MA 105 : Calculus
Division 1, Lecture 02

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Recap of the previous lecture

- Generalities about the course
- Text and References
- Basic notations: $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$.
- The set $\mathbb{R}$ of real numbers as a set containing $\mathbb{Q}$ and satisfying the
  - Algebraic Properties
  - Order Properties
  - Completeness Property
- Upper bound, lower bound, supremum,and infimum
- Consequences of the Completeness Property:
  - Existence of infimum of a nonempty set bounded below;
  - Archimedean Property;
  - Existence and uniqueness of positive nth roots of positive real numbers;
  - Existence of a rational as well as an irrational between any two real numbers $a < b$.

Recap of the previous lecture Contd.

- Intervals: open, closed, semi-open, infinite
- Functions. Notions of one-one, onto, and bijective functions; Inverse functions
- Examples of functions. Mention of:
  - Polynomials functions
  - Rational functions
  - Algebraic functions
  - Transcendental functions.
  - Functions obtained by piecing together known functions
- Sequences: Basic definitions and examples
- Boundedness. Examples of bounded and unbounded sequences
- Notion of convergence of a sequence. A few examples

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $a \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \text{ for all } n \geq n_0.$$

In this case, we say that (a_n) **converges** to a, or that a is a **limit** of (a_n), and we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$ as $n \to \infty$).

The part in red may be stated as: $|a_n - a| < \epsilon$ for all large n. In general, when we say that a statement holds for all large n, it means that there is $n_0 \in \mathbb{N}$ such that it holds for all $n \geq n_0$.

If a sequence is not convergent, then it is said to be **divergent**, *or we say that the sequence* **diverges**.

The convergence of a sequence is unaltered if a finite number of its terms are replaced by some other terms.

Examples:

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a$. We can let $n_0 := 1$.

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

Let $\epsilon > 0$ be given. We want to find $n_0 \in \mathbb{N}$ such that $|(1/n) - 0| < \epsilon$ for all $n \geq n_0$.

Choose any $n_0 \in \mathbb{N}$ which is greater than $1/\epsilon$. This is possible because of the Archimedean property of $\mathbb{R}$. For example, we can let $n_0 := [1/\epsilon] + 1$.

(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > \sqrt{2}/\sqrt{\epsilon}$. For example, let $n_0 := [\sqrt{2}/\sqrt{\epsilon}] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(iv) $a_n := 5/(3n+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\frac{5}{3n+1} < \frac{5}{3n} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > 5/3\epsilon$. For example, let $n_0 := [5/3\epsilon] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(v) $a_n := (-1)^n$ for all $n \in \mathbb{N}$. Then the sequence (a_n) is divergent, that is, it is not convergent.

Suppose (a_n) is convergent. Then there is a real number a such that $a_n \to a$. Let $\epsilon := 1/2$. Find $n_0 \in \mathbb{N}$ such that

$$|(-1)^n - a| < \frac{1}{2} \text{ for all } n \geq n_0.$$

Since $(-1)^{2n_0} = 1$ and $(-1)^{2n_0+1} = -1$,

$$\begin{aligned} 2 &= |(-1)^{2n_0} - (-1)^{2n_0+1}| \\ &\leq |(-1)^{2n_0} - a| + |a - (-1)^{2n_0+1}| \\ &< \frac{1}{2} + \frac{1}{2} = 1, \end{aligned}$$

which is a contradiction.

Uniqueness of limit

Theorem

A convergent sequence has a unique limit.

Proof: Let (a_n) be a sequence. Assume for a moment that $a_n \to a$ and $a_n \to b$, where $a \neq b$. Let $\epsilon := |a-b|/2 > 0$.

Let $n_0 \in \mathbb{N}$ be such that $n \geq n_0 \implies |a_n - a| < \epsilon$, and let $m_0 \in \mathbb{N}$ be such that $n \geq m_0 \implies |a_n - b| < \epsilon$. Consider $n := \max\{n_0, m_0\}$. Then

$$|a-b| \leq |a-a_n| + |a_n - b| < \epsilon + \epsilon = |a-b|,$$

which is a contradiction. Hence $a = b$. $\square$

A useful result

Theorem

Every convergent sequence is bounded.

Proof: Suppose $a_n \to a$. Let $\epsilon := 1$. There is $n_0 \in \mathbb{N}$ such that

$$|a_n - a| < 1 \text{ for all } n > n_0.$$

Hence

$$|a_n| \leq |a_n - a| + |a| < 1 + |a| \text{ for all } n > n_0.$$

Define $\alpha := \max\{|a_1|, \ldots, |a_{n_0}|, |a| + 1\}$. Then $|a_n| \leq \alpha$ for all $n \in \mathbb{N}$. Hence (a_n) is bounded. □

- A bounded sequence need not be convergent.
  Example: $a_n := (-1)^n$ for $n \in \mathbb{N}$.
- If $a_n := (-1)^n n$ for $n \in \mathbb{N}$, then (a_n) is divergent since it is not bounded.

