MA 105 : Calculus
Division 1, Lecture 21

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Recap of the previous lecture

- Vector algebra
- Notions of cross product, scalar triple product and vector triple product
- Scalar fields and vector fields
- Smooth scalar fields and vector fields
- Gradient, divergence and curl
- The GCD sequence
- Laplacian of a smooth scalar field
- Paths or parametrized curves
- Notions of closed/simple/smooth/regular/piecewise smooth/piecewise regular paths. Examples
- Path-connected and convex subsets
- Arc length
- Line integrals of scalar fields along piecewise smooth paths

Line Integral of a Scalar Field

Let $m \in \mathbb{N}$. Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and let $C := \gamma([\alpha, \beta])$. Let $f : C \to \mathbb{R}$ be a bounded scalar field. We define the **line integral** of f **along** γ by

$$\int_\gamma f \, ds := \int_\alpha^\beta f(\gamma(t)) \|\gamma'(t)\| dt,$$

provided the Riemann integral on the right side exists. In particular, if f is continuous, then $\int_\gamma f \, ds$ is well-defined.

Letting $f := 1$ on C, we see that $\int_\gamma ds = \ell(\gamma)$.

Example: Let $f(x, y) := x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$. If γ is the standard parametrized circle of radius a, then

$$\int_\gamma f \, ds = \int_{-\pi}^{\pi} (a^2 \cos^2 t + a^2 \sin^2 t) a \, dt = 2\pi a^3.$$

The algebraic and the order properties of the Riemann integral imply similar properties of the line integral of a scalar field.

Next, let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and consider $t_0 \in (\alpha, \beta)$. Define $\gamma_\ell : [\alpha, t_0] \to \mathbb{R}^m$ by $\gamma_\ell(t) := \gamma(t)$ for all $t \in [\alpha, t_0]$ and $\gamma_r : [t_0, \beta] \to \mathbb{R}^m$ by $\gamma_r(t) := \gamma(t)$ for all $t \in [t_0, \beta]$. Then we write $\gamma = \gamma_\ell + \gamma_r$.

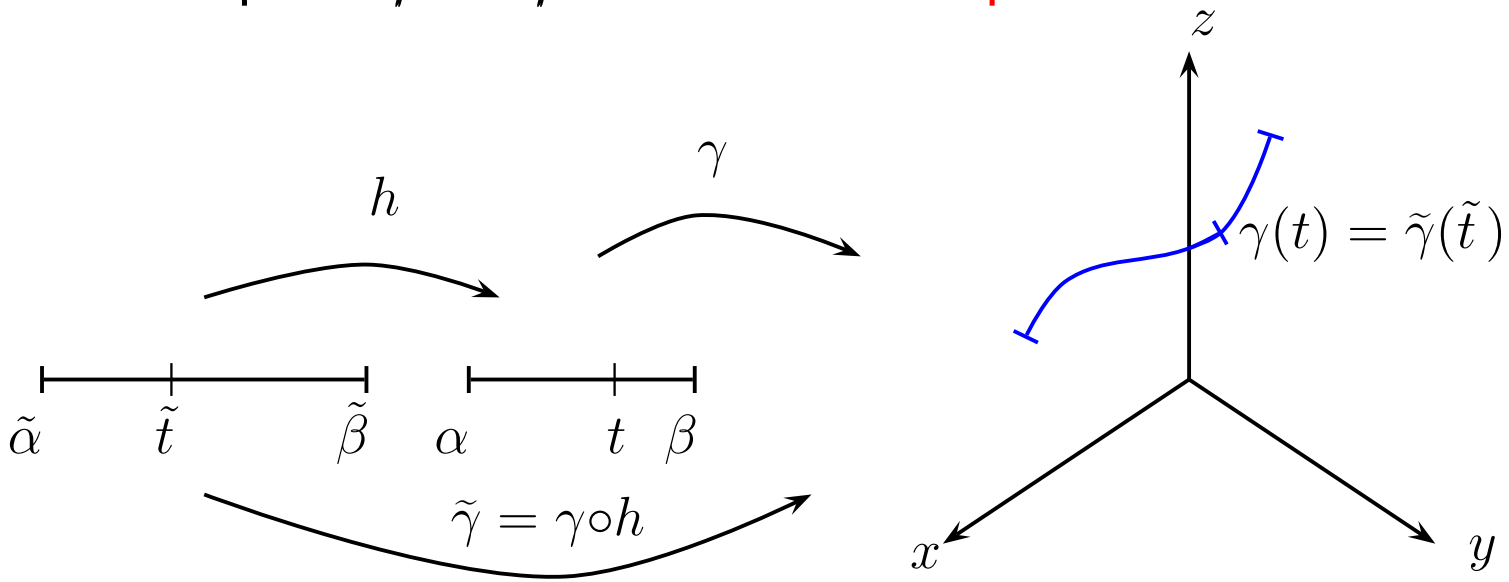
By the Domain Additivity of the Riemann integral, we obtain

$$\int_{\gamma_\ell + \gamma_r} f \, ds = \int_{\gamma_\ell} f \, ds + \int_{\gamma_r} f \, ds$$

for every continuous function $f : \gamma([\alpha, \beta]) \to \mathbb{R}$.

