MA 105 : Calculus
Division 1, Lecture 20

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Recap of the previous lecture

- Change of Variables in a Double Integral. Example
- Polar transformation, Example
- Triple integrals
- Sets of (three-dimensional) content zero. Examples
- Criterion for the integrability of a function of 3 variables
- Elemenentary regions. Cavalieri Principle
- Volume of a bouded subset of the 3-space
- Domain additivity for triple integrals
- Change of Variables in a Triple Integral. Example
- Cylindrical and Spherical coordinates. Examples.

Theorem (Change of Variables Formula)

Let D be a closed and bounded subset of $\mathbb{R}^3$ which has a volume. Let $f : D \to \mathbb{R}$ be continuous. Suppose Ω is an open subset of $\mathbb{R}^3$ and $\Phi : \Omega \to \mathbb{R}^3$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2, \phi_3)$, where ϕ_1, ϕ_2, ϕ_3 have continuous partial derivatives in Ω and $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D$. Then E is a closed and bounded subset of Ω, and E has a volume. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iiint_D f = \iiint_E (f \circ \Phi)(u, v, w) |J(\Phi)(u, v, w)| d(u, v, w).$$

Note: The hypothesis $J(\Phi)(u, v, w) \neq 0$ for all $(u, v, w) \in \Omega$ can be weakened by assuming only that $J(\Phi)(u, v, w) \geq 0$ for all $(u, v, w) \in \Omega$ or $J(\Phi)(u, v, w) \leq 0$ for all $(u, v, w) \in \Omega$, and $J(\Phi)(u, v, w) = 0$ only for (u, v, w) in a subset of Ω of (three-dimensional) content zero.

Two important cases involving a change of variables in triple integrals are given by switching to cylindrical coordinates, and to spherical coordinates.

(i) Cylindrical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(r, \theta, z) := (r\cos\theta,\ r\sin\theta,\ z) \quad \text{for } (r, \theta, z) \in \mathbb{R}^3.$$

Then for all $(r, \theta, z) \in \mathbb{R}^3$,

$$J(\Phi)(r, \theta, z) = \det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r.$$

The Jacobian of Φ is nonzero on $\{(r, \theta, z) \in \mathbb{R}^3 : r \neq 0\}$.

The function $\Phi : \{(r, \theta, z) \in \mathbb{R}^3 : r > 0 \text{ and } -\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

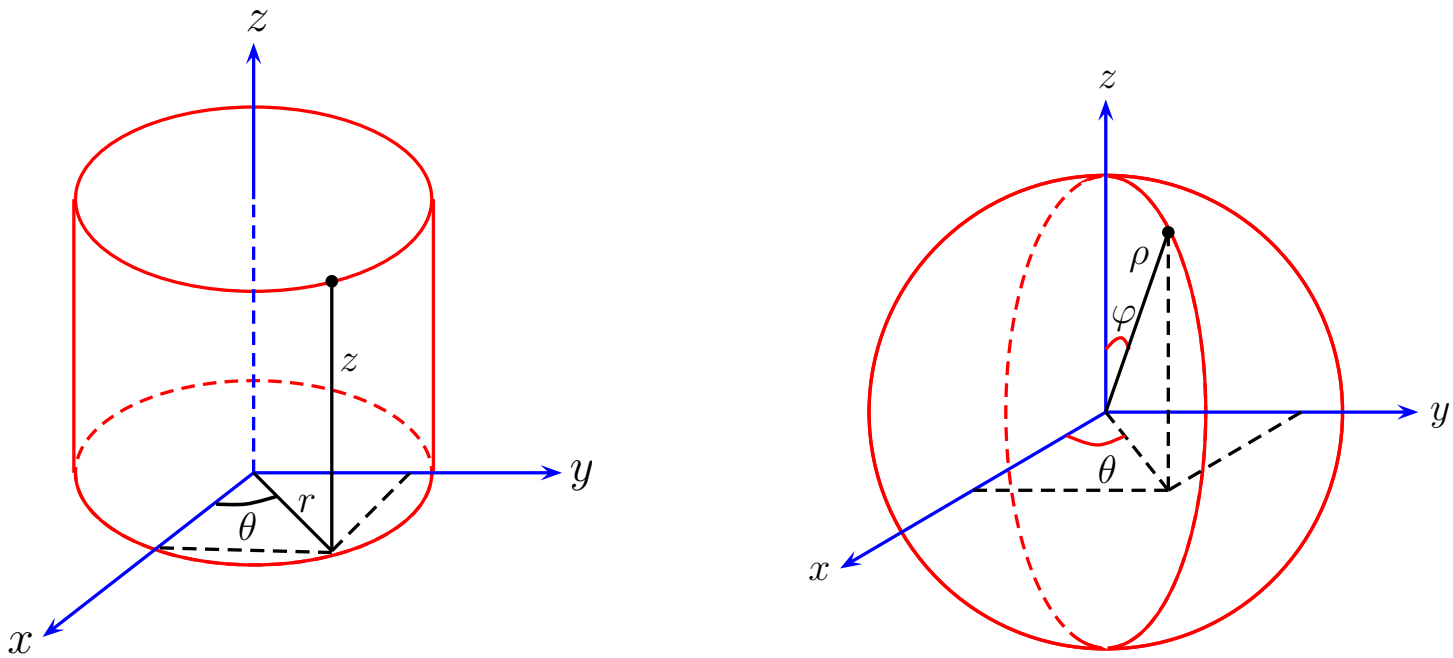


Figure: Illustrations of cylindrical and spherical coordinates.

