MA 105 : Calculus
Division 1, Lecture 25

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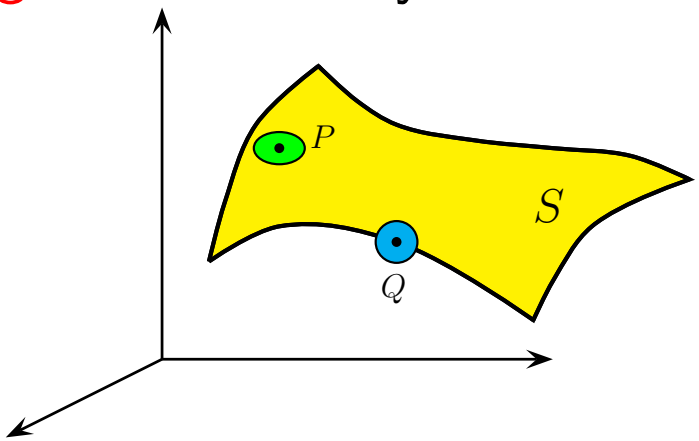
Recap of the previous lecture

- Surface integral of a scalar field. Examples
- Reparametrization of a surface
- Invariance of surface integrals of scalar fields under reparametrixations
- Notion of a geometric surface
- Surface integral of a vector field.
- Differential notaion (involving wedge products) and physical interpretation. Examples
- Invariance of surface integrals of vector fields up to a sign under reparametrizations
- Opposite of a parametrized surface
- Oriented surfaces
- Orientation-preserving/reversing parametrizations
- Examples, including Mobius strip and its paramerization
- Intrinsic boundary of a surface

Intrinsic Boundary of a Surface

Let S be a (geometric) surface in $\mathbb{R}^3$, and suppose S is a closed subset of $\mathbb{R}^3$. Typically, no point of S is an interior point of S as a subset of $\mathbb{R}^3$, and so $\partial S = S$.

Let P be a point on S. It is called an **intrinsic interior point** of S if there is a two dimensional disk-like neighbourhood of P which is contained in S, that is, there is a curvilinear patch on S around P which can be flattened into a disk. Otherwise, the point P is called an **intrinsic boundary point** of S. We shall denote the set of all intrinsic boundary points of S by $\eth S$. It consists of the 'edges' of S, if any.



In the figure on the previous page, P is an intrinsic interior point of S, whereas Q is an intrinsic boundary point of S.

If $\eth S = \emptyset$, then S is called a **surface without edges**. It is also known as a **closed surface**. (Compare the definition of a closed geometric curve.)

If S is a planar surface, that is, $S := D$, a subset of $\mathbb{R}^2$, then $\eth S$ of S coincides with the usual boundary ∂D of D.

Examples

(i) If $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$, then $\eth S = \emptyset$.

(ii) If S is the union of the three sets $[a, b]\times[c, d]\times\{p, q\}, [a, b]\times\{c, d\}\times[p, q], \{a, b\}\times[c, d]\times[p, q]$, then $\eth S = \emptyset$.

