MA 105 : Calculus
Division 1, Lecture 26

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Recap of the previous lecture

- Orientation of the intrinsic boundary
- The Stokes Theorem
- Sketch of proof of Stokes Theorem
- Vanishing of surface integrals of curl fields as a Corollary
- Surface-independence. Examples
- Surface-independence of flux integrals of continuous curl fields
- Necessary condition for a smooth vector field to be a curl field
- Example to show that this necessary condition is not sufficient.
- Sufficiency in case of domains of the type I x J x K, where I, J, K are intervals

Let us recall the statement of Stokes' Theorem.

Let S be an oriented (geometric) surface in $\mathbb{R}^3$ which is a closed and bounded subset of $\mathbb{R}^3$, which is piecewise C^2 and whose piecewise smooth intrinsic boundary ∂S consists of a finite number of nonintersecting simple closed curves along with the induced orientation. Let $\boldsymbol{F}$ be a smooth vector field on an open subset containing S. Then

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}.$$

Computational Uses of the Stokes Theorem

(i) Calculation of the flux integral of a curl field

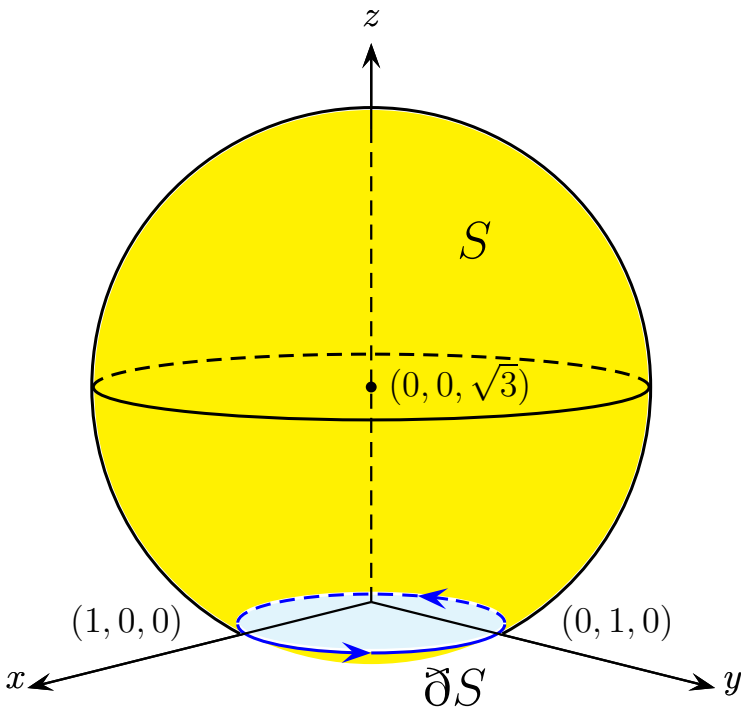
Example:

Let $\boldsymbol{F}(x,y,z) := (y, -x, e^{x\,z})$ for $(x,y,z) \in \mathbb{R}^3$, and let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-\sqrt{3})^2 = 4 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors.

In order to find $\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S}$, we note that the induced orientation on $\partial S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$ is anticlockwise as seen from the point $(0,0,4)$.

By the Stokes theorem,

$$\begin{aligned}
\iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \\
&= \int_{-\pi}^{\pi} (\sin t, -\cos t, e^{\cos t \cdot 0}) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_{-\pi}^{\pi} -(\sin^2 t + \cos^2 t)\, dt = -2\pi.
\end{aligned}$$



