# MA 105 : Calculus
# Division 1, Lecture 22

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## Proposition (Criterion for Path Independence)

Let $\boldsymbol{F}$ be a continuous vector field on $D \subset \mathbb{R}^m$. Then line integrals of $\boldsymbol{F}$ are path-independent in $D \iff$ the line integral of $\boldsymbol{F}$ along every closed path that lies in $D$ is zero.

$\Longrightarrow$ ) Let $\boldsymbol{\gamma} : [\alpha, \beta] \to \mathbb{R}^m$ be a closed path that lies in $D$. Then $\boldsymbol{\gamma}(\alpha) = \boldsymbol{\gamma}(\beta) \in D$. Define $\boldsymbol{\gamma}_0(t) := \boldsymbol{\gamma}(\alpha)$ for all $t \in [\alpha, \beta]$. Then both $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_0$ lie in $D$, and they have the same initial point $\boldsymbol{\gamma}(\alpha)$ as well as the same final point $\boldsymbol{\gamma}(\alpha)$. By the path-independence of $\boldsymbol{F}$ in $D$,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}_0} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} \boldsymbol{F}(\boldsymbol{\gamma}(\alpha)) \cdot \boldsymbol{0}\, dt = 0.$$

$\Longleftarrow$ ) Let $\boldsymbol{\gamma}$ and $\widetilde{\boldsymbol{\gamma}}$ be paths lying in $D$ which have the same initial point, and also the same final point. Then $\boldsymbol{\gamma}_c := \boldsymbol{\gamma} + (-\widetilde{\boldsymbol{\gamma}})$ is a closed path in $D$, and so

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{-\widetilde{\boldsymbol{\gamma}}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{\gamma}_c} \boldsymbol{F} \cdot d\boldsymbol{s} = 0. \quad \square$$

# FTC for Line Integrals

Consider the following versions of Part I and Part II of the Fundamental Theorem of Calculus for Riemann integrals of scalar fields on an open interval in $\mathbb{R}$.

Part I: Let $I$ be an open interval in $\mathbb{R}$, and let $g$ be a continuous scalar field on $I$. If $a \in I$, and we let $f(x) := \int_a^x g(t)dt$ for $x \in I$, then $f' = g$ on $I$.

Part II: Let $I$ be an open interval in $\mathbb{R}$, and let $f$ be a smooth scalar field on $I$. Then

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{for all } a, b \in I.$$

We shall now attempt to find analogues of the above results for line integrals of vector fields. In this process, we shall obtain another important criterion for the path-independence of line integrals of a vector field.

## Theorem (FTC for Line Integrals: Part I)

Let $D$ be an open subset of $\mathbb{R}^m$, and let $\boldsymbol{G}$ be a continuous vector field on $D$. Suppose line integrals of $\boldsymbol{G}$ are path-independent in $D$. If $D$ is path-connected and for a fixed $\boldsymbol{x}_0$ in $D$, we define

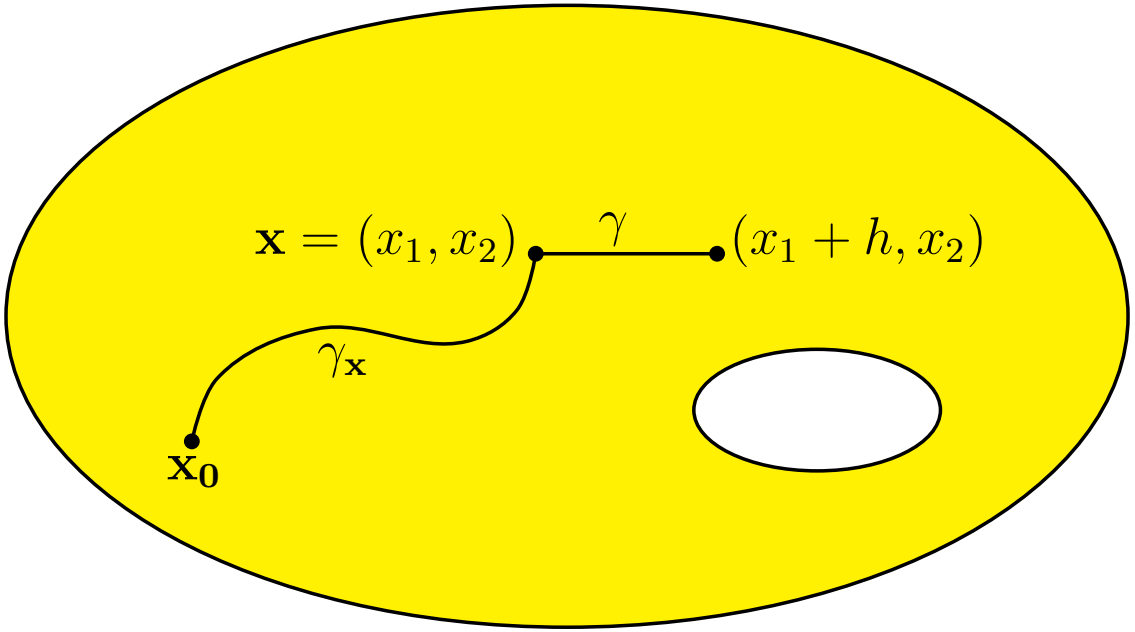
$$f(\boldsymbol{x}) := \int_{\gamma_{\boldsymbol{x}}} \boldsymbol{G} \cdot d\boldsymbol{s}, \quad \text{where } \gamma_{\boldsymbol{x}} \text{ is a path in } D \text{ from } \boldsymbol{x}_0 \text{ to } \boldsymbol{x},$$