Reparametrization of a Curve

Let $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m) : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth parametrized curve. Let $h : [\tilde{\alpha}, \tilde{\beta}] \to \mathbb{R}$ be continuously differentiable, let $h'(\tilde{t}) \neq 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, and let $h([\tilde{\alpha}, \tilde{\beta}]) = [\alpha, \beta]$. Then the smooth path $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma} \circ h$ is called a reparametrization of $\boldsymbol{\gamma}$.



Proposition

The line integral of a continuous scalar field along a smooth path is invariant under reparametrization. In particular, the length of a smooth path is invariant under reparametrization.

Proof. Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth path, and let f be a continuous scalar field on $C := \gamma([\alpha, \beta])$.

Let $\widetilde{\gamma} := \gamma \circ h$ be a reparametrization of γ. By the chain rule, $\widetilde{\gamma}'(\tilde{t}) = \big(\gamma_1'(h(\tilde{t}))h'(\tilde{t}), \ldots, \gamma_m'(h(\tilde{t}))h'(\tilde{t})\big)$, which is equal to $\big(\gamma_1'(h(\tilde{t})), \ldots, \gamma_m'(h(\tilde{t}))\big)h'(\tilde{t})$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$.

Then our result on Integration by Substitution gives

$$\begin{aligned}
\int_{\widetilde{\gamma}} f\, ds &= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\widetilde{\gamma}(\tilde{t}))\|\widetilde{\gamma}'(\tilde{t})\|\, d\tilde{t} \\
&= \int_{\tilde{\alpha}}^{\tilde{\beta}} f(\gamma(h(\tilde{t})))\|\gamma'(h(\tilde{t}))\|\,|h'(\tilde{t})|\, d\tilde{t} \\
&= \int_{\alpha}^{\beta} f(\gamma(t))\|\gamma'(t)\|\, dt = \int_{\gamma} f\, ds.
\end{aligned}$$

Considering $f := 1$, we see that the length of a smooth path is invariant under reparametrization. □

Examples: Define $\gamma(t) := (\cos t, \sin t)$ for $t \in [-\pi, \pi]$.

If $\widetilde{\gamma}(\tilde{t}) := (\cos \tilde{t}, -\sin \tilde{t})$ for $\tilde{t} \in [-\pi, \pi]$, then $\widetilde{\gamma}$ is a reparametrization of γ since $\widetilde{\gamma}(\tilde{t}) = \gamma(h(\tilde{t}))$, where $h(\tilde{t}) := -\tilde{t}$ for $\tilde{t} \in [-\pi, \pi]$.

Similarly, if $\widetilde{\gamma}(\tilde{t}) := (\cos 2\tilde{t}, \sin 2\tilde{t})$ for $\tilde{t} \in [-\pi/2, \pi/2]$, then $\widetilde{\gamma}$ is a reparametrization of γ since $\widetilde{\gamma}(\tilde{t}) = \gamma(h(\tilde{t}))$, where $h(\tilde{t}) := 2\tilde{t}$ for $\tilde{t} \in [-\pi/2, \pi/2]$.

Hence in both these cases, $\int_{\widetilde{\gamma}} f\, ds = \int_{\gamma} f\, ds$ for each continuous function on $C := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

On the other hand, if $\widetilde{\gamma}(\tilde{t}) = (\cos 2\tilde{t}, \sin 2\tilde{t})$ for $\tilde{t} \in [-\pi, \pi]$, then $\widetilde{\gamma}$ cannot be a reparametrization of γ since $\ell(\widetilde{\gamma}) = 4\pi \neq 2\pi = \ell(\gamma)$. Note: If we let $h(\tilde{t}) := 2\tilde{t}$ for $\tilde{t} \in [-\pi, \pi]$, then $h([-\pi, \pi]) = [-2\pi, 2\pi] \neq [-\pi, \pi]$.