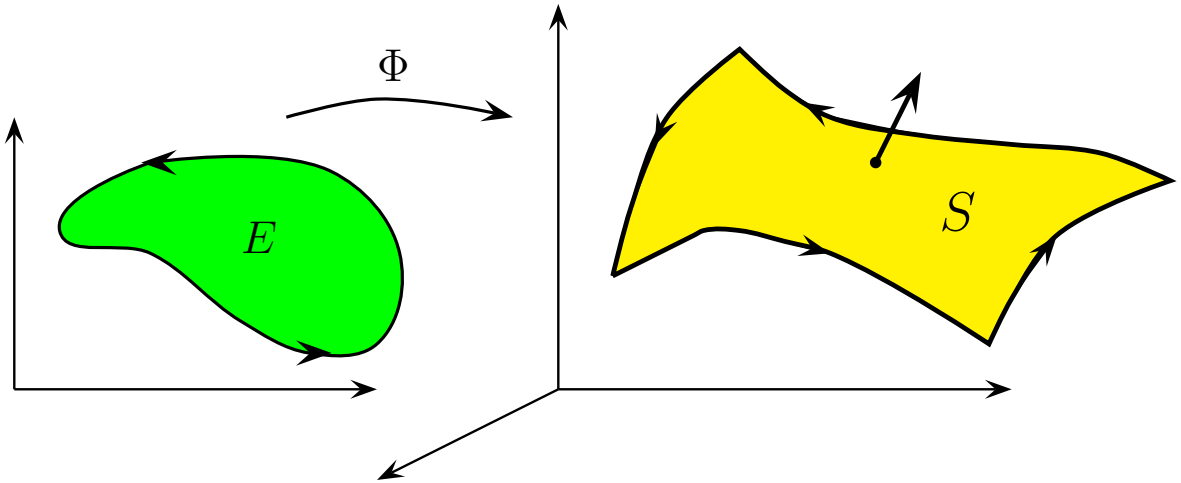
(iii) If $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2,\ 0 \le z \le h\}$, then $\eth S$ is the union of the two sets $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2$ and $z = 0\}$ and $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = a^2$ and $z = h\}$.

Orientation of the Intrinsic Boundary

An orientation of a surface induces an orientation on its intrinsic boundary as follows.

Suppose S is an oriented surface, and let $\boldsymbol{n}(P)$ be the specified unit normal vector at a point P on S. As we travel along any part of the intrinsic boundary $\eth S$ of the surface S with $\boldsymbol{n}(P)$ as our upright direction, suppose the surface S lies on our left. This gives the **induced orientation** on $\eth S$.

Note: If E is a subset of $\mathbb{R}^2$, and $\Phi : E \to \mathbb{R}^3$ is a smooth orientation-preserving parametrization of the surface S, then $\Phi(\partial E) = \eth S$, and the induced orientation on the intrinsic boundary $\eth S$ of S corresponds to the positive orientation of the boundary ∂E of the parameter set $E \subset \mathbb{R}^2$ defined earlier.



Example

Suppose the unit upper hemisphere $S := \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$ is oriented by the unit normal vector $\boldsymbol{n}(P) := (x, y, z)$ for $P := (x, y, z) \in S$.

Let $E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $\Phi(x, y) := \left(x, y, \sqrt{1 - x^2 - y^2}\right)$ for $(x, y) \in E$. Then Φ is an orientation-preserving parametrization of S, and the induced orientation on $\partial S = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \text{ and } z = 0\}$ corresponds to the counterclockwise orientation of $\partial E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

The Stokes Theorem

An important result of Stokes relates line integrals along the edges of a surface to flux integrals across the surface. It is a 'curved version' of the Green theorem, and can be viewed as an analogue of Part II of the Fundamental Theorem of Calculus. The proof of the Stokes theorem heavily depends on the proof of the Green theorem.

Theorem

Let S be an oriented (geometric) surface in $\mathbb{R}^3$ which is a closed and bounded subset of $\mathbb{R}^3$, which is piecewise C^2 and whose piecewise smooth intrinsic boundary ∂S consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field on an open subset containing S. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

Sketch of Proof:

Let the (geometric) surface S correspond to $\Phi : E \to \mathbb{R}^3$, where E is a closed and bounded subset of $\mathbb{R}^2$. Suppose $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in E$, and $\boldsymbol{F} := (P, Q, R)$ on an open subset of $\mathbb{R}^3$ containing S. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial S} P\, dx + Q\, dy + R\, dz.$$

Let ∂E be given by a path $\gamma := (u, v)$. Then ∂S is given by the path $\Phi \circ \gamma$. Since, by the chain rule, $(x \circ \gamma)' = x_u u' + x_v v'$,

$$\int_{\partial S} P\, dx = \int_{\partial E} (P \circ \Phi)\, (x_u du + x_v dv).$$

Let us write $g := P \circ \Phi$ on E. Then

$$\int_{\partial S} P\, dx = \int_{\partial E} (g\, x_u) du + (g\, x_v) dv.$$

The Green theorem shows that

$$\begin{aligned}\int_{\partial E} (g\,x_u)du + (g\,x_v)dv &= \iint_E \big((g\,x_v)_u - (g\,x_u)_v\big)\,d(u,v)\\ &= \iint_E (g_u\,x_v - g_v\,x_u)d(u,v)\end{aligned}$$

since $g\,x_{vu} - g\,x_{uv} = 0$ by the Mixed Partials Theorem.
(Note: $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$ is piecewise C^2.)