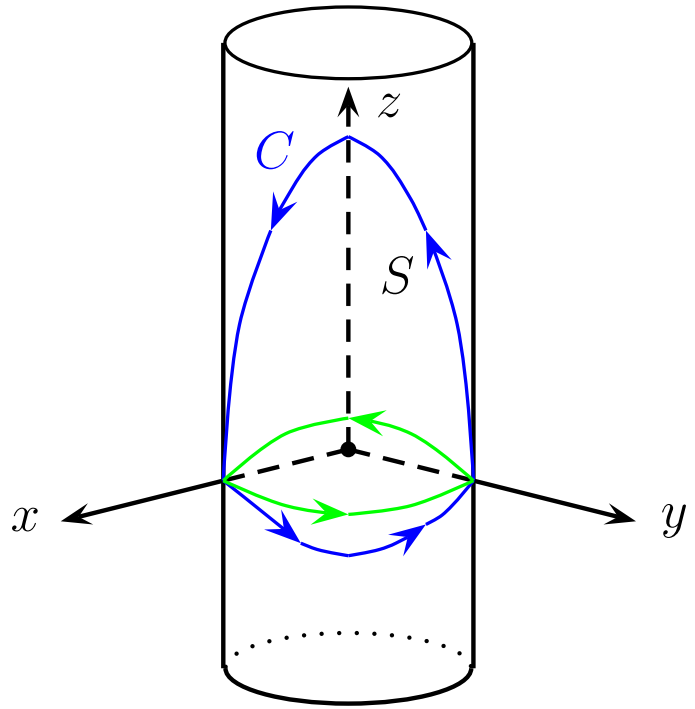


Figure: Examples (i) and (ii) of the use of the Stokes theorem

(ii) Calculation of the line integral along an oriented boundary

Example:
Let $\boldsymbol{F}(x,y,z) := (-y^3, x^3, -z^3)$ for $(x,y,z) \in \mathbb{R}^3$, and let C denote the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$, oriented by the anticlockwise motion on the projection $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ of C on the xy-plane.
In order to find $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$, consider the surface

$$S := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq 1 \text{ and } x+y+z=1\}.$$

Let us think of S as the graph of the function $f : D_0 \to \mathbb{R}$, where $D_0 := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ and for $(x,y) \in D_0$, $f(x,y) := 1-x-y$. Let S be oriented by the upward unit normal vectors. Then $\eth S = C$, and the induced orientation is the same as the given orientation of C. Hence

$$d\boldsymbol{S} = (-f_x, -f_y, 1)d(x,y) = (1,1,1)d(x,y).$$

Also,

$$\operatorname{curl} \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (0, 0, 3x^2 + 3y^2),$$

and so $(\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3(x^2 + y^2)d(x, y)$.

By the Stokes theorem,

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\boldsymbol{s} &= \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = 3 \iint_{D_0} (x^2 + y^2) d(x, y) \\ &= 3 \int_0^1 \int_{-\pi}^{\pi} r^2 r \, dr \, d\theta = \frac{3\pi}{2}. \end{aligned}$$

Consequences of the Stokes Theorem

Proposition

Let $\boldsymbol{F}$ be a smooth vector field on an open subset D of $\mathbb{R}^3$ such that $\operatorname{curl} \boldsymbol{F} = \boldsymbol{0}$ on D.

(i) Suppose S is a bounded oriented piecewise C^2 surface in D, and let ∂S denote its intrinsic boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

In particular, if $\partial S = C_1 \cup (-C_2)$, so that C_1 and $-C_2$ have the induced orientation, then $\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$.

(ii) If D is simply connected, then $\boldsymbol{F}$ is a gradient field on D.

Proof:

(i) By the Stokes theorem,

$$\int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = \iint_S (\operatorname{curl} \boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_S \boldsymbol{0} \cdot d\boldsymbol{S} = 0.$$

Next, suppose $\partial S = C_1 \cup (-C_2)$, so that C_1 and $-C_2$ have the induced orientation. Then

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_1 - C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

(ii) Suppose D is simply connected. Let C denote a simple closed smooth curve in D. Then there is a smooth surface S in D such that $\partial S = C$, and so $\int_C \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

This also holds if a smooth closed curve C in D intersects itself, since we can break up C into several simple closed smooth curves. As a result, line integrals of $\boldsymbol{F}$ are path independent in D, and so $\boldsymbol{F}$ is a gradient field on D. □

Remark:
The last statement in part (i) above is known as the invariance of line integrals, or as the deformation principle for line integrals along paths in $\mathbb{R}^3$. (See the figure on the next slide.)