(ii) Spherical coordinates: Let $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\Phi(\rho, \varphi, \theta) := (\rho \sin\varphi \cos\theta,\ \rho \sin\varphi \sin\theta,\ \rho \cos\varphi) \quad \text{for } (\rho, \varphi, \theta) \in \mathbb{R}^3.$$

Then for all $(\rho, \varphi, \theta) \in \mathbb{R}^3$,

$$\begin{aligned} J(\Phi)(\rho, \varphi, \theta) &= \det \begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix} \\ &= \rho^2 \sin\varphi. \end{aligned}$$

The Jacobian of Φ is nonzero on the set
$\{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \neq 0$ and $\varphi \neq m\pi$ for any $m \in \mathbb{Z}\}$.

The function $\Phi : \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho > 0,\ 0 < \varphi < \pi$ and $-\pi < \theta \leq \pi\} \to \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is one-one and onto.

In these cases the following results hold.

Proposition

Let D be a closed and bounded subset of $\mathbb{R}^3$ which has a volume, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $E := \{(r, \theta, z) \in \mathbb{R}^3 : r \geq 0, -\pi \leq \theta \leq \pi$ and $(r\cos\theta,\ r\sin\theta,\ z) \in D\}$, and if E has a volume, then the triple integral of f over D is equal to

$$\iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, d(r, \theta, z).$$

(ii) If $E := \{(\rho, \varphi, \theta) \in \mathbb{R}^3 : \rho \geq 0,\ 0 \leq \varphi \leq \pi,\ -\pi \leq \theta \leq \pi$ and $(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \in D\}$, and if E has a volume, then the triple integral of f over D is equal to

$$\iiint_E f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \rho^2 \sin\varphi\ d(\rho, \varphi, \theta).$$

Vector Algebra

We begin a part of this course known as '**Vector Analysis**'.
It deals with scalar fields and vector fields.
Initially, we recall some concepts in '**Vector Algebra**'.
In Lecture 17, we have defined the **Euclidean space**

$$\mathbb{R}^m := \{\boldsymbol{x} = (x_1, \ldots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \ldots, m\},$$

where $m \in \mathbb{N}$. An element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar**, and an element of $\mathbb{R}^m$ is called a **vector** if $m \geq 2$.

Let $\boldsymbol{x} := (x_1, \ldots, x_m)$, $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$.
We have already defined the sum $\boldsymbol{x} + \boldsymbol{y}$ and the scalar multiple $a\boldsymbol{x}$. Also, we have studied the **dot product** or the **scalar product** of $\boldsymbol{x}$ and $\boldsymbol{y}$ defined by

$$\boldsymbol{x} \cdot \boldsymbol{y} := x_1 y_1 + \cdots + x_m y_m \in \mathbb{R}.$$

Vector Product

Let $m := 3$. Consider $\boldsymbol{x} := (x_1, x_2, x_3)$ and $\boldsymbol{y} := (y_1, y_2, y_3)$. The **cross product** or the **vector product** of $\boldsymbol{x}$ with $\boldsymbol{y}$ is defined by

$$\boldsymbol{x} \times \boldsymbol{y} := (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$

$$\text{Let} \quad \boldsymbol{i} := (1,0,0), \quad \boldsymbol{j} := (0,1,0), \quad \boldsymbol{k} := (0,0,1).$$

Then $\boldsymbol{x} := x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$, $\boldsymbol{y} := y_1\boldsymbol{i} + y_2\boldsymbol{j} + y_3\boldsymbol{k}$, and

$$\begin{aligned} \boldsymbol{x} \times \boldsymbol{y} &= (x_2y_3 - x_3y_2)\boldsymbol{i} + (x_3y_1 - x_1y_3)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k} \\ &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}. \end{aligned}$$

Note: $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$, $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$, $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$.

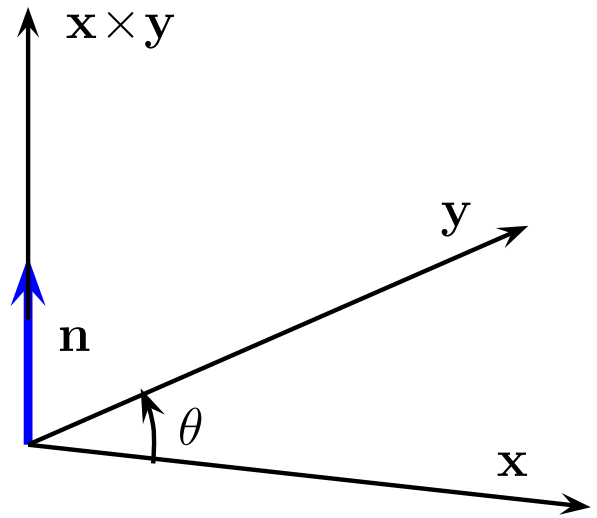
Properties of determinants show that for every $\boldsymbol{z} \in \mathbb{R}^3$,

$$(\boldsymbol{x}+\boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z}) \quad \text{and} \quad \boldsymbol{y} \times \boldsymbol{x} = -(\boldsymbol{x} \times \boldsymbol{y}).$$

Let $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$. It can be shown that

$$\boldsymbol{x} \times \boldsymbol{y} = \|\boldsymbol{x}\| \, \|\boldsymbol{y}\| (\sin\theta)\, \boldsymbol{n},$$

where $\theta \in [0, \pi]$ is the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, and $\boldsymbol{n}$ is the unit vector which is perpendicular to the plane containing $\boldsymbol{x}$ and $\boldsymbol{y}$, and obeys the 'right-hand rule'.