then $\nabla f = \boldsymbol{G}$ on $D$. Consequently, $\boldsymbol{G}$ is a gradient field on $D$.

Proof: Suppose $D$ is path-connected. Then for every $\boldsymbol{x} \in D$, there is a (piecewise smooth) path $\gamma_{\boldsymbol{x}}$ with initial point $\boldsymbol{x}_0$ and final point $\boldsymbol{x}$. Further, as line integrals of $\boldsymbol{G}$ are path-independent on $D$, the function $f : D \to \mathbb{R}$ mentioned above is well-defined.

For simplicity, let $m = 2$, and consider $\boldsymbol{x} := (x_1, x_2) \in D$.

Since the set $D$ is open in $\mathbb{R}^2$, the line segment joining $(x_1, x_2)$ to $(x_1 + h, x_2)$ lies in $D$ for all $h \in \mathbb{R}$ if $|h|$ is sufficiently small.




Let $\gamma(t) := (x_1 + th, x_2)$ for $t \in [0, 1]$. Now $\gamma_{\boldsymbol{x}}$ followed by $\gamma$ is a (piecewise smooth) path from $\boldsymbol{x}_0$ to $(x_1 + h, x_2)$. Hence

$$f(x_1+h, x_2)-f(x_1, x_2) = \int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s} = \int_0^1 \boldsymbol{G}(x_1+th, x_2)\cdot(h, 0)dt.$$

Let $\boldsymbol{G} := (G_1, G_2)$. It follows that for all $h \neq 0$ and small $|h|$,

$$\frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h} = \int_0^1 G_1(x_1 + th, x_2)dt.$$

Also, since $G_1$ is continuous at $(x_1, x_2)$,

$$\int_0^1 G_1(x_1+th, x_2)dt \to \int_0^1 G_1(x_1, x_2)dt = G_1(x_1, x_2) \text{ as } h \to 0.$$

Thus the partial of $f$ at any $(x_1, x_2) \in D$ with respect to the first variable is equal to $G_1(x_1, x_2)$. Similarly, the partial of $f$ at $(x_1, x_2)$ with respect to the second variable equals $G_2(x_1, x_2)$. Hence $\nabla f = (f_{x_1}, f_{x_2}) = (G_1, G_2) = \boldsymbol{G}$.

Now if $D$ is any open subset of $\mathbb{R}^2$, then by considering components of $D$, each of which is open and path-connected, we see that $\nabla f = \boldsymbol{G}$. □

## Theorem (FTC for Line Integrals: Part II)

Let $D$ be an open subset of $\mathbb{R}^m$, and let $f$ be a smooth scalar field on $D$. Let $\gamma$ be a path that lies in $D$, $A$ be the initial point of $\gamma$, and $B$ be the final point of $\gamma$. Then

$$\int_{\gamma} (\nabla f) \cdot d\boldsymbol{s} = f(B) - f(A).$$

Consequently, line integrals of a continuous gradient field on $D$ are path-independent in $D$.