Limit theorems for sequences

Suppose (a_n) and (b_n) are convergent sequences. Then

- $\lim_{n\to\infty}(a_n \pm b_n) = (\lim_{n\to\infty} a_n) \pm (\lim_{n\to\infty} b_n)$ resp.,
- $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

  In particular, $\lim_{n\to\infty}(r\, a_n) = r(\lim_{n\to\infty} a_n)$ for $r \in \mathbb{R}$.
- If $\lim_{n\to\infty} b_n \neq 0$, then there is $\delta > 0$ such that $|b_n| \geq \delta > 0$ for all large n and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- If $a_n \leq b_n$ for all large $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.
- (Sandwich theorem) If $a_n \leq c_n \leq b_n$ for all large $n \in \mathbb{N}$, and $\lim_{n\to\infty} a_n = c = \lim_{n\to\infty} b_n$, then $\lim_{n\to\infty} c_n = c$.

Examples

(i) Let $a \in \mathbb{R}$, and $a_n := a^n$ for $n \in \mathbb{N}$.
Then (a_n) is convergent $\iff$ $-1 < a \leq 1$.

Clearly, if $a := 0$, then $a_n \to 0$, and if $a := 1$, then $a_n \to 1$.
Also, if $a := -1$, then we have seen that (a_n) is divergent.

Let $0 < |a| < 1$, and $r := 1/|a|$. Then $r > 1$, and so $r = 1 + h$ with $h > 0$. By the binomial theorem,

$$r^n = (1+h)^n = 1 + nh + \cdots + h^n > nh \quad \text{for all } n \in \mathbb{N}.$$

Hence $0 \leq |a_n| = |a|^n = (1/r^n) \leq (1/nh) \to 0$. Thus $a_n \to 0$.

Let $s := |a| > 1$. Then $s = 1 + h$ with $h > 0$, and for all $n \in \mathbb{N}$, $|a_n| = s^n > nh$. Hence (a_n) is unbounded, and so it is divergent.

Examples (continued)

(ii) Let $a_n := \dfrac{n^3 + 3n^2 + 1}{n^4 + 8n^2 + 2}$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

since $0 \leq a_n \leq \dfrac{1}{n} + \dfrac{3}{8n^2} + \dfrac{1}{n^4} \to 0.$

(iii) Let $a_n := \dfrac{1}{n} \sin\left(\dfrac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $a_n \to 0$,

since $|a_n| \leq \dfrac{1}{n} \to 0.$

Monotonic sequences

A sequence (a_n) is said to be **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $a_1 \leq a_2 \leq a_3 \leq \cdots$.

A sequence (a_n) is said to be **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $a_1 \geq a_2 \geq a_3 \geq \cdots$.

A sequence is **monotonic** if it is either increasing or decreasing.

Examples:

- The sequences (n) and $(-1/n)$ are increasing.
- The sequences $(-n)$ and $(1/n)$ are decreasing.
- The sequences $((-1)^n)$ and $((-1)^n n)$ are neither increasing nor decreasing. Thus they are not monotonic.

Convergence theorems

- An increasing sequence (a_n) that is bounded above is convergent, and $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.
- A decreasing sequence (a_n) that is bounded below is convergent, and $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.

Thus, a monotonic bounded sequence is convergent.

Proof: Suppose (a_n) is increasing and bounded above. Let $a := \sup\{a_n : n \in \mathbb{N}\}$. Let $\epsilon > 0$. Then $a - \epsilon$ is not an upper bound of $\{a_n : n \in \mathbb{N}\}$, and so there is $n_0 \in \mathbb{N}$ such that $a - \epsilon < a_{n_0}$. But then

$$a - \epsilon < a_{n_0} \leq a_n \leq a \quad \text{for all } n \geq n_0.$$

Thus $|a_n - a| < \epsilon$ for all $n \geq n_0$. Hence $a_n \to a$. A similar proof holds for a decreasing sequence that is bounded below.

Examples: 1. The sequence (a_n) defined by $a_n := 1 + (1/2^2) + \cdots + (1/n^2)$ for $n \in \mathbb{N}$ is clearly increasing. Further, since

$$\begin{aligned} a_n &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \end{aligned}$$

we see that $a_n \leq 2 - (1/n) < 2$. Thus (a_n) is bounded. Hence by the result in the previous slide, (a_n) is convergent.