But since $g(u,v) = P(x(u,v), y(u,v), z(u,v))$, we obtain

$$g_u = P_x\,x_u + P_y\,y_u + P_z\,z_u, \quad g_v = P_x\,x_v + P_y\,y_v + P_z\,z_v,$$

by the chain rule. Hence

$$\begin{aligned}g_u\,x_v - g_v\,x_u &= -P_y(x_u\,y_v - x_v\,y_u) + P_z(z_u\,x_v - z_v\,x_u)\\ &= -P_y\,\frac{\partial(x,y)}{\partial(u,v)} + P_z\,\frac{\partial(z,x)}{\partial(u,v)}.\end{aligned}$$

It follows that

$$\int_{\partial S} P\,dx = \iint_E \left(-P_y \frac{\partial(x,y)}{\partial(u,v)} + P_z \frac{\partial(z,x)}{\partial(u,v)}\right) d(u,v).$$

In a similar manner, we obtain

$$\int_{\partial S} Q\,dy = \iint_E \left(-Q_z \frac{\partial(y,z)}{\partial(u,v)} + Q_x \frac{\partial(x,y)}{\partial(u,v)}\right) d(u,v).$$

and

$$\int_{\partial S} R\,dz = \iint_E \left(-R_x \frac{\partial(z,x)}{\partial(u,v)} + R_y \frac{\partial(y,z)}{\partial(u,v)}\right) d(u,v).$$

Hence $\displaystyle\int_{\partial S} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{\partial S} P\,dx + Q\,dy + R\,dz$ is equal to

$$\iint_E \left((R_y - Q_z)\frac{\partial(y,z)}{\partial(u,v)} + (P_z - R_x)\frac{\partial(z,x)}{\partial(u,v)} + (Q_x - P_y)\frac{\partial(x,y)}{\partial(u,v)}\right) d(u,v).$$

On the other hand,

$$\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_E \operatorname{curl} \boldsymbol{F}(\Phi(u,v)) \cdot (\Phi_u \times \Phi_v)\, d(u,v),$$

where

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y),$$

and

$$\Phi_u \times \Phi_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

Thus

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}, \quad \text{as desired.} \quad \square$$

The conclusion of the Stokes theorem is also written as follows:

$$\int_{\partial S} P\,dx + Q\,dy + R\,dz$$
$$= \iint_S (R_y - Q_z)\,dy \wedge dz + (P_z - R_x)\,dz \wedge dx + (Q_x - P_y)\,dx \wedge dy.$$

Here is a version of the Stokes theorem when S does not have intrinsic boundary.

Corollary

Let S be a closed oriented smooth surface in $\mathbb{R}^3$ (and so $\partial S = \emptyset$). Suppose $\boldsymbol{F}$ is a smooth vector field on an open subset containing S. Then $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 0$.

Proof: Introduce a hole in S by cutting out a small piece along a smooth simple closed curve C on S. Let S_1 denote the part of S cut out, and let S_2 denote the remaining part of S.

