x := \phi_1(u,v,w)$, $y := \phi_2(u,v,w)$ and $z := \phi_3(u,v,w)$ for $(u,v,w) \in E$, and the functions x, y and z have partial derivatives at Q_0, then the **Jacobian** of Φ at Q_0 is given by

$$J(\Phi)(Q_0) := \frac{\partial(x,y,z)}{\partial(u,v,w)}(Q_0) := \det \begin{bmatrix} \frac{\partial x}{\partial u}(Q_0) & \frac{\partial x}{\partial v}(Q_0) & \frac{\partial x}{\partial w}(Q_0) \\ \frac{\partial y}{\partial u}(Q_0) & \frac{\partial y}{\partial v}(Q_0) & \frac{\partial y}{\partial w}(Q_0) \\ \frac{\partial z}{\partial u}(Q_0) & \frac{\partial z}{\partial v}(Q_0) & \frac{\partial z}{\partial w}(Q_0) \end{bmatrix}.$$

Theorem (Change of Variables Formula)

Let D be a closed and bounded subset of $\mathbb{R}^3$ which has a volume. Let $f : D \to \mathbb{R}$ be continuous. Suppose Ω is an open subset of $\mathbb{R}^3$ and $\Phi : \Omega \to \mathbb{R}^3$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2, \phi_3)$, where ϕ_1, ϕ_2, ϕ_3 have continuous partial derivatives in Ω and $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D$. Then E is a closed and bounded subset of Ω, and E has a volume. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iiint_D f = \iiint_E (f \circ \Phi)(u, v, w) |J(\Phi)(u, v, w)| d(u, v, w).$$

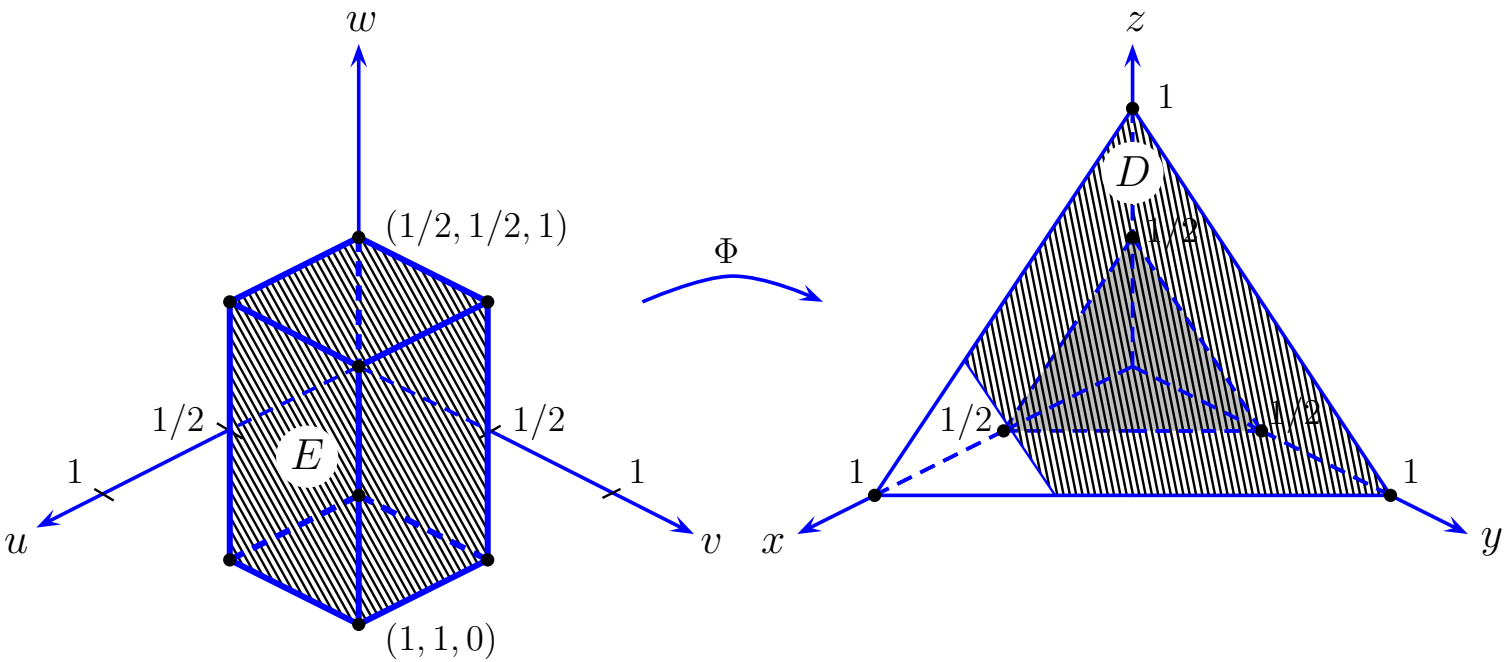
Note: The hypothesis $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$ can be weakened by assuming only that $J(\Phi)(u, v, w) \geq 0$ for all $(u, v, w) \in \Omega$ or $J(\Phi)(u, v, w) \leq 0$ for all $(u, v, w) \in \Omega$, and $J(\Phi)(u, v, w) = 0$ only for (u, v, w) in a subset of Ω of (three-dimensional) content zero.