2. Consider the sequence (a_n) defined recursively by

$$a_1 := \frac{3}{2} \quad \text{and} \quad a_{n+1} := \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \quad \text{for } n \in \mathbb{N}.$$

Note that if $1 \leq a_n \leq 2$, then it is easily deduced from the recurrence relation that $1 \leq a_{n+1} \leq 2$. Since $1 \leq a_1 \leq 2$, we see by induction that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$. In particular, (a_n) is bounded. Also, $a_2 = 17/12 < a_1$.

Let us check whether the sequence (a_n) is decreasing. Since

$$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{a_n^2 - 2}{2\,a_n} \quad \text{for all } n \in \mathbb{N},$$

(a_n) is decreasing if and only if $a_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$. But

$$a_1^2 \geq 2 \quad \text{and} \quad a_{n+1}^2 - 2 = \frac{(a_n^2 - 2)^2}{4a_n^2} \geq 0 \text{ for all } n \in \mathbb{N}.$$

Hence the sequence (a_n) is decreasing. It follows that (a_n) is convergent. Let $a_n \to a$. Then $a_{n+1} \to a$ as well. Moreover, since $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$, we see that $1 \leq a \leq 2$. Now

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) \to \frac{1}{2}\left(a + \frac{2}{a}\right).$$

Since the limit of a sequence is unique, we see that

$$\frac{1}{2}\left(a + \frac{2}{a}\right) = a, \quad \text{that is,} \quad a^2 = 2.$$

Thus $a = \pm\sqrt{2}$. But since $1 \leq a \leq 2$, it follows that $a = \sqrt{2}$.

Continuity of a function of a real variable

Let D be a subset of $\mathbb{R}$, and f be a real-valued function defined on D. Then we write $f : D \to \mathbb{R}$. The subset $f(D) := \{f(x) : x \in D\}$ of $\mathbb{R}$ is called the **range** of f.

Definition

Let $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $c \in D$. We say that f is **continuous** at c if
(x_n) is a sequence in D, $x_n \to c \implies f(x_n) \to f(c)$.

We say that f is **continuous on** D if f is continuous at each point of D.

We say that f is **discontinuous** on D if it is not continuous on D, that is, if there is $c \in D$ at which f is not continuous.

Examples

(i) Let $D := \mathbb{R}$ and $f(x) := x$ for $x \in \mathbb{R}$. Clearly, f is continuous on $\mathbb{R}$.

(ii) Let $D := [0, \infty)$ and $f(x) := \sqrt{x}$ for $x \in [0, \infty)$. First we show that f is continuous at 0. Let $x_n \to 0$, where $x_n \geq 0$ for each $n \in \mathbb{N}$. Consider $\epsilon > 0$. Find $n_0 \in \mathbb{N}$ such that $x_n < \epsilon^2$ for all $n \geq n_0$. Then $0 \leq f(x_n) = \sqrt{x_n} < \epsilon$ for all $n \geq n_0$, and so $f(x_n) \to 0 = f(0)$. Thus f is continuous at 0.

Next, let $c \in (0, \infty)$, and $x_n \to c$, where $x_n \geq 0$ for each $n \in \mathbb{N}$. It is easy to see that $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x - y|}$ for all $x, y \in [0, \infty)$ by considering $x \geq y$ and $x < y$. Since $|x_n - c| \to 0$, we obtain $|\sqrt{x_n} - \sqrt{c}| \leq \sqrt{|x_n - c|} \to 0$, that is, $f(x_n) \to f(c)$. Thus f is continuous at c.

More generally, if $k \in \mathbb{N}$, and $f(x) := x^{1/k}$ for $x \in [0, \infty)$, then f is continuous on $[0, \infty)$. We shall later give a proof of this result.

Basic Properties of Continuous functions

- Let $D \subset \mathbb{R}$, and let $f, g : D \to \mathbb{R}$ be functions. Let $c \in D$, and let f and g be continuous at c. Then $f + g$ and $f \cdot g$ are continuous at c. Further, f/g is continuous at c if $g(c) \neq 0$. These results follow from the limit theorems for sequences.

  Consequences:

  (i) A polynomial is continuous on $\mathbb{R}$.

  (ii) A **rational function** $r(x) := p(x)/q(x)$, where p and q are polynomials, is continuous at $c \in \mathbb{R}$ if $q(c) \neq 0$.

- Composite of continuous functions: Let $D \subset \mathbb{R}$, $E \subset \mathbb{R}$. If $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions such that $f(D) \subset E$, f is continuous at $c \in D$, and g is continuous at $f(c) \in E$, then $g \circ f : D \to \mathbb{R}$ is continuous at c.