We now give an intrinsic parametrization of a regular path $\gamma : [\alpha, \beta] \to \mathbb{R}^m$. For $t \in [\alpha, \beta]$, let

$$s(t) := \int_\alpha^t \|\gamma'(u)\| du.$$

Then $s(\alpha) = 0$, $s(\beta) = \ell(\gamma)$, and $s'(t) = \|\gamma'(t)\|$ for all $t \in [\alpha, \beta]$ by the FTC (Part I). Since γ is a regular path, $s'(t) > 0$ for all $t \in [\alpha, \beta]$, and so $t \longmapsto s(t)$ is a strictlty increasing differentiable function from $[\alpha, \beta]$ onto $[0, \ell(\gamma)]$. Let $h : [0, \ell(\gamma)] \to [\alpha, \beta]$ denote its inverse. Then h is differentiable and its derivative does not vanish on $[0, \ell(\gamma)]$. Define $\widetilde{\gamma}(s) := \gamma(h(s))$ for $s \in [0, \ell(\gamma)]$.

Then $\widetilde{\gamma}$ is a reparametrization of the regular path γ; it is known as the **arc length parametrization**, and s is called the **arc length parameter**.

Example: Let γ denote the standard parametrized circle in $\mathbb{R}^2$ of radius a. Then

$$s(t) = \int_{-\pi}^{t} \|(-a\sin t, a\cos t)\|dt = a(t+\pi) \quad \text{for } t \in [-\pi, \pi].$$

The inverse of the function $t \longmapsto a(t+\pi)$ is given by the function $s \longmapsto (s/a) - \pi$ for $s \in [0, 2\pi a]$. Hence the arc length parametrization of γ is given by

$$\widetilde{\gamma}(s) = \left(a\cos\left(\frac{s}{a} - \pi\right),\ a\sin\left(\frac{s}{a} - \pi\right)\right) = \left(-a\cos\frac{s}{a},\ a\sin\frac{s}{a}\right)$$

for $s \in [0, 2\pi a]$.

In view of the equality $\dfrac{ds}{dt} = s'(t) = \|\gamma'(t)\|$ for $t \in [\alpha, \beta]$ stated earlier, we introduce the following differential notation.

For a regular path γ, we let $ds := \|\gamma'(t)\|dt$. (Compare our definition of $\int_\gamma f\, ds := \int_\alpha^\beta f(\gamma(t))\|\gamma'(t)\|dt$.) We note that

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = \frac{d\gamma}{ds}\|\gamma'(t)\|, \quad \text{and so } \hat{\boldsymbol{t}} = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{d\gamma}{ds}.$$

Geometric Curve

Let $[\alpha, \beta]$ be an interval in $\mathbb{R}$, and let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path. Then $C := \gamma([\alpha, \beta])$ is called a **geometric curve**; γ is said to give a **parametrization** of C.

Suppose γ is simple. In view of the reparametrization result for continuous scalar fields on paths proved above, we define

$$\int_C f \, ds := \int_\gamma f \, ds = \int_\alpha^\beta f(\gamma(t)) \|\gamma'(t)\| dt$$

for a continuous scalar field f on C. In particular, we write

$$\ell(C) := \ell(\gamma) = \int_\alpha^\beta \|\gamma'(t)\| dt.$$

Line Integral of a Vector Field

Let $m \in \mathbb{N}$. Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a (piecewise smooth) path, and let $C := \boldsymbol{\gamma}([\alpha, \beta])$. Let $\boldsymbol{F} : C \to \mathbb{R}^m$ be a bounded vector field. We define the **line integral** of $\boldsymbol{F}$ **along** $\boldsymbol{\gamma}$ by

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} := \int_{\alpha}^{\beta} \boldsymbol{F}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t) dt,$$

provided the Riemann integral on the right side exists. In particular, if $\boldsymbol{F}$ is continuous, that is, if $\boldsymbol{F} = (F_1, \ldots, F_m)$ and each $F_i : C \to \mathbb{R}$ is continuous, then $\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is well-defined.

In analogy with our differential notation $ds := \|\boldsymbol{\gamma}'(t)\| dt$, we let $d\boldsymbol{s} := \boldsymbol{\gamma}'(t) dt$.

Thus if $\boldsymbol{\gamma} := (\gamma_1, \ldots, \gamma_m)$, then $d\boldsymbol{s} := (\gamma_1'(t)dt, \ldots, \gamma_m'(t)dt)$.

Let $m = 3$, and $\gamma(t) := (x(t), y(t), z(t))$ for $t \in [\alpha, \beta]$. Then

$$\gamma'(t) := (x'(t), y'(t), z'(t)) \quad \text{for } t \in [\alpha, \beta].$$

Let $\boldsymbol{F} := (P, Q, R)$. Then we write the line integral

$$\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

as $\displaystyle\int_{\gamma} P\,dx + Q\,dy + R\,dz$. If the path γ is regular, then

$$d\boldsymbol{s} = \gamma'(t)dt = \hat{\boldsymbol{t}}\, \|\gamma'(t)\| dt = \hat{\boldsymbol{t}}\, ds,$$

and so we may write the line integral $\displaystyle\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s}$ as $\displaystyle\int_{\gamma} \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds$.