Example: Let

$D := \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0,\ x \leq y+z,\ 1 \leq 2(x+y+z) \leq 2\}$,

and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z/(y + z)$.

(Note: $(x, y, z) \in D \implies 0 \leq x \leq x/2 + (y + z)/2 \leq 1/2$.)



Let $u := x + y + z$, $\quad v := \dfrac{y + z}{x + y + z}$, $\quad w := \dfrac{z}{y + z}$,

that is, $x := u(1 - v)$, $\quad y := u\,v(1 - w)$, $\quad z := u\,v\,w$.

Consider $\Omega := \{(u,v,w) \in \mathbb{R}^3 : u > 0 \text{ and } v > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^3$ by $\Phi(u,v,w) := \big(u(1-v), u\,v(1-w), u\,v\,w\big)$. Now Φ maps Ω onto $\{(x,y,z) \in \mathbb{R}^2 : x+y+z > 0 \text{ and } y+z > 0\}$ and is one-one on Ω. Also, if $\Phi = (\phi_1, \phi_2, \phi_3)$, then the partials of ϕ_1, ϕ_2, and ϕ_3 are continuous, and for all $(u,v,w) \in \Omega$,

$$J(\Phi)(u,v,w) = \det \begin{bmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -u\,v \\ v\,w & u\,w & u\,v \end{bmatrix} = u^2 v \neq 0.$$

Let $E := [1/2, 1] \times [1/2, 1] \times [0, 1]$. Then $\Phi(E) = D$, and $(f \circ \Phi)(u,v,w) = w$ for $(u,v,w) \in E$. By the CoV formula,

$$\begin{aligned} \iiint_D f &= \iiint_E f(u(1-v), u\,v(1-w), u\,v\,w)\,|u^2 v|\,d(u,v,w) \\ &= \iiint_E u^2 v\,w\,d(u,v,w) \\ &= \left(\int_{1/2}^1 u^2\,du\right)\left(\int_{1/2}^1 v\,dv\right)\left(\int_0^1 w\,dw\right) = \frac{7}{128}. \end{aligned}$$

Two important cases involving a change of variables in triple integrals are given by switching to cylindrical coordinates, and to spherical coordinates.

(i) Cylindrical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(r, \theta, z) := (r\cos\theta,\ r\sin\theta,\ z) \quad \text{for } (r, \theta, z) \in \mathbb{R}^3.$$

Then for all $(r, \theta, z) \in \mathbb{R}^3$,

$$J(\Phi)(r, \theta, z) = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.$$

The Jacobian of Φ is nonzero on $\{(r, \theta, z) \in \mathbb{R}^3 : r \neq 0\}$.

The function $\Phi : \{(r, \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } -\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