Hence $\|\boldsymbol{x} \times \boldsymbol{y}\|$ = the area of the parallelogram with sides $\boldsymbol{x}$, $\boldsymbol{y}$.

Scalar Triple Product and Vector triple Product

Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^3$. Then $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}$ and $\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) \in \mathbb{R}^3$ are called the **scalar triple product** and the **vector triple product** of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ respectively. It is easy to see that if $\boldsymbol{x} := (x_1, x_2, x_3)$, $\boldsymbol{y} := (y_1, y_2, y_3)$, $\boldsymbol{z} := (z_1, z_2, z_3)$, then

$$\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Geometrically, $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z})$ can be interpreted as the (signed) volume of the parallelopiped defined by the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

One can prove the **Lagrange formula**

$$\boldsymbol{x} \times (\boldsymbol{y} \times \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{z})\, \boldsymbol{y} - (\boldsymbol{x} \cdot \boldsymbol{y})\, \boldsymbol{z}$$

by considering each component of the LHS and the RHS.

Scalar Fields and Vector Fields

A **scalar field** is an assignment of a scalar to each point in a region in the space. For example, the temperature at a point on the earth is a scalar field (defined on a subset of $\mathbb{R}^3$).

A **vector field** is an assignment of a vector to each point in a region in the space. For example, the velocity field of a moving fluid is a vector field that associates a velocity vector to each point in the fluid.

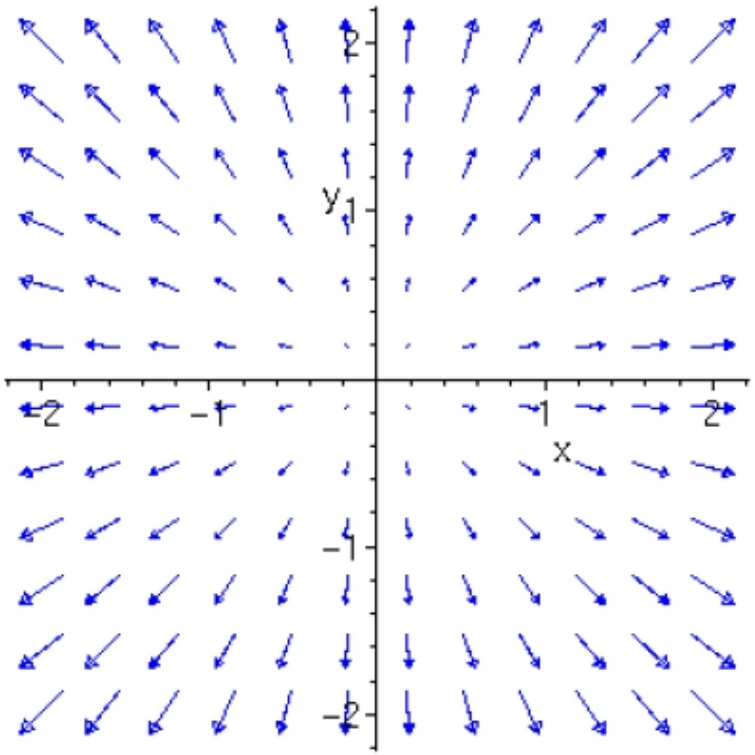
Definition

Let $m \in \mathbb{N}$, and let D be a subset of $\mathbb{R}^m$.

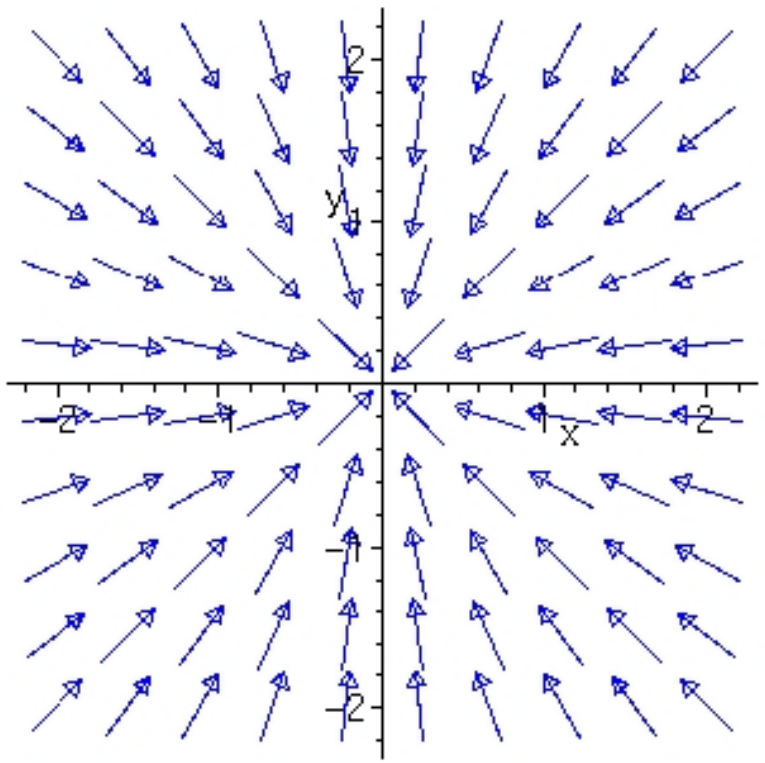
A **scalar field** is a function from D to $\mathbb{R}$.