Proof: By part (ii) of the chain rule,

$$\int_{\gamma} (\nabla f) \cdot d\boldsymbol{s} = \int_{\alpha}^{\beta} (\nabla f)(\gamma(t)) \cdot \gamma'(t) dt = \int_{\alpha}^{\beta} (f \circ \gamma)'(t) dt.$$

So by Part II of the FTC for the function $f \circ \gamma : [\alpha, \beta] \to \mathbb{R}$,

$$\int_{\gamma} \nabla f \cdot d\boldsymbol{s} = f(\gamma(\beta)) - f(\gamma(\alpha)) = f(B) - f(A). \quad \square$$

The previous result shows that if $\boldsymbol{G}$ is a continuous gradient field on an open subset of $\mathbb{R}^m$, then the evaluation of a line integral $\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s}$ can be simplified on replacing $\gamma$ by a suitable path having the same initial point and the same final point.

If we actually know a scalar field $f$ such that $\nabla f = \boldsymbol{G}$, then the evaluation of $\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s}$ is immediate even when the path $\gamma$ is very complicated. This process is similar to the evaluation of a Riemann integral by finding an antiderivative of its integrand.

Example: Let $\boldsymbol{G}(x,y,z) := (yz, zx, xy)$ for $(x,y,z) \in \mathbb{R}^3$, and let $\gamma(t) := (\cos^4 t, \sin^4 t, \tan^4 t)$, $t \in [0, \pi/4]$. If we let $f(x,y,z) := xyz$ for $(x,y,z) \in \mathbb{R}^3$, then $\nabla f = \boldsymbol{G}$ on $\mathbb{R}^3$, and

$$\int_\gamma \boldsymbol{G}\cdot d\boldsymbol{s} = f(\gamma(\pi/4)) - f(\gamma(0)) = f\left(\frac{1}{4}, \frac{1}{4}, 1\right) - f(1,0,0) = \frac{1}{16}.$$

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Parts I and II of the FTC for line integrals show that line integrals of a continuous vector field $\boldsymbol{G}$ on an open subset of $\mathbb{R}^m$ are path-independent if and only if $\boldsymbol{G}$ is a gradient field.

In view of the results proved above, it is important to know whether a continuous vector field $\boldsymbol{F}$ is indeed a gradient field. First we prove a necessary condition when $\boldsymbol{F}$ is smooth.

**Proposition (Cross-Derivative Test)**

Let $D \subset \mathbb{R}^m$ be open, and let $\boldsymbol{F} := (F_1, \ldots, F_m) : D \to \mathbb{R}^m$ be a smooth vector field. Suppose $\boldsymbol{F}$ is a gradient field. Then $\dfrac{\partial F_i}{\partial x_j} = \dfrac{\partial F_j}{\partial x_i}$ for all $i, j = 1, \ldots, m$.

Proof: Let $\boldsymbol{F} = \nabla f$ on $D$, where $f : D \to \mathbb{R}$ is smooth. Since the vector field $\boldsymbol{F}$ is also smooth, the partial derivatives of $f$ of orders 1 and 2 are continuous on $D$. By the Mixed Partials Theorem,

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial F_j}{\partial x_i} \quad \text{for } i, j = 1, \ldots, m. \qquad \square$$

The above test says that 'cross partials' are equal.

## Special Cases of the Cross-Derivative Test

$m := 2$, $\boldsymbol{F} := (P, Q)$: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

$m := 3$, $\boldsymbol{F} := (P, Q, R)$: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}, \dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$,

that is, $\operatorname{curl} \boldsymbol{F} = \operatorname{curl}(\operatorname{grad} f) = \boldsymbol{0}$.

Examples:

(i) Let $D := \mathbb{R}^2$, and let $\boldsymbol{F} := (P, Q)$, where $P(x, y) := -y$ and $Q(x, y) := x$ for $(x, y) \in D$. Then $P_y = -1$ and $Q_x = 1$. By the cross-derivative test, $\boldsymbol{F}$ is not a gradient field on $D$.