The algebraic properties of the Riemann integral imply similar properties of the line integral of a vector field. Also, as for the line integral of a scalar field, $\int_{\gamma_\ell + \gamma_r} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\gamma_\ell} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\gamma_r} \boldsymbol{F} \cdot d\boldsymbol{s}$.

Examples:

(i) Let $\gamma(t) := (\cos t, \sin t, \cos t \sin t) \in \mathbb{R}^3$ for $t \in [-\pi, \pi]$, and $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$. Noting that $\cos t \sin t = (\sin 2t)/2$ for $t \in [-\pi, \pi]$, we obtain

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_\gamma y\, dx + z\, dy + x\, dz$$

$$= \int_{-\pi}^{\pi} \big((\sin t)(-\sin t) + \frac{1}{2}(\sin 2t)(\cos t) + (\cos t)(\cos 2t)\big)\, dt$$

$$= -\pi + 0 + 0 = -\pi.$$

(ii) Let $\gamma(t) := (t, 2t^2)$ for $t \in [1, 2]$, and $\boldsymbol{F}(x, y) := (x\, y, y^2)$ for $(x, y) \in \mathbb{R}^2$. Then

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_1^2 (2t^3, 4t^4) \cdot (1, 4t) dt = \int_1^2 (2t^3 + 16t^5) dt = \frac{351}{2}.$$

On the other hand, if $\widetilde{\gamma}(t) := (-t, 2t^2)$ for $t \in [-2, -1]$, then

$$\int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{-2}^{-1} (-2t^3, 4t^4) \cdot (-1, 4t) dt = \int_{-2}^{-1} (2t^3 + 16t^5) dt = -\frac{351}{2}.$$

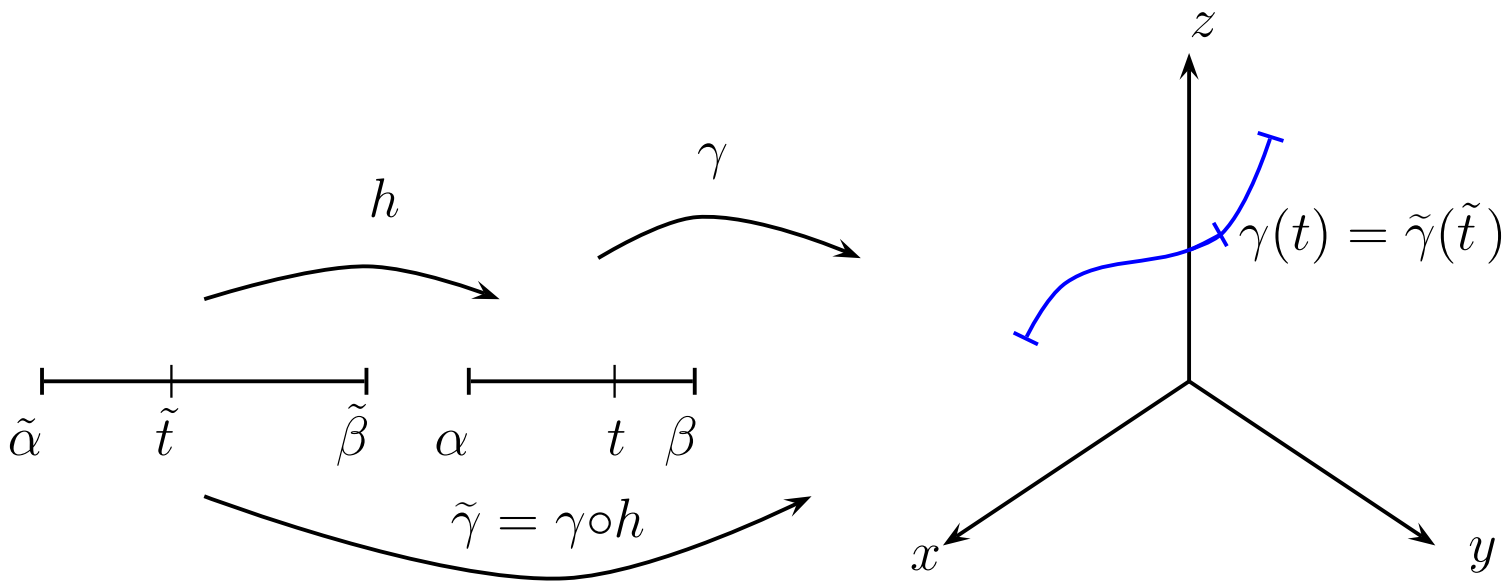
In the example (ii) above, the path $\widetilde{\gamma}$ is a reparametrization of the path γ, and the line integral of the vector field $\boldsymbol{F}$ along $\widetilde{\gamma}$ is the negative of the line integral of $\boldsymbol{F}$ along γ.