A **vector field** is a function from D to $\mathbb{R}^m$.
If $m = 2$, then it is called a **vector field in the plane**, and if $m = 3$, then it is called a **vector field in the space**.

Depiction of Vector Fields

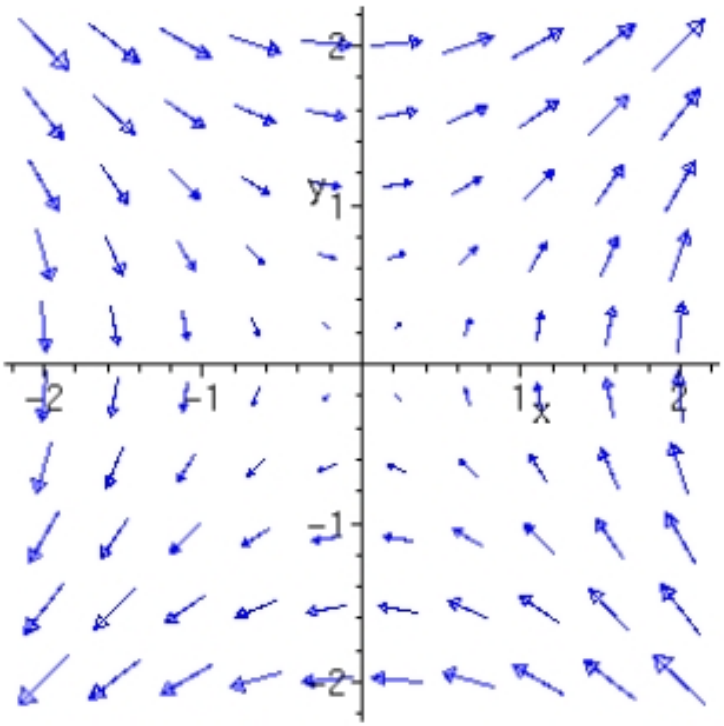


$\boldsymbol{F}(x, y) := (2x, 2y)$

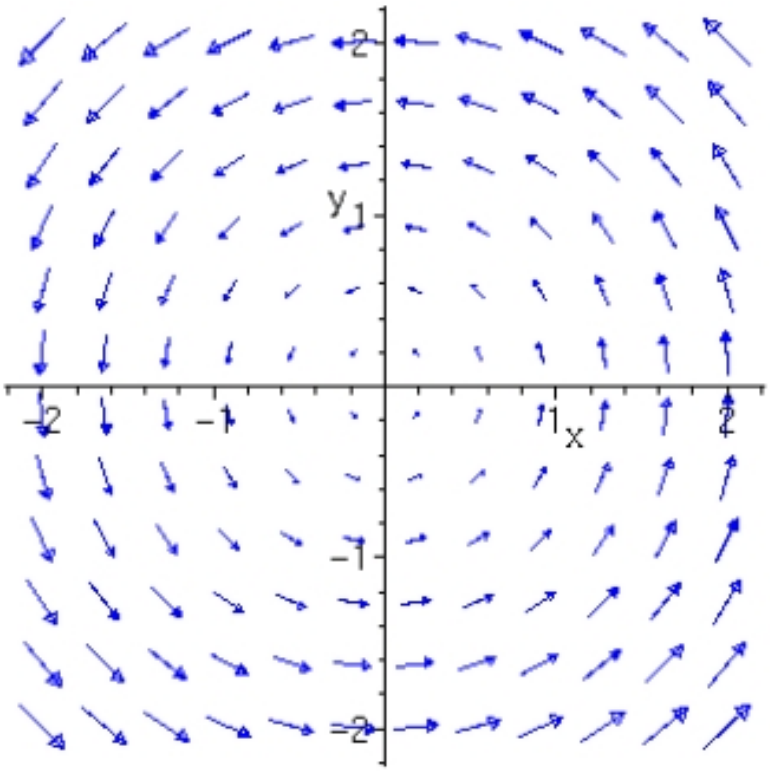


$\boldsymbol{F}(x, y) := (-x, -y)/\sqrt{x^2 + y^2}$

Depiction of Vector Fields

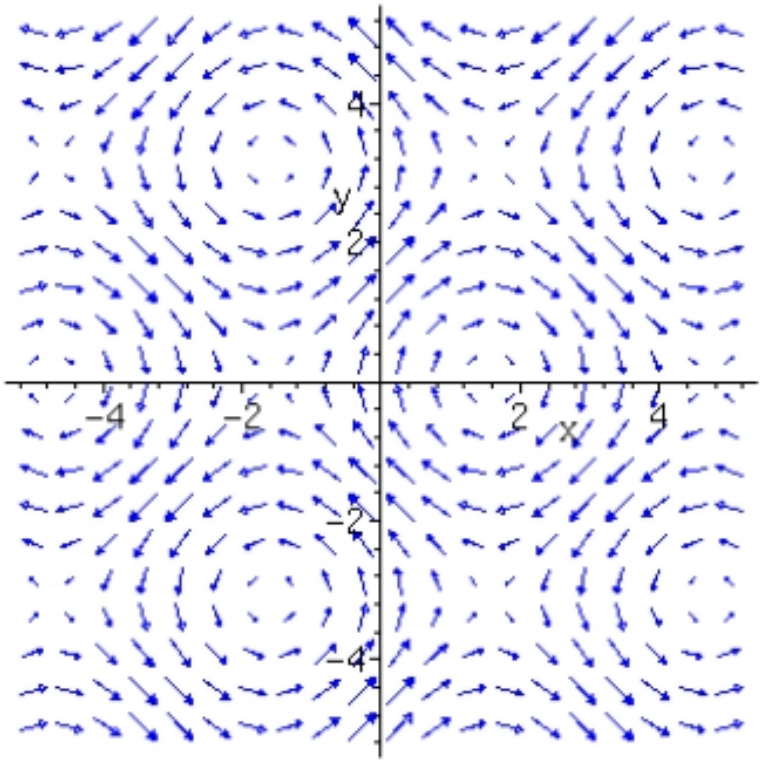


$\boldsymbol{F}(x, y) := (y, x)$



$\boldsymbol{F}(x, y) := (-y, x)$

Depiction of Vector Fields



$$\boldsymbol{F}(x, y) := (\sin y, \cos x)$$

Smooth Scalar and Vector Fields

Suppose D is an **open** subset of $\mathbb{R}^m$, that is, every point in D is an interior point of D.

A scalar field $f : D \to \mathbb{R}$ is called **smooth** if $\dfrac{\partial f}{\partial x_j}$ exists and is continuous on D for $j = 1, \ldots, m$. The set of all smooth scalar fields on D is denoted by $C^1(D)$. Similarly, the set of all scalar fields on D having continuous partial derivatives of the first and second order is denoted by $C^2(D)$.

Let $\boldsymbol{F} : D \to \mathbb{R}^m$ be a vector field on D, and let

$$\boldsymbol{F}(\boldsymbol{x}) := (F_1(\boldsymbol{x}), \ldots, F_m(\boldsymbol{x})) \quad \text{for } \boldsymbol{x} \in D,$$

where $F_i : D \to \mathbb{R}$ is the ith **component scalar field** on D for $i = 1, \ldots, m$. The vector field $\boldsymbol{F}$ is called **smooth** on D if each F_i is smooth on D, that is, $\dfrac{\partial F_i}{\partial x_j}$ exists and is continuous on D for $i, j = 1, \ldots, m$.

Gradient, Divergence and Curl

Let D be an open subset of $\mathbb{R}^3$.
If f is a smooth scalar field defined on D, then the vector field

$$\operatorname{grad} f := \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

defined on D is called the **gradient field** of f.

If $\boldsymbol{F} := (P, Q, R)$ is a smooth vector field defined on D, then the **divergence field** of $\boldsymbol{F}$ is the scalar field on D defined by

$$\operatorname{div} \boldsymbol{F} := \nabla \cdot \boldsymbol{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

and the **curl field** of $\boldsymbol{F}$ is the vector field on D defined by

$$\operatorname{curl} \boldsymbol{F} := \nabla \times \boldsymbol{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

As in the case of the cross product $\boldsymbol{x} \times \boldsymbol{y}$, we may write

