MA 105 : Calculus
Division 1, Lecture 05

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Recap of the previous lecture

- Higher order derivatives
- Equivalent condition for Differentiability: C-Lemma
- Consequences of the Carathéodory Lemma (C-Lemma):
  - Differentiability implies Continuity
  - Algebraic Rules (concerning sums, products, and quotients of differentiable functions)
  - Chain Rule.
  - Differentiability of the inverse function. Examples
- Local extremum. Basic properties.
- Rolle's theorem.

Rolle's Theorem

Theorem

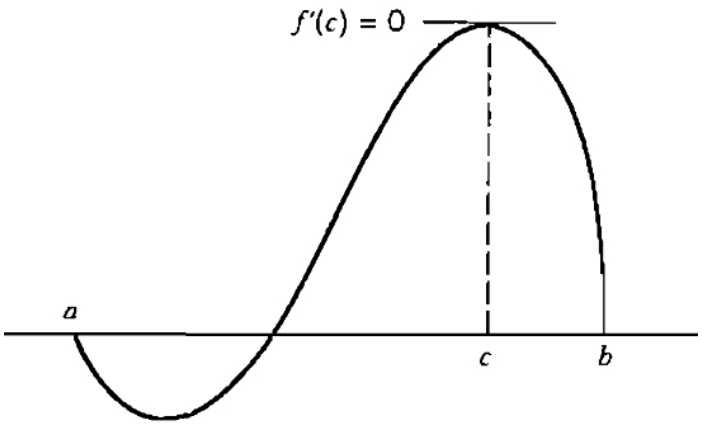
Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$,

(ii) f is differentiable on (a, b), and

(iii) $f(a) = f(b)$.

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.



Thus there exists a point $c \in (a, b)$ such that the tangent to the graph at $(c, f(c))$ is parallel to the x-axis.

Proof. Since f is continuous on $[a, b]$, it is bounded and attains its bounds on $[a, b]$. Let $c_1, c_2 \in [a, b]$ be such that

$$f(c_1) = \min\{f(x) : x \in [a, b]\} \text{ and } f(c_2) = \max\{f(x) : x \in [a, b]\}.$$

If c_1 or c_2 is an interior point of $[a, b]$, then $f'(c_1) = 0$ or $f'(c_2) = 0$ by the previous proposition, and we are done.

If both c_1 and c_2 are end-points of $[a, b]$, then $f(c_1) = f(a) = f(b) = f(c_2)$. Since the minimum and the maximum values of f on $[a, b]$ are the same, f is a constant function on $[a, b]$, and so $f'(c) = 0$ for every $c \in (a, b)$. □

Example:
Let $f(x) := |x|$ for $x \in [-1, 1]$. Then f is continuous on $[-1, 1]$ and $f(-1) = 1 = f(1)$. But there is no $c \in (-1, 1)$ such that $f'(c) = 0$. Note: f is not differentiable at 0.

Application of the Rolle theorem

Let $f(x) := x^4 + 2x^3 - 2$ for $x \in \mathbb{R}$. We show that f has a unique root in $(0, \infty)$, and it is in $(0, 1)$.

- Using the IVP, we have seen that f has a root in (0,1).
- Suppose there are two roots in $(0, \infty)$. Then there are $a, b \in \mathbb{R}$ with $0 < a < b$ such that $f(a) = 0 = f(b)$.
- By Rolle's theorem, there is $c \in (a, b) \subseteq (0, \infty)$ such that $f'(c) = 0$.
- But $f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) \neq 0$ for $x \in (0, \infty)$.
- Hence f has exactly one (real) root in $(0, \infty)$.

Lagrange's Mean Value Theorem

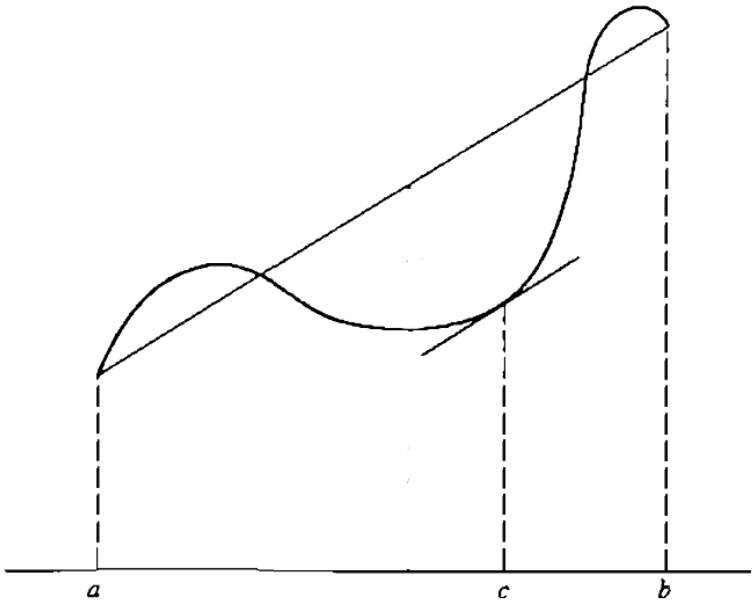
Theorem (MVT)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that