Thus line integrals of a vector field are not, in general, invariant under reparametrization. However, they can differ only in sign, as we shall see in the next proposition.

Proposition

The line integral of a continuous vector field along a smooth path is invariant under a reparametrization only up to its sign.

Proof. Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a smooth path, and let $\boldsymbol{F}$ be a continuous vector field on $C := \boldsymbol{\gamma}([\alpha, \beta])$. Suppose $\widetilde{\boldsymbol{\gamma}} := \boldsymbol{\gamma} \circ h$ is a reparametrization of $\boldsymbol{\gamma}$.



As before, by the chain rule,
$\widetilde{\boldsymbol{\gamma}}'(\tilde{t}) = \big(\gamma_1'(h(\tilde{t})), \ldots, \gamma_m'(h(\tilde{t})\big) h'(\tilde{t})$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, and so

$$\begin{aligned}\int_{\widetilde{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_{\tilde{\alpha}}^{\tilde{\beta}} \boldsymbol{F}(\widetilde{\gamma}(\tilde{t}))\cdot\widetilde{\gamma}'(\tilde{t})\,d\tilde{t}\\ &= \int_{\tilde{\alpha}}^{\tilde{\beta}} \boldsymbol{F}(\gamma(h(\tilde{t})))\cdot\gamma'(h(\tilde{t}))h'(\tilde{t})d\tilde{t}.\end{aligned}$$

Since $h'(\tilde{t}) \neq 0$ for $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$ and h' is continuous on $[\tilde{\alpha}, \tilde{\beta}]$, either $h'(\tilde{t}) > 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, or $h'(\tilde{t}) < 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$.

In the former case, $h'(\tilde{t}) = |h'(\tilde{t})|$, and so our result on Integration by Substitution shows that $\displaystyle\int_{\widetilde{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\gamma} \boldsymbol{F}\cdot d\boldsymbol{s}$.

In the latter case, $h'(\tilde{t}) = -|h'(\tilde{t})|$, and so our result on Integration by Substitution shows that

$$\int_{\widetilde{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s} = -\int_{\gamma} \boldsymbol{F}\cdot d\boldsymbol{s}. \qquad \square$$

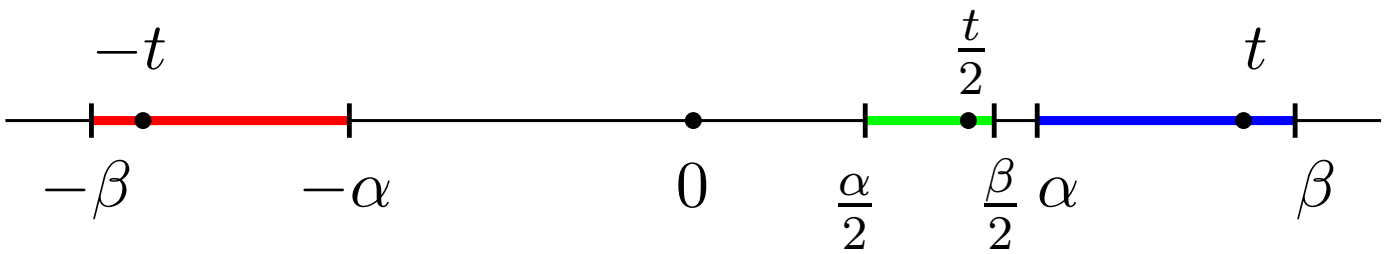
Examples Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a path, and let $\boldsymbol{F} : \gamma([\alpha, \beta]) \to \mathbb{R}^m$ be a continuous vector field.

(i) Let $\tilde{\alpha} := -\beta$, $\tilde{\beta} := -\alpha$, and define $h(\tilde{t}) := -\tilde{t}$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Then $\widetilde{\gamma}(\tilde{t}) := \gamma(-\tilde{t})$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Since $h'(\tilde{t}) = -1 < 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, we obtain

$$\int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

(ii) Let $\tilde{\alpha} := \alpha/2$, $\tilde{\beta} := \beta/2$, and define $h(\tilde{t}) := 2\tilde{t}$, $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Then $\widetilde{\gamma}(\tilde{t}) := \gamma(h(\tilde{t})) = \gamma(2\tilde{t})$ for $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$. Since $h'(\tilde{t}) = 2 > 0$ for all $\tilde{t} \in [\tilde{\alpha}, \tilde{\beta}]$, we obtain