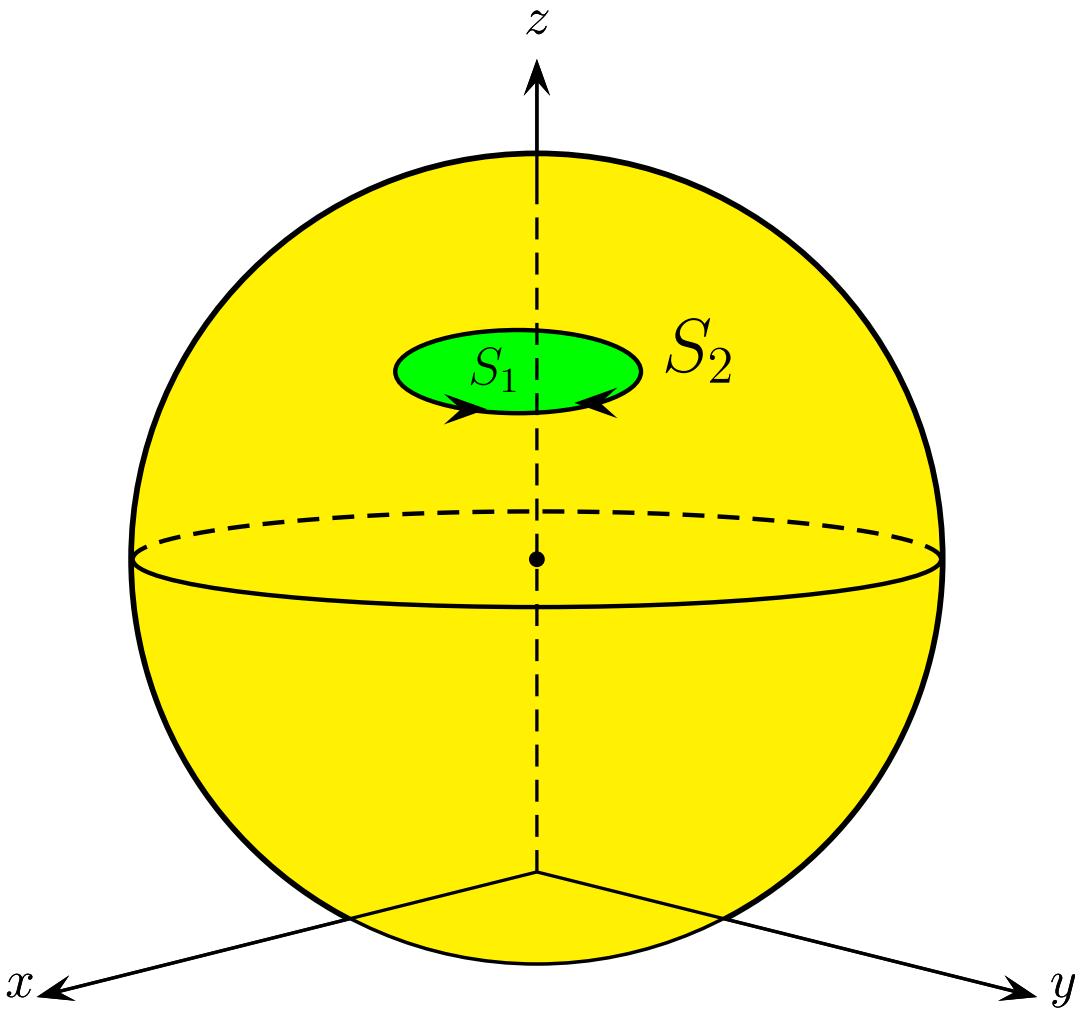


Figure: The Stokes theorem for S with $\partial S = \emptyset$.

Then $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_{S_1} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} + \iint_{S_2} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$

by the domain additivity. Now the Stokes theorem shows that

$$\iint_{S_1} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} \text{ and } \iint_{S_2} (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

We observe that the intrinsic boundary ∂S_1 of S_1 is the closed curve C with the orientation induced by the orientation on S_1. Since $\partial S = \emptyset$, the intrinsic boundary ∂S_2 of S_2 is also C. But the orientations induced on C by the orientations on S_1 and on S_2 are opposite. Hence

$$\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} - \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