(ii) Even if $\boldsymbol{F}$ satisfies the cross-derivative test on an open subset $D$ of $\mathbb{R}^m$, $\boldsymbol{F}$ may not be a gradient field. For instance, let $D := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$, and $\boldsymbol{F} := (P, Q)$, where $P := -y/(x^2 + y^2)$ and $Q := x/(x^2 + y^2)$, $(x, y) \in D$. Then $P_y = (y^2 - x^2)/(x^2 + y^2)^2 = Q_x$. But $\boldsymbol{F}$ is not a gradient field on $D$: If $\gamma$ is the standard unit circle in $D$, then

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{-\pi}^{\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)dt = 2\pi \neq 0.$$

Next, let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$, and let $\boldsymbol{F} := (P, Q, R)$, where $P := -y/(x^2 + y^2)$, $Q := x/(x^2 + y^2)$ and $R := 1$ for $(x, y, z) \in D$. As above, $P_y = Q_x$, and so $\text{curl}\,(\boldsymbol{F}) = (R_y - Q_z, P_z - R_x, Q_x - P_y) = \boldsymbol{0}$. But $\boldsymbol{F}$ is not a gradient field on $D$: If $\gamma(t) := (\cos t, \sin t, 0)$, $t \in [-\pi, \pi]$, then

$$\int_\gamma \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{-\pi}^{\pi} (-\sin t, \cos t, 1)\cdot(-\sin t, \cos t, 0)dt = 2\pi \neq 0.$$

Remark

If $D := \mathbb{R}^m$ and if $\boldsymbol{F}$ satisfies the cross-derivative test on $\mathbb{R}^m$, then it can be proved that $\boldsymbol{F}$ is indeed a gradient field on $D$. In fact, this also holds if $D$ is an open simply connected subset of $\mathbb{R}^m$ and $\boldsymbol{F}$ satisfies the cross-derivative test on $D$. Roughly speaking, $D$ is simply connected if it does not have any holes!

Let $D \subset \mathbb{R}^2$. Then $D$ is called **simply connected** if every simple closed path lying in $D$ encloses points of $D$ only. If $D$ is bounded, then it is simply connected if and only if the complement $\mathbb{R}^2 \setminus D$ of $D$ in $\mathbb{R}^2$ is path-connected.
For example, a disk or a rectangle in $\mathbb{R}^2$ is simply connected, but an annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is not.

Let $D \subset \mathbb{R}^3$. Then $D$ is called **simply connected** if every simple closed path lying in $D$ is the 'edge' of a bounded surface lying entirely in $D$. For example, a ball or a cuboid in $\mathbb{R}^3$ is simply connected, and so is the solid body $D := \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 4\}$, but the set $\{(x, y, z) \in \mathbb{R}^3 : (x, y) \neq (0, 0)\}$ is not.

Let a smooth vector field $\boldsymbol{F}$ satisfy the cross-derivative test on an open simply connected subset $D$ of $\mathbb{R}^2$ or $\mathbb{R}^3$. The theorems of Green and Stokes will show that $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{s} = 0$ for every simple closed path lying in $D$, and hence for every closed path lying in $D$. This implies that $\boldsymbol{F}$ is a gradient field on $D$.

As a special case, let $D := I \times J \times K$, where $I, J, K$ are open intervals in $\mathbb{R}$. Suppose a smooth vector field $\boldsymbol{F} := (P, Q, R)$ on $D$ satisfies the cross-derivative test, that is, $\operatorname{curl}(\boldsymbol{F}) = \boldsymbol{0}$ on $D$. Then $\boldsymbol{F}$ is a gradient field. How to find a scalar field $f$ on $D$ such that $\boldsymbol{F} = \nabla f$, that is, $(P, Q, R) = (f_x, f_y, f_z)$? This can be accomplished by repeated integration as follows. Fix $x_0 \in I$.

Solve $f_x = P$ by integrating with respect to $x$: Let

$$f(x, y, z) := \int_{x_0}^{x} P(u, y, z)\,du + g(y, z) \quad \text{for } (x, y, z) \in D,$$

where $g$ is an arbitrary function of $(y, z) \in J \times K$. Use this expression for $f$ in the equation $f_y = Q$ to obtain an equation for $g_y$. Solve it by integrating with respect to $y$. Its solution will involve an arbitrary function $h$ of $z \in K$. Use this expression for $g$ in the equation $f_z = R$ to obtain an equation for $h'$. Solve it by integrating with respect to $z$. Its solution will involve an arbitrary constant. This yields $f$ with $\nabla f = \boldsymbol{F}$.

Further, if $\boldsymbol{F} = \nabla \tilde{f}$ for a scalar field $\tilde{f}$ on $D$ as well, then $\nabla(\tilde{f} - f) = 0$ on $D$. By the Trivariate Mean Value Theorem, $\tilde{f} - f = c$, that is, $\tilde{f} = f + c$, where $c$ is a constant.