$$\nabla \times \boldsymbol{F} = \nabla \times (P, Q, R) \;=\; \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

For any $m \in \mathbb{N}$, we can define the gradient field grad f of a scalar field f defined on an open subset of $\mathbb{R}^m$, and also the divergence field div $\boldsymbol{F}$ of a vector field $\boldsymbol{F}$ defined on an open subset of $\mathbb{R}^m$ in a similar manner. But the curl field curl $\boldsymbol{F}$ is defined only for a vector field $\boldsymbol{F}$ on an open subset of $\mathbb{R}^3$.

If $D \subset \mathbb{R}$ is open, and f is a smooth scalar field on D, then for $(x, y, z) \in \mathbb{R} \times D \times \mathbb{R}$, we can let $\boldsymbol{F}(x, y, z) := (0, f(x), 0)$ and define $\operatorname{curl} f := \operatorname{curl} \boldsymbol{F} = (0, 0, f')$. Also, if $D \subset \mathbb{R}^2$ is open and $\boldsymbol{\Phi} := (P, Q)$ is a smooth vector field on D, then for $(x, y, z) \in D \times \mathbb{R}$, we can let $\boldsymbol{F}(x, y, z) := (P(x, y), Q(x, y), 0)$ and define $\operatorname{curl} \boldsymbol{\Phi} := \operatorname{curl} \boldsymbol{F} = (0, 0, Q_x - P_y)$.

GCD Sequence

Let D be an open subset of $\mathbb{R}^3$. Suppose the first and the second order partial derivatives of $f, P, Q, R : D \to \mathbb{R}$ exist and are continuous on D. By the Mixed Partials Theorem,

(i) $\operatorname{curl}(\operatorname{grad} f) = \nabla \times (\nabla f) = \mathbf{0}$:

$$\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) = (0, 0, 0),$$

(ii)and if $\boldsymbol{F} := (P, Q, R)$, then $\operatorname{div}(\operatorname{curl} \boldsymbol{F}) = \nabla \cdot (\nabla \times \boldsymbol{F}) = 0$:

$$\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

Gradient, Curl and Divergence are three **operators**

$$\left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{grad}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{div}} \left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\}$$

which satisfy $\operatorname{curl}(\operatorname{grad} f) = \mathbf{0}$ and $\operatorname{div}(\operatorname{curl} \boldsymbol{F}) = 0$, so that the successive composites are zero.

The above phenomenon raises the following basic questions:

(i) If $\boldsymbol{G}$ is a smooth vector field such that $\operatorname{curl} \boldsymbol{G} = \boldsymbol{0}$, then must $\boldsymbol{G}$ be a **gradient field**, that is, is there a scalar field f such that $\boldsymbol{G} = \operatorname{grad} f$? (Then f is called a **potential field**.)