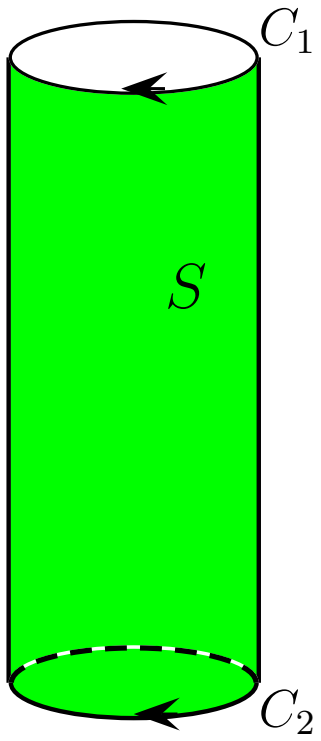


Figure: If curl $\boldsymbol{F} = \boldsymbol{0}$ on S, then $\displaystyle\int_{C_1} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{s}$.

Examples: (i) Let $\boldsymbol{F} := (P, Q, R)$, where $P := -y/(x^2 + y^2)$, $Q := x/(x^2 + y^2)$ and $R := 1$ for $(x, y, z) \in \mathbb{R}^3$ with $(x, y) \neq 0$. We have seen in Lecture 33 that $Q_x = P_y$, and so $\operatorname{curl}(\boldsymbol{F}) = (0, 0, Q_x - P_y) = \boldsymbol{0}$. Consider the upper hemisphere $H := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$. Let C_1 denote the equator given by $x^2 + y^2 = 1$ and $z = 0$. Also, let a plane in $\mathbb{R}^3$ intersect H in a circle C_2 such that $C_1 \cap C_2 = \emptyset$ and $(0, 0, 1) \notin C_2$.

Let S denote the part of the hemisphere H bounded by C_1 and C_2, and let S be oriented by the outward unit normal vectors. Suppose C_1 and $-C_2$ have the induced orientation, so that both C_1 and C_2 are traced anticlockwise as seen from the point $(0, 0, 2)$. Then

$$\int_{C_2} -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy + dz = \int_{C_1} -y\,dx + x\,dy = 2\pi,$$

as we had calculated earlier.

(ii) Let $D := \mathbb{R}^3$, and let $\boldsymbol{F}(x, y, z) := (y\, e^z,\ x\, e^z,\ x\, y\, e^z)$ for $(x, y, z) \in D$. Is $\boldsymbol{F}$ a gradient field? Now

$$(\operatorname{curl} \boldsymbol{F})(x, y, z) = (xe^z - xe^z, ye^z - ye^z, e^z - e^z) = (0, 0, 0)$$

for all $(x, y, z) \in D$. Also, D is simply connected. Hence $\boldsymbol{F}$ must be a gradient field. In fact, if we let $f(x, y, z) := x\, y\, e^z$ for $(x, y, z) \in D$, then $\nabla f = \boldsymbol{F}$ on D.

Recall the Green theorem for a multiply connected region D in $\mathbb{R}^2$: If ∂D denotes the positively oriented boundary of D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)d(x,y).$$

(i) (Circulation-Curl form) Let $\boldsymbol{F} := (P, Q, 0)$. Then

$$\int_{\partial D} \boldsymbol{F}\cdot d\boldsymbol{s} = \iint_D (\operatorname{curl}\boldsymbol{F})\cdot \boldsymbol{k}\, d(x,y).$$

(ii) (Flux-Divergence form) Let $\boldsymbol{F} := (Q, -P, 0)$. Then

$$\int_{\partial D} \boldsymbol{F}\cdot \hat{\boldsymbol{n}}\, ds = \iint_D (\operatorname{div}\boldsymbol{F})\, d(x,y).$$

We have seen that the Stokes theorem generalizes the form (i) above from a region D in $\mathbb{R}^2$ to a surface in $\mathbb{R}^3$. We shall now show that the Gauss divergence theorem generalizes the form (ii) above from a region D in $\mathbb{R}^2$ to a solid in $\mathbb{R}^3$.

Let D be a closed and bounded subset of $\mathbb{R}^3$. Suppose that the (usual) boundary ∂D of D consists of a finite number of orientable, piecewise smooth, nonintersecting closed geometric surfaces. (Thus all these surfaces are without any 'edges', that is, their intrinsic boundary is empty.) If each of these surfaces is so oriented that the unit normal vectors point out of the solid D, then we say that the boundary ∂D of D is **positively oriented**.