Example:
Let $\boldsymbol{F} := (P, Q, R)$ where $P := x^2 + y\,z$, $Q := y^2 + z\,x$, $R := z^2 + x\,y$ for $(x, y, z) \in \mathbb{R}^3$. It is easy to check that $\text{curl}\,(\boldsymbol{F}) = \boldsymbol{0}$. Let $f$ be a scalar field such that $\nabla f = \boldsymbol{F}$. Then

$f_x = P = x^2 + y\,z \implies f(x, y, z) = x^3/3 + x\,y\,z + g(y, z)$.
$f_y = Q = y^2 + z\,x \implies x\,z + g_y = y^2 + z\,x$, and so $g_y = y^2$.
Thus $g(y, z) = y^3/3 + h(z)$, and as a consequence,

$$f(x, y, z) = \frac{x^3 + y^3}{3} + x\,y\,z + h(z).$$

$f_z = R = z^2 + x\,y \implies x\,y + h' = z^2 + x\,y$, and so $h' = z^2$.
Thus $h(z) = z^3/3 + c$, and as a consequence,

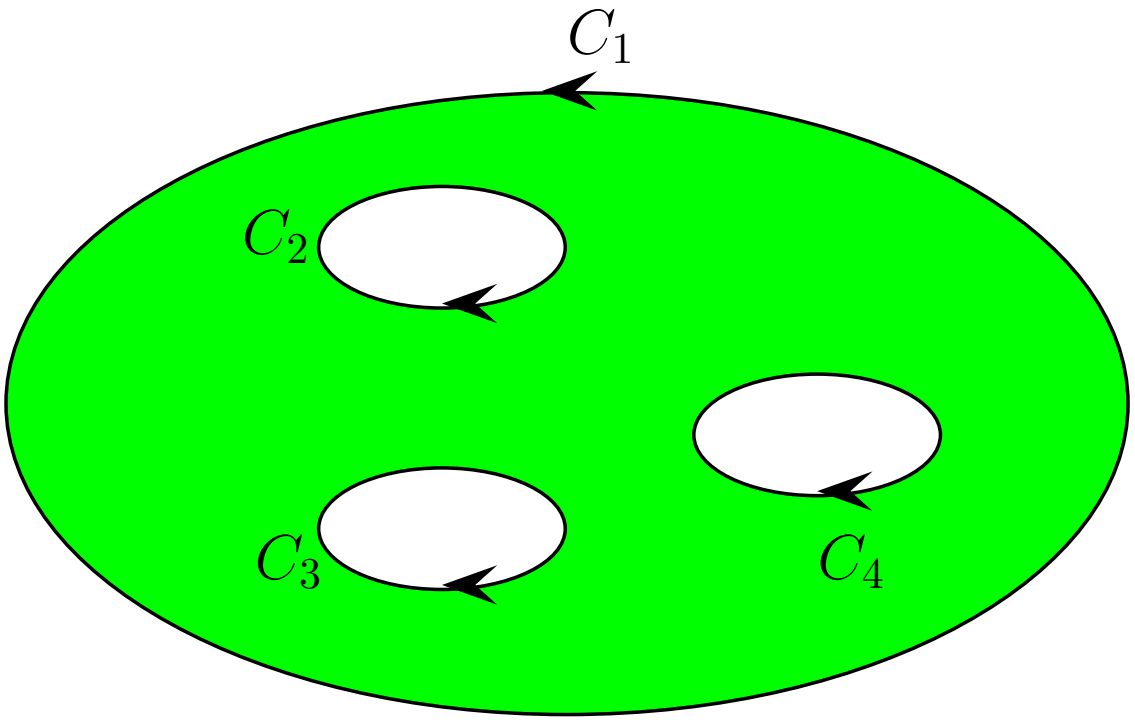
$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} + x\,y\,z + c, \quad \text{where } c \text{ is a constant.}$$

Let $\gamma$ be a simple closed piecewise smooth path in $\mathbb{R}^2$, and let $C := \gamma([\alpha, \beta])$ denote the geometric curve in $\mathbb{R}^2$ defined by $\gamma$. Suppose $\boldsymbol{F}$ is a continuous vector field defined on $C$.