Green's Theorem: Let D be a closed and bounded subset of $\mathbb{R}^2$ such that the boundary ∂D of D consists of a finite number of nonintersecting simple closed piecewise smooth curves, and suppose it is positively oriented. Let $\boldsymbol{F} := (P, Q)$ be a smooth vector field on an open subset Ω of $\mathbb{R}^2$ containing D. Then

$$\int_{\partial D} P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d(x, y).$$

Stokes' Theorem: Let S be an oriented (geometric) surface in $\mathbb{R}^3$ which is a closed and bounded subset of $\mathbb{R}^3$, which is piecewise C^2 and whose piecewise smooth intrinsic boundary $\eth S$ consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field on an open subset containing S. Then

$$\int_{\eth S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

Surface-Independence

Let $E \subset \mathbb{R}^2$, and consider a parametrized surface $\Phi : E \to \mathbb{R}^3$. Let $D \subset \mathbb{R}^3$. We say that Φ lies in D if $\Phi(E) \subset D$.

Let $\boldsymbol{F} : D \to \mathbb{R}^3$ be a continuous vector field. We say that flux integrals of $\boldsymbol{F}$ are **surface-independent** in D if

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\widetilde{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S}$$

whenever the piecewise C^2 parametrized surfaces $\Phi : E \to \mathbb{R}^3$ and $\widetilde{\Phi} : \widetilde{E} \to \mathbb{R}^3$ lie in D, and if $S := \Phi(E)$, $\widetilde{S} := \widetilde{\Phi}(\widetilde{E})$, then S and $\widetilde{S}$ are oriented (geometric) surfaces and have the same intrinsic boundary which consists of a finite number of nonintersecting simple closed curves and on which the induced orientation is the same. Note that the geometric surfaces S and $\widetilde{S}$ need not be the same.

(Compare the definition of path-independence of a vector field given in Lecture 31.)

Examples: Let $E := [0, \pi/2] \times [-\pi, \pi]$, and let $\Phi(\varphi, \theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi, \theta) \in E$. Also, let $\widetilde{E} := \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$, and let $\widetilde{\Phi}(u, v) := (u, v, 0)$ for $(u, v) \in \widetilde{E}$. Then the oriented intrinsic boundaries of $S := \Phi(E)$ and $\widetilde{S} := \widetilde{\Phi}(E)$ are the same, namely $\{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$, traced counterclockwise as seen from $(0, 0, 2)$.

(i) Let $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$. Then $\iint_\Phi \boldsymbol{F} \cdot d\boldsymbol{S}$ is equal to

$$\iint_E (\sin\varphi\sin\theta, \cos\varphi, \sin\varphi\cos\theta) \cdot (\sin\varphi)\Phi(\varphi, \theta)d(\varphi, \theta)$$

which is equal to 0 since

$$\int_{-\pi}^{\pi} \sin\theta\cos\theta\, d\theta = \int_{-\pi}^{\pi} \sin\theta\, d\theta = \int_{-\pi}^{\pi} \cos\theta\, d\theta = 0.$$

Also, since $\widetilde{\Phi}_u \times \widetilde{\Phi}_v = (1, 0, 0) \times (0, 1, 0) = (0, 0, 1)$, we obtain

$$\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\widetilde{E}} (v,0,u)\cdot(0,0,1)d(u,v) = \iint_{\widetilde{E}} u\,d(u,v)$$
$$= \int_{-\pi}^{\pi}\int_{0}^{1} (r\cos\theta) r\,dr\,d\theta = 0.$$

In fact, we shall presently see that flux integrals of $\boldsymbol{F}$ are surface-independent in $\mathbb{R}^3$.

(ii) Let $\boldsymbol{F}(x,y,z) := (x,y,z)$ for $(x,y,z) \in \mathbb{R}^3$. Then

$$\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{E} \Phi(\varphi,\theta)\cdot(\sin\varphi)\Phi(\varphi,\theta)d(\varphi,\theta) = 2\pi.$$

On the other hand,

$$\iint_{\widetilde{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\widetilde{E}} (u,v,0)\cdot(0,0,1)d(u,v) = 0.$$

Thus flux integrals of $\boldsymbol{F}$ are not surface-independent in $\mathbb{R}^3$.