(Compare the definition of the positively oriented boundary of a multiply connected domain in $\mathbb{R}^2$.)

Example: Let $D := \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4\}$. Then $\partial D = S_1 \cup S_2$ is positively oriented if
$S_1 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is oriented by the inward unit normal vectors, and
$S_2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\}$ is oriented by the outward unit normal vectors. Note that $\eth S_1 = \emptyset = \eth S_2$.

Theorem (Gauss Divergence Theorem)

Let D be a closed and bounded subset of $\mathbb{R}^3$ whose boundary ∂D consists of a finite number of orientable, piecewise smooth, nonintersecting closed geometric surfaces. Suppose ∂D is positively oriented. Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing D. Then

$$\iint_{\partial D} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_{D} (\operatorname{div} \boldsymbol{F}) d(x, y, z).$$

Proof: Let $\boldsymbol{F} := (P, Q, R)$. Then the surface integral $\iint_{\partial D} (P, Q, R) \cdot d\boldsymbol{S}$ on the left side is well-defined since P, Q, R are continuous on ∂D. The triple integral $\iiint_D (P_x + Q_y + R_z)\, d(x, y, z)$ on the right side is well-defined since P_x, Q_y and R_z are continuous on D and since ∂D is of three-dimensional content zero. (See Lecture 27.)

We show that

$$\iint_{\partial D} (0,0,R)\cdot d\boldsymbol{S} = \iiint_D R_z \, d(x,y,z).$$

The other two equalities would follow similarly.

Let us suppose that D is an elementary region given as follows.

Let $D_0 \subset \mathbb{R}^2$, and let $\psi_1, \psi_2 : D_0 \to \mathbb{R}$ be continuous functions. Let

$$D := \{(x,y,z) \in \mathbb{R}^3 : \psi_1(x,y) \le z \le \psi_2(x,y)\},$$

and suppose ∂D consists of a single piecewise smooth, positively oriented, closed geometric surface S, where $S := S_1 \cup S_2$, with

$S_1 := \{(x,y,z) \in \mathbb{R}^3 : z = \psi_1(x,y)\}$ is its lower part, and

$S_2 := \{(x,y,z) \in \mathbb{R}^3 : z = \psi_2(x,y)\}$ is its upper part.