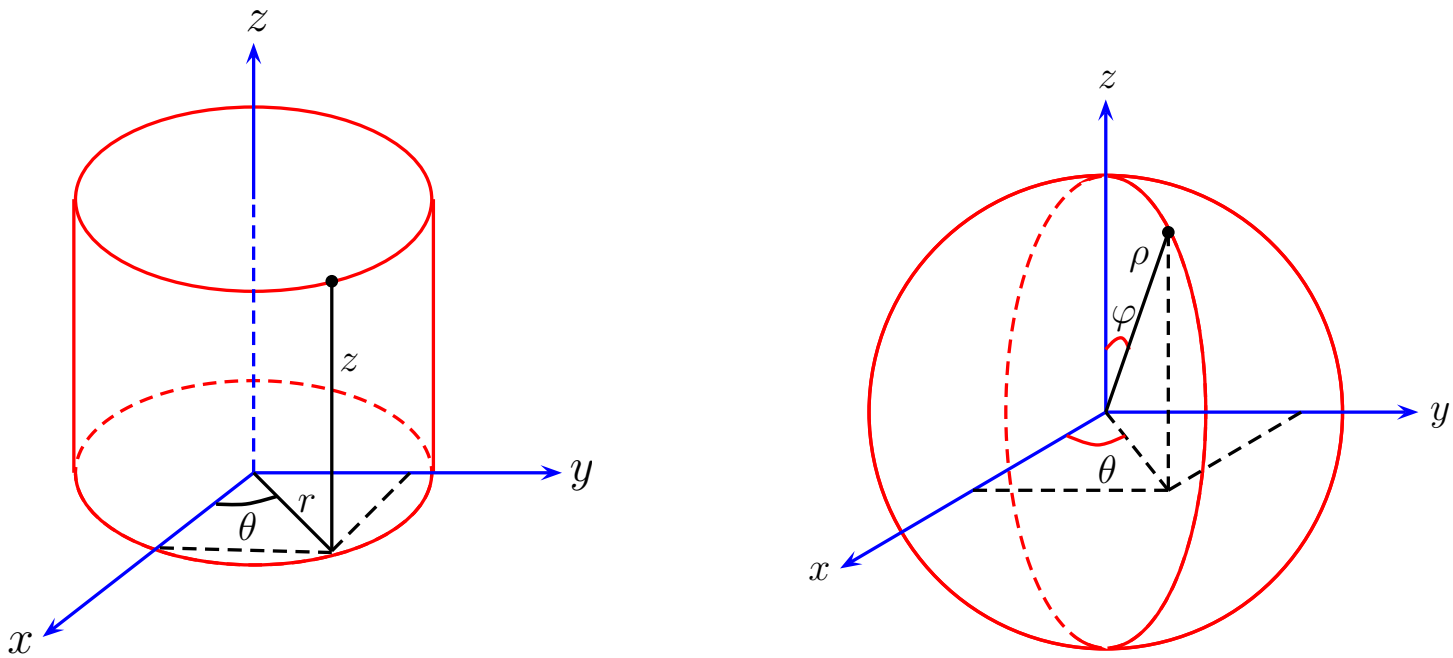


Figure: Illustrations of cylindrical and spherical coordinates.

(ii) Spherical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(\rho, \varphi, \theta) := (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi) \quad \text{for } (\rho, \varphi, \theta) \in \mathbb{R}^3.$$

Then for all $(\rho, \varphi, \theta) \in \mathbb{R}^3$,

$$\begin{aligned} J(\Phi)(\rho, \varphi, \theta) &= \det \begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix} \\ &= \rho^2 \sin\varphi. \end{aligned}$$

The Jacobian of Φ is nonzero on the set
$\{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \neq 0$ and $\varphi \neq m\pi$ for any $m \in \mathbb{Z}\}$.

The function $\Phi : \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho > 0,\ 0 < \varphi < \pi$ and $-\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

In these cases the following results hold.

Proposition

Let D be a closed and bounded subset of $\mathbb{R}^3$ which has a volume, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $E := \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0, -\pi \leq \theta \leq \pi$ and $(r\cos\theta,\ r\sin\theta,\ z) \in D\}$, and if E has a volume, then the triple integral of f over D is equal to

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z).$$

(ii) If $E := \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi$ and $(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \in D\}$, and if E has a volume, then the triple integral of f over D is equal to

$$\iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi\ d(\rho, \varphi, \theta).$$

Examples

(i) Let $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1 \text{ and } 0 \leq z \leq 1\}$, and define $f : D \to \mathbb{R}$ by $f(x, y, z) := z\sqrt{1 - x^2 - y^2}$. Then

$$\begin{aligned} E &:= \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0,\ -\pi \leq \theta \leq \pi,\ (r\cos\theta, r\sin\theta, z) \in D\} \\ &= [0, 1] \times [-\pi, \pi] \times [0, 1]. \end{aligned}$$

Hence

$$\begin{aligned} \iiint_D f &= \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z) \\ &= \int_0^1 \left[\int_{-\pi}^{\pi} \left(\int_0^1 z\sqrt{1 - r^2}\, r\, dz \right) d\theta \right] dr \\ &= \pi \int_0^1 \sqrt{1 - r^2}\, r\, dr = \frac{\pi}{2} \int_0^1 \sqrt{s}\, ds = \frac{\pi}{3}. \end{aligned}$$

(ii) Let $a > 0$ and $D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\}$. Define $f : D \to \mathbb{R}$ by $f(x, y, z) = z^2$. Then

$$\begin{aligned} E &:= \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi \text{ and} \\ &\qquad (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \in D\} \\ &= [0, a] \times [0, \pi] \times [-\pi, \pi]. \end{aligned}$$

Hence the triple integral of f over D is equal to

$$\begin{aligned} &\iiint_E f(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi)\, \rho^2 \sin\varphi \, d(\rho, \varphi, \theta) \\ &= \int_0^a \left[\int_0^\pi \left(\int_{-\pi}^\pi (\rho^2 \cos^2\varphi)\, \rho^2 \sin\varphi \, d\theta \right) d\varphi \right] d\rho \\ &= 2\pi \int_0^a \rho^4 \left(\int_0^\pi \cos^2\varphi \sin\varphi \, d\varphi \right) d\rho = \frac{2\pi a^5}{5} \cdot \frac{2}{3} = \frac{4\pi a^5}{15}. \end{aligned}$$