Theorem

Let $D \subset \mathbb{R}^3$ be open, and let $\boldsymbol{F}$ be a continuous curl field on D. Then flux integrals of $\boldsymbol{F}$ are surface-independent in D.

Proof: Let $\boldsymbol{G}$ be a smooth vector field on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on D. Let $\Phi : E \to \mathbb{R}^3$ be a piecewise C^2 parametrized surface in D, and let the intrinsic boundary $\eth S$ of the oriented surface $S := \Phi(E)$ consist of a finite number of nonintersecting simple closed curves.

If $\eth S$ is given the induced orientation, then by the Stokes theorem,

$$\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{S} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\eth S} \boldsymbol{G} \cdot d\boldsymbol{s},$$

which clearly depends only on $\boldsymbol{G}$, and on $\eth S$ and its orientation, but not on Φ. Hence the result follows. □

If $\boldsymbol{F}$ is a curl field, then the evaluation of a surface integral $\iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S}$ can be simplified by replacing Φ by a parametrized surface having the same oriented intrinsic boundary. If we actually know a vector field $\boldsymbol{G}$ such that $\operatorname{curl} \boldsymbol{G} = \boldsymbol{F}$, then we can reduce our problem to the calculation of a line integral.

Example: Let $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$.
Consider $\boldsymbol{G}(x, y, z) := (z^2/2, x^2/2, y^2/2)$ for $(x, y, z) \in \mathbb{R}^3$.
Then $\operatorname{curl} \boldsymbol{G} = \boldsymbol{F}$ on $\mathbb{R}^3$, and so flux integrals of $\boldsymbol{F}$ are surface-independent in $\mathbb{R}^3$.
Let $E := [0, \pi/2] \times [-\pi, \pi]$, and for $(\varphi, \theta) \in E$, let $\Phi(\varphi, \theta) := (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi)$. Let $S := \Phi(E)$. Then

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{S} \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{G} \cdot d\boldsymbol{s} \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (0, \cos^2 t, \sin^2 t) \cdot (-\sin t, \cos t, 0)\, dt = 0 \end{aligned}$$

since $\cos^3 t = (\cos 3t + 3\cos t)/4$ for $t \in [-\pi, \pi]$.

It is important to know whether a continuous vector field $\boldsymbol{F}$ is in fact a curl field, that is, whether there is a smooth vector field $\boldsymbol{G}$ such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

We already know a necessary condition when $\boldsymbol{F}$ is smooth:

If $\boldsymbol{F}$ is smooth and $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$ on an open subset D of $\mathbb{R}^3$, then $\operatorname{div}(\boldsymbol{F}) = 0$ on D.

Examples:

(i) Let $D := \mathbb{R}^3$, and $\boldsymbol{F} := (P, Q, R)$, where $P(x, y, z) := x$, $Q(x, y, z) := y$ and $R(x, y, z) := z$ for $(x, y, z) \in D$. Since $\operatorname{div} \boldsymbol{F} = 3 \neq 0$, the vector field $\boldsymbol{F}$ is not a curl field on D.

(ii) Even if $\operatorname{div} \boldsymbol{F} = 0$ on an open subset D of $\mathbb{R}^3$, the vector field $\boldsymbol{F}$ may not be a curl field on D. For instance, let $D := \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, and let $\boldsymbol{F} := (P, Q, R)$, where for $(x, y, z) \in D$,

$$P := \frac{x}{(x^2+y^2+z^2)^{3/2}}, \; Q := \frac{y}{(x^2+y^2+z^2)^{3/2}}, \; R := \frac{z}{(x^2+y^2+z^2)^{3/2}}.$$

Then $(x^2+y^2+z^2)^3 \operatorname{div} \boldsymbol{F} = (x^2+y^2+z^2)^3(P_x+Q_y+R_z) = 3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2) = 0.$