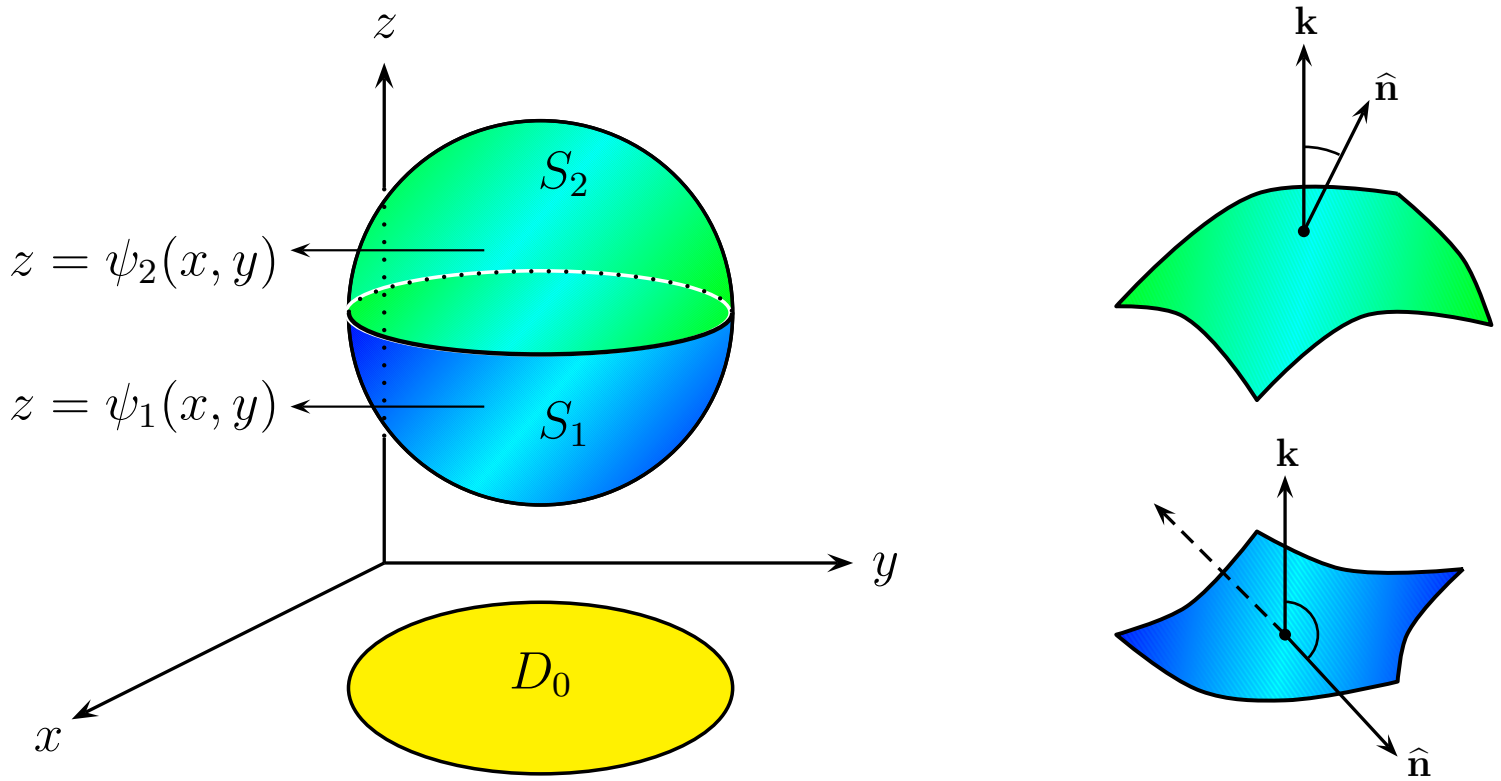


Figure: An outward normal $\hat{\boldsymbol{n}}$ to the upper part S_2 is upward, and an outward normal $\hat{\boldsymbol{n}}$ to the lower part S_1 is downward.

The upper part S_2 is given by $\Phi(x,y) := (x, y, \psi_2(x,y))$ for $(x,y) \in D_0$, and so the normal vector

$$\Phi_x \times \Phi_y = \big(1, 0, (\psi_2)_x\big) \times \big(0, 1, (\psi_2)y\big) = \big(-(\psi_2)_x, -(\psi_2)_y, 1\big),$$

points 'upward', that is, it goes out of the solid D. Hence Φ is an orientation-preserving parametrization of S_2, and so

$$\begin{aligned} \iint_{S_2} (0,0,R) \cdot d\boldsymbol{S} &= \iint_{D_0} \big(0,0,R(\Phi(x,y))\big) \cdot (\Phi_x \times \Phi_y) d(x,y) \\ &= \iint_{D_0} R(x,y,\psi_2(x,y)) d(x,y). \end{aligned}$$

The lower part S_1 is given by $\Psi(x,y) := (x, y, \psi_1(x,y))$ for $(x,y) \in D_0$. Now the 'upward' normal vector $\Psi_x \times \Psi_y = \big(-(\psi_1)_x, -(\psi_1)_y, 1\big)$ points inward to the solid D.

Hence Ψ is an orientation-reversing parametrization of S_1, and so

$$\iint_{S_1} (0,0,R)\cdot d\boldsymbol{S} = -\iint_{D_0} R(x,y,\psi_1(x,y))d(x,y).$$

Combining the above two equalities, and using domain additivity (since $S_1 \cup S_2 = S = \partial D$), we obtain

$$\begin{aligned}\iint_{\partial D} (0,0,R)\cdot d\boldsymbol{S} &= \iint_{D_0} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} R_z \, dz \right) d(x,y) \\ &= \iiint_{D} R_z \, d(x,y,z)\end{aligned}$$

by Part II of the FTC, and by the Fubini theorem for the triple integrals, which is also known as the Cavalieri Principle.

The above proof can be modified to treat the general case. □