(ii) If $\boldsymbol{H}$ is a smooth vector field such that $\operatorname{div} \boldsymbol{H} = 0$, then must $\boldsymbol{H}$ be a **curl field**, that is, is there a vector field $\boldsymbol{F}$ such that $\boldsymbol{H} = \operatorname{curl} \boldsymbol{F}$?

These questions remind us of the following situation. Consider the operator $\operatorname{der} : C^1([a,b]) \to C([a,b])$ given by $\operatorname{der}(f) = f'$. If g is a continuous function on $[a,b]$, then is there a continuously differentiable function f on $[a,b]$ such that $g = f'$? The Fundamental theorem of Calculus (Part I) answers this question in the affirmative: If we let $f(x) := \int_a^x g(t)dt$ for $x \in [a,b]$, then $f' = g$ on $[a,b]$.

In view of this, each of the two questions raised above call for a suitable theory of integration, to which we now turn.

Laplacian

Let D be an open subset of $\mathbb{R}^3$, and let f be a smooth scalar field defined on D. Suppose the second order partial derivatives f_{xx}, f_{yy}, f_{zz} exist on D. The **Laplacian field** of f is the scalar field defined on D by

$$\operatorname{div}(\operatorname{grad} f) := \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian field is the composition of the operators grad and div:

$$\left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{grad}} \left\{\begin{matrix}\text{vector}\\ \text{fields}\end{matrix}\right\} \xrightarrow{\text{div}} \left\{\begin{matrix}\text{scalar}\\ \text{fields}\end{matrix}\right\}.$$

The Laplacian plays a very important role in the theory of partial differential equations, and its various applications.

Paths

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and let $m \in \mathbb{N}$. A **path** or a **parametrized curve** in $\mathbb{R}^m$ is a continuous function $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$, that is, if $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m)$, then the function $\gamma_j : [\alpha, \beta] \to \mathbb{R}$ is continuous for each $j = 1, \ldots, m$.

A path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ is called **closed** if $\boldsymbol{\gamma}(\alpha) = \boldsymbol{\gamma}(\beta)$. A path $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ is called **simple** if $\boldsymbol{\gamma}(t_1) \neq \boldsymbol{\gamma}(t_2)$ for $t_1 < t_2$ in $[\alpha, \beta]$ unless $t_1 = \alpha$ and $t_2 = \beta$.

Let $t \in [\alpha, \beta]$ and suppose $\gamma_1'(t), \ldots, \gamma_m'(t)$ exist. Then

$$\frac{d\boldsymbol{\gamma}}{dt} = \boldsymbol{\gamma}'(t) := (\gamma_1'(t), \ldots, \gamma_m'(t))$$

is called the **tangent vector** to $\boldsymbol{\gamma}$ at t. If $\boldsymbol{\gamma}'(t) \neq \mathbf{0}$, then $\hat{\boldsymbol{t}} := \boldsymbol{\gamma}'(t)/\|\boldsymbol{\gamma}'(t)\|$ is called the **unit tangent vector** to $\boldsymbol{\gamma}$ at t. (We write $\hat{\boldsymbol{t}}$ instead of $\hat{\boldsymbol{t}}_{\boldsymbol{\gamma}}(t)$ for brevity.)

Further, a path γ in $\mathbb{R}^m$ is called **smooth** if each $\gamma_j : [\alpha, \beta] \to \mathbb{R}$ is continuously differentiable for $j = 1, \ldots, m$; in case γ is a closed curve, that is, if $\gamma(\alpha) = \gamma(\beta)$, we also require $\gamma'(\alpha) = \gamma'(\beta)$. A smooth path is also called a C^1-path.

A smooth path γ in $\mathbb{R}^m$ is called **regular** if $\gamma'(t) \neq \mathbf{0}$ for each $t \in [\alpha, \beta]$, that is, if the unit tangent vector to γ exists at each $t \in [\alpha, \beta]$.

Examples: For $t \in [-1, 1]$, let $\gamma_1(t) := (t, t^2)$ and $\gamma_2(t) := (t^2, t^3)$. Then γ_1 is regular, but γ_2 is not.

A path γ in $\mathbb{R}^m$ is called **piecewise smooth** if there are $\alpha := t_0 < t_1 < \cdots < t_n =: \beta$ such that γ is smooth on each $[t_{i-1}, t_i]$, $i = 1, \ldots, n$, and it is called **piecewise regular**, if γ is regular on each $[t_{i-1}, t_i]$, $i = 1, \ldots, n$.

We shall assume hereafter that all paths are piecewise smooth, unless otherwise stated.

Example Let $a > 0$. Define $\gamma(t) := (a\cos t, a\sin t)$ for $t \in [-\pi, \pi]$. This path is called the **standard parametrized circle in $\mathbb{R}^2$ of radius a**. Since $\gamma(-\pi) = (-1, 0) = \gamma(\pi)$ and $\gamma'(t) := (-a\sin t, a\cos t)$ for $t \in [-\pi, \pi]$, we see that the path γ is closed, simple, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $\hat{\boldsymbol{t}} := (-\sin t, \cos t)$.

If we let $\widetilde{\gamma}(t) := (a\cos t, -a\sin t)$ for $t \in [-\pi, \pi]$, then it is easy to see that the path $\widetilde{\gamma}$ is also closed, simple, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $(-\sin t, -\cos t)$.

Note that $\widetilde{\gamma}([-\pi, \pi]) = \gamma([-\pi, \pi])$, that is, the functions $\widetilde{\gamma}$ and γ have the same range, although they are clearly different paths: one goes around the circle anticlockwise, but the other goes around the circle clockwise.