Thus $\operatorname{div} \boldsymbol{F} = 0$ on D. But $\boldsymbol{F}$ is not a curl field on D since the standard unit sphere $S := \Phi([0,\pi]\times[-\pi,\pi])$ lies in D, and

$$\begin{aligned} \iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_{E} \Phi(\varphi,\theta)\cdot(\sin\varphi)\Phi(\varphi,\theta)d(\varphi,\theta) \\ &= \int_{-\pi}^{\pi}\left(\int_{0}^{\pi}\sin\varphi\, d\varphi\right)d\theta = 4\pi \neq 0. \end{aligned}$$

But if $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$, then $\iint_{\Phi} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S} \operatorname{curl} \boldsymbol{G}\cdot d\boldsymbol{S} = 0$ since $\partial S = \emptyset$. (Recall the corollary of the Stokes theorem.)

If $D := I\times J\times K$, where I, J, K are open intervals in $\mathbb{R}$, and $\boldsymbol{F}$ is a smooth vector field on D such that $\operatorname{div} \boldsymbol{F} = 0$ on D, then $\boldsymbol{F}$ is indeed a curl field. In fact, this also holds if D is an open subset of $\mathbb{R}^3$ such that every closed surface in D is the boundary of a solid lying entirely in D.

Let $D := I \times J \times K$, where I, J, K are open intervals in $\mathbb{R}$. Suppose $\boldsymbol{F} := (P, Q, R)$ is a smooth vector field on D such that $\operatorname{div} \boldsymbol{F} = 0$ on D. We find a vector field $\boldsymbol{G} := (L, M, N)$ on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$, that is,

$$(P, Q, R) = (N_y - M_z, L_z - N_x, M_x - L_y).$$

Let $x_0 \in I$, $z_0 \in K$.

Define $L := 0$ and $N(x, y, z) := -\int_{x_0}^{x} Q(u, y, z)du$.

Then $L_z - N_x = 0 + Q = Q$. Next, define

$M(x, y, z) := \int_{x_0}^{x} R(u, y, z)du - \int_{z_0}^{z} P(x_0, y, v)dv$. Then

$M_x - L_y = R - 0 = R$.

Finally, by differentiating under the integral sign,

$$\begin{aligned} N_y(x, y, z) &= -\int_{x_0}^{x} Q_y(u, y, z)du = \int_{x_0}^{x} (P_x + R_z)(u, y, z)du \\ &= P(x, y, z) - P(x_0, y, z) + \int_{x_0}^{x} R_z(u, y, z)du, \end{aligned}$$

and

$$M_z(x, y, z) = \int_{x_0}^{x} R_z(u, y, z)du - P(x_0, y, z).$$

Hence $N_y - M_z = P$. Thus $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$.

Further, if there is a vector field $\boldsymbol{H}$ on D such that $\boldsymbol{F} = \operatorname{curl} \boldsymbol{H}$ as well, then $\operatorname{curl}(\boldsymbol{H} - \boldsymbol{G}) = 0$ on D, and so $\boldsymbol{H} = \boldsymbol{G} + \nabla f$, where f is a smooth scalar field on D, as we have mentioned while considering 'path-independence' of vector fields in Lecture 32. This will follow as a consequence of the Stokes theorem since D is simply connected.

Example:

Let $\boldsymbol{F} := (P, Q, R)$, where $P := x$, $Q = -2\,y$ and $R := z$ for $(x, y, z) \in \mathbb{R}^3$. Clearly, $\operatorname{div} \boldsymbol{F} = 0$ on $\mathbb{R}^3$.
Let $x_0 := 0 =: z_0$. Define $L := 0$, $N := 2\,x\,y$ and $M := z\,x$ as above. If we let $\boldsymbol{G} := (0, z\,x, 2\,x\,y)$, then $\boldsymbol{F} = \operatorname{curl} \boldsymbol{G}$. Also, if $\operatorname{curl} \boldsymbol{H} = \boldsymbol{F}$, then $\boldsymbol{H} = \boldsymbol{G} + (f_x, f_y, f_z)$, where f is a smooth scalar field on $\mathbb{R}^3$.