(i) f is continuous on $[a, b]$, and

(ii) f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.



Thus there exists $c \in (a, b)$ such that the tangent to the graph at $(c, f(c))$ is parallel to the line joining $(a, f(a))$ and $(b, f(b))$.

Proof. For $x \in [a, b]$, define

$$F(x) := f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then F is continuous on $[a, b]$, differentiable on (a, b) and $F(a) = f(a) = F(b)$. By Rolle's theorem, there is $c \in (a, b)$ such that $F'(c) = 0$, that is, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Note: We have used the Rolle theorem to prove a more general result, namely the Lagrange mean value theorem.

Physical Interpretation of MVT: For $t \in [a, b]$, let $f(t)$ denote the distance travelled by a particle at time t. Then the average speed of the particle is $\dfrac{f(b) - f(a)}{b - a}$, and its instantaneous speed at time $t \in [a, b]$ is $f'(t)$. Thus there is a time $c \in [a, b]$ at which the average speed equals the instantaneous speed.

Consequences of the Mean Value Theorem (MVT)

Proposition

Let I be an interval and $f : I \to \mathbb{R}$. If $f'(x)$ exists and equals 0 for all $x \in I$, then f is constant on I.

Proof. Let x_1, x_2 in I with $x_1 < x_2$. Then $[x_1, x_2] \subseteq I$.
By the MVT,

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \quad \text{for some } c \in (x_1, x_2).$$

Since $f'(c) = 0$, we obtain $f(x_2) = f(x_1)$. Thus f has the same value at every point of I. □

We have already seen that the converse is also true.

Example: Let $D := \mathbb{R} \setminus \{0\}$, and $f(x) := x/|x|$ for $x \in D$. Then $f' = 0$ on D, but f is not constant on D; this is because D is not an interval.

Consequences of MVT (continued)

Monotonicity

Let I be an interval, and let $f : I \to \mathbb{R}$. We say that f is

- **(monotonically) increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \leq f(x_2)$.
- **(monotonically) decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) \geq f(x_2)$.
- **monotonic** on I if it is increasing on I, or it is decreasing on I.
- **strictly increasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) < f(x_2)$.
- **strictly decreasing** on I if
  $x_1, x_2 \in I,\ x_1 < x_2 \implies f(x_1) > f(x_2)$.
- **strictly monotonic** on I if either it is strictly increasing on I, or it is strictly decreasing on I.

Derivatives and Monotonicity

Proposition

Let I be an interval, and let $f : I \to \mathbb{R}$ be differentiable. Then

(i) $f' \geq 0$ on $I \iff f$ is increasing on I.

(ii) $f' \leq 0$ on $I \iff f$ is decreasing on I.

(iii) $f' > 0$ on $I \implies f$ is strictly increasing on I.

(iv) $f' < 0$ on $I \implies f$ is strictly decreasing on I.

All forward ($\implies$) implications follow from the MVT.

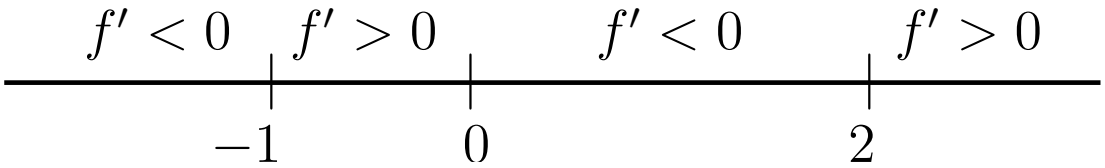
The backward ($\impliedby$) implication in (i) follows since $\big(f(x+h) - f(x)\big)/h \geq 0$ for all $h \neq 0$ with $x + h \in I$.
Similarly, the backward ($\impliedby$) implication in (ii) follows.