Since the line integral of a continuous vector field along a smooth path is invariant under a reparametrization only up to its sign, $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$ is determined only up to its sign. To determine it completely, we specify the sense or the direction in which the geometric curve $C$ is traversed. For example, we may specify that as we travel along the curve $C$ with $\boldsymbol{k}$ as our upright direction, the region in $\mathbb{R}^2$ enclosed by $C$ lies on our left. In this case, the curve $C$ is traversed **anticlockwise** (as viewed from high above the curve).

Suppose $D$ is a bounded region in $\mathbb{R}^2$ whose boundary $\partial D$ consists of a finite number of nonintersecting simple closed piecewise smooth geometric curves $C_1, \ldots, C_k$.

Suppose the curve $C_1$ is the outer boundary of $D$.




If the outer boundary curve $C_1$ of $D$ is traversed anticlockwise, but the inner boundary curves $C_2, \ldots, C_k$ are traversed clockwise, then we say that the boundary $\partial D$ of the region $D$ is **positively oriented**. In this case, the region $D$ is on our left as we travel along any part of its boundary (with $\boldsymbol{k}$ as our upright direction).

# The Green Theorem

We have defined line integrals in terms of Riemann integrals. An important result of Green relates line integrals along the boundary of a domain to double integrals over the domain. It can be viewed as a two-dimensional analogue of Part II of the Fundamental Theorem of Calculus for Riemann integrals and for line integrals.

## Theorem

Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ such that the boundary $\partial D$ of $D$ consists of a finite number of nonintersecting simple closed piecewise smooth curves, and suppose it is positively oriented. Let $\boldsymbol{F} := (P, Q)$ be a smooth vector field on an open subset $\Omega$ of $\mathbb{R}^2$ containing $D$. Then

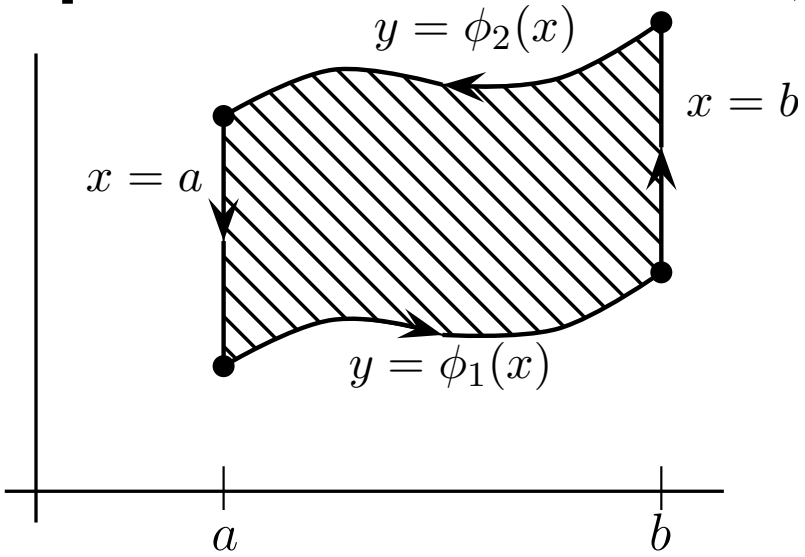
$$\int_{\partial D} P\, dx + Q\, dy = \iint_D (Q_x - P_y) d(x, y).$$

Proof. The LHS is well-defined since $P$ and $Q$ are continuous on $\partial D$, while the RHS is well-defined since $Q_x$ and $P_y$ are continuous on $D$ and since $\partial D$ is of two-dimensional content zero (as shown in Lecture 25).

First suppose $D$ is an elementary region of type I, and

$$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\},$$

where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous and $\phi_1 \leq \phi_2$.




Clearly,

$$\int_{\partial D} P\, dx = \int_a^b P(x, \phi_1(x))dx + 0 + \int_b^a P(x, \phi_2(x))dx + 0$$

which is equal to $\int_a^b \big(P(x,\phi_1(x)) - P(x,\phi_2(x))\big)\,dx$.