Further, the path given by $\widetilde{\gamma}(t) := (a\cos 2t, a\sin 2t)$ for $t \in [-\pi, \pi]$ is closed and regular, its unit tangent vector at $t \in [-\pi, \pi]$ is $(-\sin 2t, \cos 2t)$, and has the same range as γ.

Path-connected and Convex Subsets

A subset D of $\mathbb{R}^m$ is called **path-connected** if for all $\boldsymbol{u}, \boldsymbol{v} \in D$, there is a path $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ such that $\gamma(\alpha) = \boldsymbol{u}$, $\gamma(\beta) = \boldsymbol{v}$ and $\gamma(t) \in D$ for all $t \in (\alpha, \beta)$.

A subset D of $\mathbb{R}^m$ is called **convex** if the line segment joining any two points in D lies in D. A convex set is path-connected since for $\boldsymbol{u}, \boldsymbol{v} \in D$, the straight-line path $\gamma : [0, 1] \to \mathbb{R}^m$ defined by $\gamma(t) := \boldsymbol{u} + t(\boldsymbol{v} - \boldsymbol{u})$ for $t \in [0, 1]$, lies in D.

Examples:

(i) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected; in fact it is convex.

(ii) The subset $\{(x, y) \in \mathbb{R}^2 : 1/2 \leq x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected, but it is not convex.

(iii) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(2, 0)\}$ of $\mathbb{R}^2$ is not path-connected.

Arc length

Let $\gamma := (\gamma_1, \ldots, \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a piecewise smooth parametrized curve. We define the **arc length** of γ by

$$\ell(\gamma) := \int_\alpha^\beta \|\gamma'(t)\| dt = \int_\alpha^\beta \sqrt{\gamma_1'(t)^2 + \cdots + \gamma_m'(t)^2}\, dt.$$

This definition agrees with our earlier definition if $m = 2, 3$.

Example

Let γ denote the standard parametrized circle in $\mathbb{R}^2$ of radius a. Then $\ell(\gamma) = \int_{-\pi}^{\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t}\, dt = 2\pi a$.

Next, let $\widetilde{\gamma}(t) := (a \cos 2t, a \sin 2t)$ for $t \in [-\pi, \pi]$. Then $\ell(\widetilde{\gamma}) = \int_{-\pi}^{\pi} \sqrt{4a^2 \sin^2 2t + 4a^2 \cos^2 2t}\, dt = 4\pi a$.

Note that $\widetilde{\gamma}([-\pi, \pi]) = \gamma([-\pi, \pi])$, but $\ell(\widetilde{\gamma}) = 2\ell(\gamma)$.

Line Integral of a Scalar Field

Let $m \in \mathbb{N}$. Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and let $C := \gamma([\alpha, \beta])$. Let $f : C \to \mathbb{R}$ be a bounded scalar field. We define the **line integral** of f **along** γ by

$$\int_\gamma f \, ds := \int_\alpha^\beta f(\gamma(t)) \|\gamma'(t)\| dt,$$

provided the Riemann integral on the right side exists. In particular, if f is continuous, then $\int_\gamma f \, ds$ is well-defined.

Letting $f := 1$ on C, we see that $\int_\gamma ds = \ell(\gamma)$.

Example: Let $f(x, y) := x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$. If γ is the standard parametrized circle of radius a, then

$$\int_\gamma f \, ds = \int_{-\pi}^{\pi} (a^2 \cos^2 t + a^2 \sin^2 t) a \, dt = 2\pi a^3.$$

The algebraic and the order properties of the Riemann integral imply similar properties of the line integral of a scalar field.

Next, let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and consider $t_0 \in (\alpha, \beta)$. Define $\gamma_\ell : [\alpha, t_0] \to \mathbb{R}^m$ by $\gamma_\ell(t) := \gamma(t)$ for all $t \in [\alpha, t_0]$ and $\gamma_r : [t_0, \beta] \to \mathbb{R}^m$ by $\gamma_r(t) := \gamma(t)$ for all $t \in [t_0, \beta]$. Then we write $\gamma = \gamma_\ell + \gamma_r$.