$$\int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s}.$$



Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a path. Define $\tilde{\boldsymbol{\gamma}} : [-\beta, -\alpha] \to \mathbb{R}^m$ by $\tilde{\boldsymbol{\gamma}}(\tilde{t}) := \boldsymbol{\gamma}(-\tilde{t})$ for $\tilde{t} \in [-\beta, -\alpha]$. Then the path $\tilde{\boldsymbol{\gamma}}$ is called the **negative** of $\boldsymbol{\gamma}$, and it is denoted by $-\boldsymbol{\gamma}$. Clearly, $-(-\boldsymbol{\gamma}) = \boldsymbol{\gamma}$.

Note that $(-\boldsymbol{\gamma})([-\beta, -\alpha]) = \boldsymbol{\gamma}([\alpha, \beta])$.

Also, $(-\boldsymbol{\gamma})(-\beta) = \boldsymbol{\gamma}(\beta)$ and $(-\boldsymbol{\gamma})(-\alpha) = \boldsymbol{\gamma}(\alpha)$, that is, the initial point of $-\boldsymbol{\gamma}$ is the final point of $\boldsymbol{\gamma}$, and the final point of $-\boldsymbol{\gamma}$ is the initial point of $\boldsymbol{\gamma}$.

Note: If $h(\tilde{t}) = -\tilde{t}$ for all $\tilde{t} \in [-\beta, -\alpha]$, then $h' = -1$.
If $\boldsymbol{F}$ is a continuous vector field on $\boldsymbol{\gamma}([\alpha, \beta])$, then we obtain

$$\int_{-\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Physical interpretation of the line integral of a vector field:

Suppose a vector field $\boldsymbol{F}$ is defined on $D \subset \mathbb{R}^m$, and γ is a regular path. If $C := \gamma([\alpha, \beta]) \subset D$, then $\boldsymbol{F} \cdot \hat{\boldsymbol{t}}$ is the component of $\boldsymbol{F}$ in the direction of the unit tangent vector $\hat{\boldsymbol{t}}$.

If $\boldsymbol{F}$ is a force field in the domain D, then the line integral

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_\gamma \boldsymbol{F} \cdot \hat{\boldsymbol{t}}\, ds$$

represents the work done by the force $\boldsymbol{F}$ along the path γ.

Similarly, if $\boldsymbol{F}$ is the velocity field of a moving fluid in the domain D, then the line integral $\displaystyle\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$ represents the flow of the fluid along the path γ; in case γ is a closed path, the line integral represents the circulation of the fluid along γ.

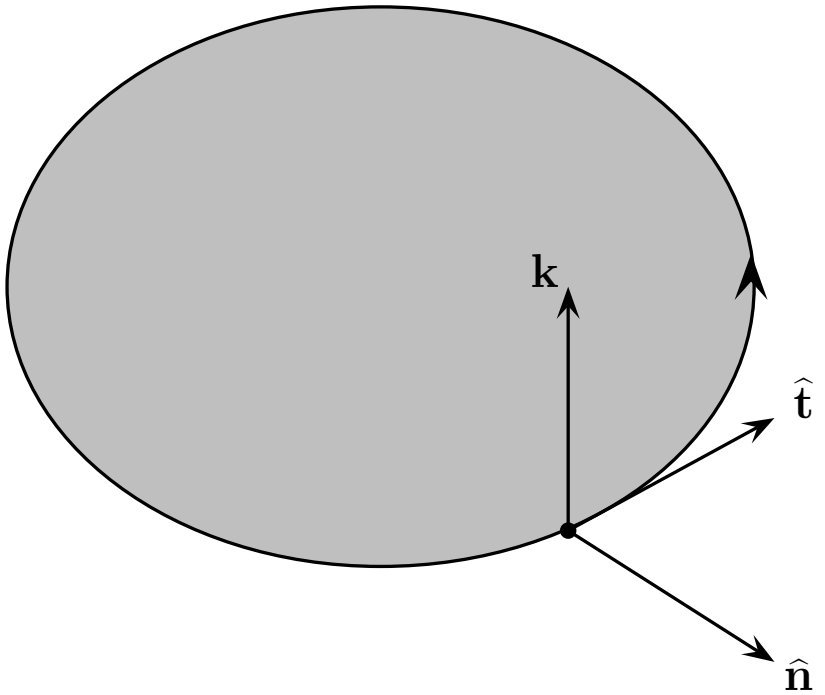
Flux Integral

There is a variant of the circulation known as the flux of a moving fluid. We shall now consider its version in $\mathbb{R}^2$.