On the other hand, the Fubini theorem and the FTC show that

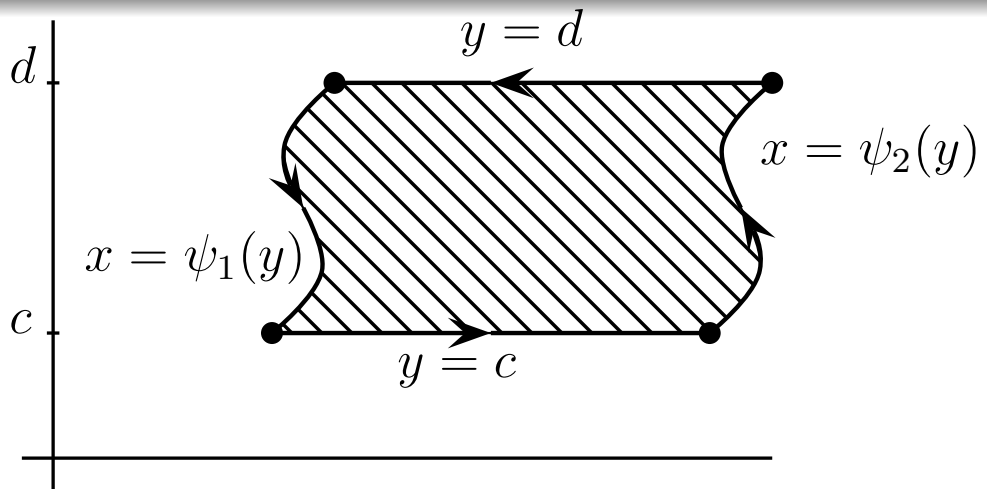
$$\begin{aligned}\iint_D P_y\, d(x,y) &= \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} P_y\, dy \right) dx \\ &= \int_a^b \big(P(x,\phi_2(x)) - P(x,\phi_1(x))\big)\,dx.\end{aligned}$$

Thus $\displaystyle\int_{\partial D} P\, dx = -\iint_D P_y\, d(x,y)$.

Next, suppose $D$ is an elementary region of type II, and

$$D := \{(x,y) \in \mathbb{R}^2 : c \le y \le d \text{ and } \psi_1(y) \le x \le \psi_2(y)\},$$

where $\psi_1, \psi_2 : [c,d] \to \mathbb{R}$ are continuous and $\psi_1 \le \psi_2$.




Clearly,

$$\int_{\partial D} Q\,dy \;=\; \int_c^d Q(\psi_2(y), y)dy + 0 + \int_d^c Q(\psi_1(y), y)dy + 0$$

which is equal to $\int_c^d \big(Q(\psi_2(y), y) - Q(\psi_1(y), y)\big)\,dy$.

On the other hand, the Fubini theorem and the FTC show that

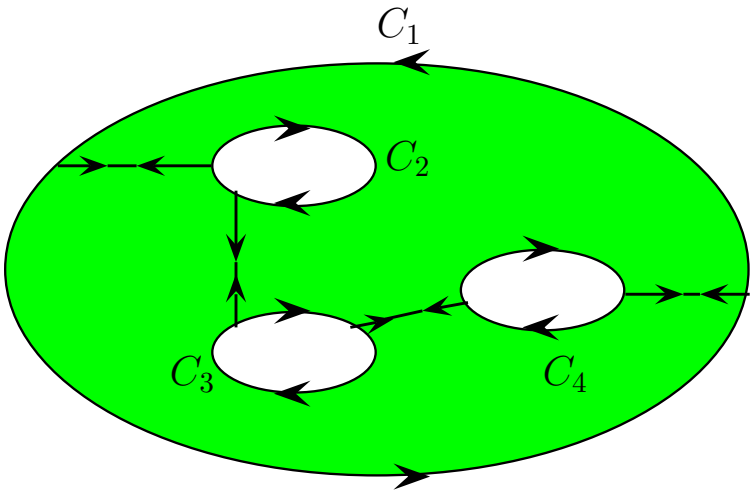
$$\begin{aligned}\iint_D Q_x\,d(x,y) &= \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} Q_x\,dx\right)dy \\ &= \int_c^d \big(Q(\psi_2(y), y) - Q(\psi_1(y), y)\big)\,dy.\end{aligned}$$

Thus $\int_{\partial D} Q\,dy = \iint_D Q_x\,d(x,y)$.

If $D$ is an elementary region of type I as well as of type II, we may combine the two equalities proved above, and obtain

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)d(x,y).$$

This proof applies when $D$ is a rectangle or a disk. It can be modified to treat the case where $D$ is simply connected, so that $\partial D$ has only the outer component. Further, the general case of a **multiply connected** domain $D$ can be treated by introducing cuts as shown below.