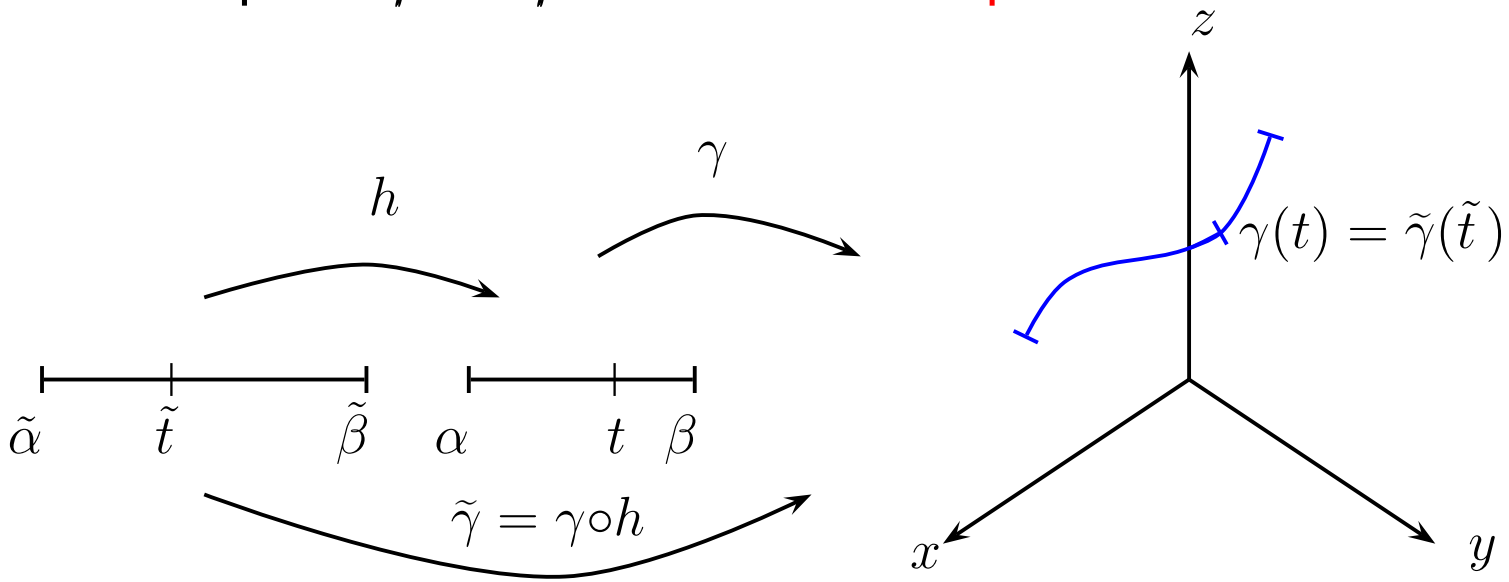
By the Domain Additivity of the Riemann integral, we obtain

$$\int_{\gamma_\ell + \gamma_r} f \, ds = \int_{\gamma_\ell} f \, ds + \int_{\gamma_r} f \, ds$$

for every continuous function $f : \gamma([\alpha, \beta]) \to \mathbb{R}$.

Reparametrization of a Curve

Let $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth parametrized curve. Let $h : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ be continuously differentiable, let $h'(\tilde{t}) \neq 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, and let $h([\tilde{\alpha}, \tilde{\beta}]) = [\alpha, \beta]$. Then the smooth path $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma} \circ h$ is called a reparametrization of $\boldsymbol{\gamma}$.



Proposition

The line integral of a continuous scalar field along a smooth path is invariant under reparametrization. In particular, the length of a smooth path is invariant under reparametrization.

Proof. Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth path, and let f be a continuous scalar field on $C := \gamma([\alpha, \beta])$.

Let $\widetilde{\gamma} := \gamma \circ h$ be a reparametrization of γ. By the chain rule, $\widetilde{\gamma}'(\tilde{t}) = \big(\gamma_1'(h(\tilde{t}))h'(\tilde{t}), \ldots, \gamma_m'(h(\tilde{t}))h'(\tilde{t})\big)$, which is equal to $\big(\gamma_1'(h(\tilde{t})), \ldots, \gamma_m'(h(\tilde{t}))\big)h'(\tilde{t})$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$.

Then our result on Integration by Substitution gives

$$\begin{aligned}
\int_{\widetilde{\gamma}} f\, ds &= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\widetilde{\gamma}(\tilde{t}))\|\widetilde{\gamma}'(\tilde{t})\| d\tilde{t} \\
&= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\gamma(h(\tilde{t})))\|\gamma'(h(\tilde{t}))\||h'(\tilde{t})| d\tilde{t} \\
&= \int_{\alpha}^{\beta} f(\gamma(t))\|\gamma'(t)\| dt = \int_{\gamma} f\, ds.
\end{aligned}$$

Considering $f := 1$, we see that the length of a smooth path is invariant under reparametrization. □

Examples: Define $\gamma(t) := (\cos t, \sin t)$ for $t \in [-\pi, \pi]$.

If $\widetilde{\gamma}(\tilde{t}) := (\cos \tilde{t}, -\sin \tilde{t})$ for $\tilde{t} \in [-\pi, \pi]$, then $\widetilde{\gamma}$ is a reparametrization of γ since $\widetilde{\gamma}(\tilde{t}) = \gamma(h(\tilde{t}))$, where $h(\tilde{t}) := -\tilde{t}$ for $\tilde{t} \in [-\pi, \pi]$.

Similarly, if $\widetilde{\gamma}(\tilde{t}) := (\cos 2\tilde{t}, \sin 2\tilde{t})$ for $\tilde{t} \in [-\pi/2, \pi/2]$, then $\widetilde{\gamma}$ is a reparametrization of γ since $\widetilde{\gamma}(\tilde{t}) = \gamma(h(\tilde{t}))$, where $h(\tilde{t}) := 2\tilde{t}$ for $\tilde{t} \in [-\pi/2, \pi/2]$.

Hence in both these cases, $\int_{\widetilde{\gamma}} f\, ds = \int_{\gamma} f\, ds$ for each continuous function on $C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

On the other hand, if $\widetilde{\gamma}(\tilde{t}) = (\cos 2\tilde{t}, \sin 2\tilde{t})$ for $\tilde{t} \in [-\pi, \pi]$, then $\widetilde{\gamma}$ cannot be a reparametrization of γ since $\ell(\widetilde{\gamma}) = 4\pi \neq 2\pi = \ell(\gamma)$. Note: If we let $h(\tilde{t}) := 2\tilde{t}$ for $\tilde{t} \in [-\pi, \pi]$, then $h([-\pi, \pi]) = [-2\pi, 2\pi] \neq [-\pi, \pi]$.