Let $\gamma : [\alpha, \beta] \to \mathbb{R}^2$ be a simple closed regular path. Let D be an open subset of $\mathbb{R}^2$ containing $C := \gamma([\alpha, \beta])$. Let $\hat{\boldsymbol{n}}$ denote the (continuous) **outward unit normal** to γ. If $\boldsymbol{F}$ is a continuous vector field on D, then

$$\int_\gamma \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \, ds$$

is called the **flux integral** of $\boldsymbol{F}$ along γ. It represents the flux of a velocity field $\boldsymbol{F}$ along a simple closed regular path γ.



Suppose $\gamma(t)$ moves in the anti-clockwise direction (as seen from high above) as the parameter t goes from α to β. Then

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{t}} \times \boldsymbol{k} = \frac{d\gamma}{ds} \times \boldsymbol{k} = \left(\frac{dx}{ds}\, \boldsymbol{i} + \frac{dy}{ds}\, \boldsymbol{j} \right) \times \boldsymbol{k} = \frac{dy}{ds}\, \boldsymbol{i} - \frac{dx}{ds}\, \boldsymbol{j}.$$

Hence if $\boldsymbol{F} := P\, \boldsymbol{i} + Q\, \boldsymbol{j}$, then the flux integral of $\boldsymbol{F}$ along γ is

$$\int_\gamma \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, ds = \int_\gamma \left(P\, \frac{dy}{ds} - Q\, \frac{dx}{ds} \right) ds = \int_\gamma P\, dy - Q\, dx.$$

Example

Let γ denote the standard unit circle in $\mathbb{R}^2$ of radius 1, and let $\boldsymbol{F} := (P, Q)$, where $P(x, y) := x - y$ and $Q(x, y) := x$ for $(x, y) \in \mathbb{R}^2$.

Then the circulation of $\boldsymbol{F}$ along γ is equal to $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s}$, that is,

$$\int_\gamma P\,dx + Q\,dy = \int_{-\pi}^{\pi} \big((\cos t - \sin t)(-\sin t) + (\cos t)(\cos t)\big)\,dt = 2\pi.$$

Also, the flux of $\boldsymbol{F}$ along γ is equal to $\int_\gamma \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\,ds$, that is,

$$\int_\gamma P\,dy - Q\,dx = \int_{-\pi}^{\pi} \big((\cos t - \sin t)(\cos t) - (\cos t)(-\sin t)\big)\,dt = \pi.$$

Path Independence of Line Integrals

Let $m \in \mathbb{N}$, $D \subset \mathbb{R}^m$, and consider a path $\gamma : [\alpha, \beta] \to \mathbb{R}^m$. We say that γ **lies in** D if $\gamma([\alpha, \beta]) \subset D$. Also, we say that $\gamma(\alpha)$ is the **initial point** and $\gamma(\beta)$ is the **final point** of γ.

Let $\boldsymbol{F} : D \to \mathbb{R}^m$ be a continuous vector field. We say that line integrals of $\boldsymbol{F}$ are **path-independent** in D if

$$\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$$

for all (piecewise smooth) paths γ and $\widetilde{\gamma}$ lying in D which have the same initial point as well as the same final point.

Note: The path independence of line integrals of $\boldsymbol{F}$ in D does not mean that $\int_{\gamma} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$ for any smooth paths γ and $\widetilde{\gamma}$ in D. It only says that this equality holds if the smooth paths γ and $\widetilde{\gamma}$ lie in D, and both have the same initial point, and both have the same final point.

Examples:

(i) Let $\boldsymbol{F}(x,y) := (x+y, x)$ for $(x,y) \in \mathbb{R}^2$. Consider a smooth path $\gamma : [\alpha, \beta] \to \mathbb{R}^2$. If $\gamma(t) := (x(t), y(t))$ for $t \in [\alpha, \beta]$, then

$$\begin{aligned}
\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_\alpha^\beta \big(x(t)+y(t), x(t)\big) \cdot (x'(t), y'(t)) dt \\
&= \int_\alpha^\beta \big(x(t)x'(t) + (xy)'(t)\big) dt \\
&= \frac{1}{2}x^2(\beta) - \frac{1}{2}x^2(\alpha) + x(\beta)y(\beta) - x(\alpha)y(\alpha).
\end{aligned}$$

Thus if smooth paths γ and $\widetilde{\gamma}$ have the same initial point, and also the same final point, then $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s}$. Hence line integrals of $\boldsymbol{F}$ are path-independent in $\mathbb{R}^2$.