# Computational Uses of the Green Theorem

## (i) Calculation of a double integral

Example: Finding the area of a subset of $\mathbb{R}^2$

Let $D$ be a closed and bounded subset of $\mathbb{R}^2$ whose positively oriented boundary $\partial D$ consists of a finite number of nonintersecting simple piecewise smooth closed curves. Then

$$\text{Area}\,(D) := \iint_D 1_D d(x,y).$$

Let $P$, $Q$ be smooth functions on an open subset containing $D$ such that $Q_x - P_y = 1$ on $D$. By the Green theorem,

$$\text{Area}\,(D) = \iint_D (Q_x - P_y) d(x,y) = \int_{\partial D} P\ dx + Q\ dy.$$

For instance, for $(x,y) \in D$, we may let

$$P(x,y) := -\frac{y}{2} \quad \text{and} \quad Q(x,y) := \frac{x}{2}.$$

Then clearly $Q_x - P_y = (1/2) + (1/2) = 1$ on $D$, and so

$$\text{Area}\,(D) = \frac{1}{2}\int_{\partial D} x\ dy - y\ dx.$$

We note that Area $(D)$ is also equal to

$$\int_{\partial D} x\,dy \quad \text{or} \quad -\int_{\partial D} y\,dx$$

if we let $Q(x,y) := x$, $P := 0$, or $P(x,y) := -y$, $Q := 0$.

Suppose the positively oriented boundary $\partial D$ of $D$ is parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Then

$$\text{Area}\,(D) = \frac{1}{2}\int_{\alpha}^{\beta} \big(x(t)y'(t) - y(t)x'(t)\big)\,dt = \frac{1}{2}\int_{\alpha}^{\beta} W(x,y)(t)dt,$$

where $W(x,y) := \det\begin{bmatrix} x & y \\ x' & y' \end{bmatrix}$ is called the **Wronskian** of $x, y$.

As a special case, suppose $\partial D$ is given by a polar equation

$$r = p(\theta) \quad \text{for } \theta \in [\alpha, \beta].$$

Let $x(\theta) := p(\theta)\cos\theta$ and $y(\theta) := p(\theta)\sin\theta$ for $\theta \in [\alpha, \beta]$. Then

$$W(x,y)(\theta) = \det \begin{bmatrix} p(\theta)\cos\theta & p(\theta)\sin\theta \\ p'(\theta)\cos\theta - p(\theta)\sin\theta & p'(\theta)\sin\theta + p(\theta)\cos\theta \end{bmatrix},$$

which is equal to $p^2(\theta)$, and so

$$\text{Area}\,(D) = \frac{1}{2}\int_\alpha^\beta p^2(\theta)d\theta,$$

as we had defined earlier. For instance, the area enclosed by the cardioid given by $r = a(1-\cos\theta)$, $\theta \in [-\pi, \pi]$, is equal to

$$\frac{1}{2}\int_{-\pi}^{\pi} a^2(1 - 2\cos\theta + \cos^2\theta)d\theta = \frac{3\pi a^2}{2}.$$

## (ii) Calculation of the line integral along an oriented boundary

Example:

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1 \text{ and } y \geq 0\}$, so that its positively oriented boundary $\partial D$ consists of the line segment from $(-1, 0)$ to $(1, 0)$ followed by the semicircle in the upper half-plane having radius 1 and centre $(0, 0)$. Let $P(x, y) := y^2$ and $Q(x, y) := 3xy$ for $(x, y) \in \mathbb{R}^2$.

To calculate $\int_{\partial D} P\,dx + Q\,dy$, we may use the Green theorem and obtain

$$\int_{\partial D} y^2 dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3x\,y) - \frac{\partial}{\partial y}(y^2)\right] d(x, y)$$

$$= \int_{-1}^{1} \left( \int_0^{\sqrt{1-x^2}} y\ dy \right) dx = \int_{-1}^{1} \frac{1 - x^2}{2} dx = \frac{2}{3}.$$

# Consequences of the Green Theorem

## Proposition

Let $\boldsymbol{F} := (P, Q)$ be a smooth vector field on an open subset containing a closed and bounded subset $D$ of $\mathbb{R}^2$ such that $Q_x = P_y$ on $D$. If $\partial D$ consists of a finite number of nonintersecting simple closed piecewise smooth curves, and if $\partial D$ is positively oriented, then

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial D} P\ dx + Q\ dy = 0.$$

Proof

By the Green theorem,

$$\int_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_D (Q_x - P_y) d(x, y) = \iint_D 0\, d(x, y) = 0. \quad \square$$