(ii) Let $\boldsymbol{F}(x,y) := (x-y, x)$ for $(x,y) \in \mathbb{R}^2$. Consider $\gamma(t) := (t, 0)$ for $t \in [-1, 1]$, and $\widetilde{\gamma}(t) := (-\cos t, \sin t)$ for $t \in [0, \pi]$. The initial point of both γ and $\widetilde{\gamma}$ is $(-1, 0)$

and the final point of both γ and $\widetilde{\gamma}$ is $(1,0)$. However,

$$\int_{\gamma} \boldsymbol{F}\cdot d\boldsymbol{s} \;=\; \int_{-1}^{1}(t-0,t)\cdot(1,0)dt = \int_{-1}^{1} t\,dt = 0,$$

whereas

$$\begin{aligned}\int_{\widetilde{\gamma}} \boldsymbol{F}\cdot d\boldsymbol{s} &= \int_{0}^{\pi}(-\cos t-\sin t,-\cos t)\cdot(\sin t,\cos t)dt\\ &= \int_{0}^{\pi}(-\cos t\sin t-\sin^2 t-\cos^2 t)dt\\ &= -\frac{1}{2}\int_{0}^{\pi}\sin 2t dt-\int_{0}^{\pi}dt = 0-\pi=-\pi.\end{aligned}$$

Hence line integrals of $\boldsymbol{F}$ are not path-independent in $\mathbb{R}^2$.

We now give a criterion for the path-independence of line integrals of a vector field.

Proposition

Let $\boldsymbol{F}$ be a continuous vector field on $D \subset \mathbb{R}^m$. Then line integrals of $\boldsymbol{F}$ are path-independent in $D \iff$ the line integral of $\boldsymbol{F}$ along every closed path that lies in D is zero.

$\Longrightarrow$) Let $\gamma : [\alpha, \beta] \to \mathbb{R}^m$ be a closed path that lies in D. Then $\gamma(\alpha) = \gamma(\beta) \in D$. Define $\gamma_0(t) := \gamma(\alpha)$ for all $t \in [\alpha, \beta]$. Then both γ and γ_0 lie in D, and they have the same initial point $\gamma(\alpha)$ as well as the same final point $\gamma(\alpha)$. By the path-independence of $\boldsymbol{F}$ in D,

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\gamma_0} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_\alpha^\beta \boldsymbol{F}(\gamma(\alpha)) \cdot \boldsymbol{0}\, dt = 0.$$

$\Longleftarrow$) Let γ and $\widetilde{\gamma}$ be paths lying in D which have the same initial point, and also the same final point. Then $\gamma_c := \gamma + (-\widetilde{\gamma})$ is a closed path in D, and so

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{-\widetilde{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\gamma_c} \boldsymbol{F} \cdot d\boldsymbol{s} = 0. \quad \square$$

FTC for Line Integrals

Consider the following versions of Part I and Part II of the Fundamental Theorem of Calculus for Riemann integrals of scalar fields on an open interval in $\mathbb{R}$.

Part I: Let I be an open interval in $\mathbb{R}$, and let g be a continuous scalar field on I. If $a \in I$, and we let $f(x) := \int_a^x g(t)dt$ for $x \in I$, then $f' = g$ on I.

Part II: Let I be an open interval in $\mathbb{R}$, and let f be a smooth scalar field on I. Then

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{for all } a, b \in I.$$

We shall now attempt to find analogues of the above results for line integrals of vector fields. In this process, we shall obtain another important criterion for the path-independence of line integrals of a vector field.

Theorem (FTC for Line Integrals: Part I)

Let D be an open subset of $\mathbb{R}^m$, and let $\boldsymbol{G}$ be a continuous vector field on D. Suppose line integrals of $\boldsymbol{G}$ are path-independent in D. If D is path-connected and for a fixed $\boldsymbol{x}_0$ in D, we define

$$f(\boldsymbol{x}) := \int_{\gamma_{\boldsymbol{x}}} \boldsymbol{G} \cdot d\boldsymbol{s}, \quad \text{where } \gamma_{\boldsymbol{x}} \text{ is a path in } D \text{ from } \boldsymbol{x}_0 \text{ to } \boldsymbol{x},$$

then $\nabla f = \boldsymbol{G}$ on D. Consequently, $\boldsymbol{G}$ is a gradient field on D.

Proof: Suppose D is path-connected. Then for every $\boldsymbol{x} \in D$, there is a (piecewise smooth) path $\gamma_{\boldsymbol{x}}$ with initial point $\boldsymbol{x}_0$ and final point $\boldsymbol{x}$. Further, as line integrals of $\boldsymbol{G}$ are path-independent on D, the function $f : D \to \mathbb{R}$ mentioned above is well-defined.

For simplicity, let $m = 2$, and consider $\boldsymbol{x} := (x_1, x_2) \in D$.