The converse of (iii) is false: Let $f(x) := x^3$ for $x \in \mathbb{R}$. Then f is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.
Similarly, the converse of (iv) is false.

Examples

(i) Let $f(x) := 3x^4 - 4x^3 - 12x^2 + 5$ for $x \in \mathbb{R}$. Let us find intervals on which f is (strictly) increasing, or (strictly) decreasing. For $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) \\ &= 12(x+1)x(x-2). \end{aligned}$$



- f is strictly decreasing on $(-\infty, -1)$.
- f is strictly increasing on $(-1, 0)$.
- f is strictly decreasing on $(0, 2)$.
- f is strictly increasing on $(2, \infty)$.

(ii) Let $f(x) := x - \sin x$ for $x \in [0, \pi/2]$. Then $f'(x) = 1 - \cos x \geq 0$ for all $x \in [0, \pi/2]$. Hence f is increasing on $[0, \pi/2]$, and so $0 = f(0) \leq f(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow \sin x \leq x$. In fact, $\sin x < x$ for $x \in (0, \pi/2]$ since $f'(x) > 0$ for $x \in (0, \pi/2]$.

(iii) Let $g(x) := x - 1 + \cos x$ for $x \in [0, \pi/2]$. Then $g'(x) = 1 - \sin x \geq 0$ for all $x \in [0, \pi/2]$. Hence g is increasing on $[0, \pi/2]$, and so $0 = g(0) \leq g(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow 1 - \cos x \leq x$. In fact, since $g'(x) > 0$ for $x \in [0, \pi/2)$, we see that $g(0) < g(x)$ for $x \in (0, \pi/2)$, that is, $1 - \cos x < x$ for $x \in (0, \pi/2)$.

(iv) Let $h(x) := \sin x - x \cos x$ for $x \in [0, \pi/2]$. Then $h'(x) = x \sin x \geq 0$ for all $x \in [0, \pi/2]$. Hence h is increasing on $[0, \pi/2]$, and so $0 = h(0) \leq h(x)$ for $x \in [0, \pi/2]$. Thus $x \in [0, \pi/2] \Longrightarrow x \cos x \leq \sin x$. In fact, $x \cos x < \sin x$ for $x \in (0, \pi/2]$ since $h'(x) > 0$ for $x \in (0, \pi/2]$.

The above 3 results yield the inequalities quoted in Lecture 3.

Convexity and Concavity

Let I be an interval. A function $f : I \to \mathbb{R}$ is called **convex** on I if the line segment joining any two points on the graph of f lies on or above the graph of f. If $x_1, x_2 \in I$, then the equation of the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by $y = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$. Thus f is **convex** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \leq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Similarly, a function $f : I \to \mathbb{R}$ is called **concave** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \geq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

A function $f : I \to \mathbb{R}$ is called **strictly convex** on I if the inequality $\leq$ in the definition of a convex function can be replaced by the strict inequality $<$.

A function $f : I \to \mathbb{R}$ is called **strictly concave** on I if the inequality $\geq$ in the definition of a concave function can be replaced by the strict inequality $>$.

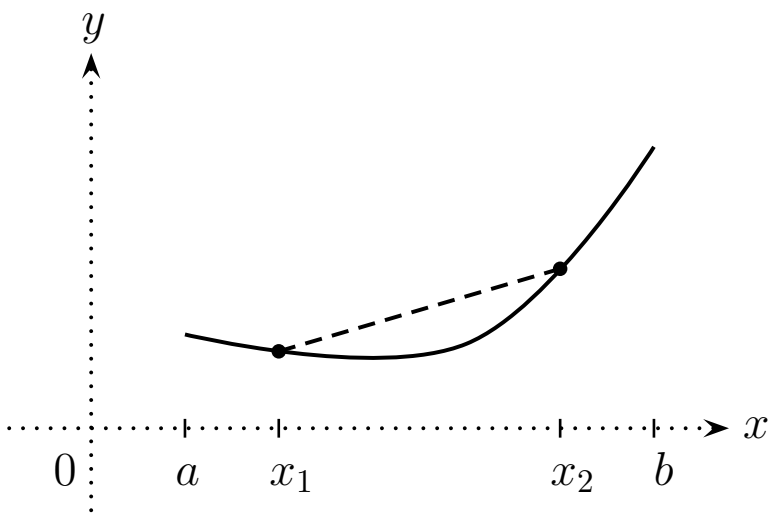
Examples:

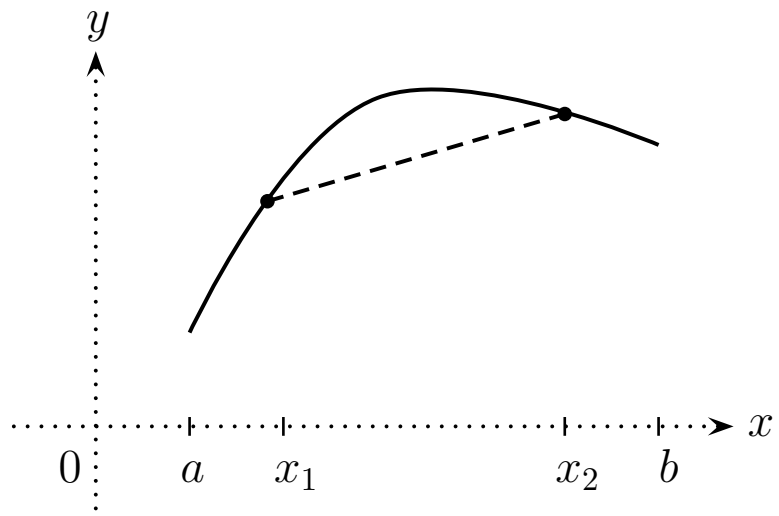
(i) Let $\alpha, \beta \in \mathbb{R}$, and let $f(x) := \alpha + \beta x$ for $x \in \mathbb{R}$. Then f is convex as well as concave on $\mathbb{R}$ (but not strictly).

(ii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha |x|$ for $x \in \mathbb{R}$. Then f is convex on $\mathbb{R}$ if $\alpha > 0$, and f is concave on $\mathbb{R}$ if $\alpha < 0$ (but not strictly). (Prove this from the first principles.)

(iii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha x^2$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$ if $\alpha > 0$, and f is strictly concave on $\mathbb{R}$ if $\alpha < 0$. (Prove this from the first principles.)

Typical graphs of convex and concave functions on $I := [a, b]$ look as follows:





Let I be an interval, and let $f : I \to \mathbb{R}$ be a function. Suppose the curve $y = f(x)$ has a (unique) nonvertical tangent at each point on it.

It is apparent that if f is convex, then the slopes of the tangents increase as we move from left to right, and if f is concave, then then the slopes of the tangents decrease as we move from left to right. This is made precise in the following.

Proposition

Let I be an interval and $f : I \to \mathbb{R}$ be differentiable. Then

(i) f' is increasing on $I \iff f$ is convex on I.

(ii) f' is decreasing on $I \iff f$ is concave on I.

(iii) f' is strictly increasing on $I \iff f$ is strictly convex on I.

(iv) f' is strictly decreasing on $I \iff f$ is strictly concave on I.

We omit proofs of the above results as they are a bit involved.

Corollary

Let I be an interval and $f : I \to \mathbb{R}$ be twice differentiable.

(i) $f'' \geq 0$ on $I \iff f$ is convex on I.

(ii) $f'' \leq 0$ on $I \iff f$ is concave on I.

(iii) $f'' > 0$ on $I \implies f$ is strictly convex on I.

(iv) $f'' < 0$ on $I \implies f$ is strictly concave on I.

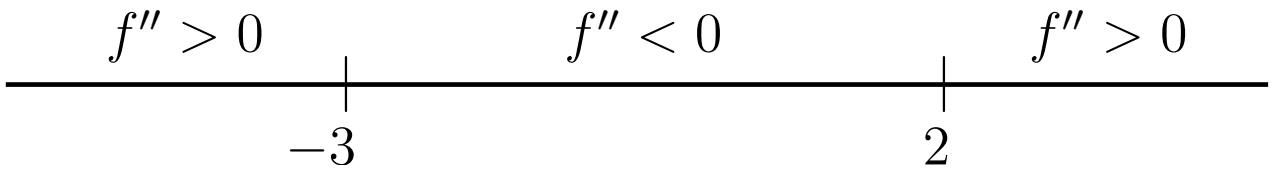
The converse of (iii) in the above corollary is false. To see this, consider $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$, but $f''(0) = 0$. Similarly, the converse of (iv) is false.

Example:

Let $f(x) := x^4 + 2x^3 - 36x^2 + 62x + 5$ for $x \in \mathbb{R}$. Let us find intervals on which f is (strictly) convex or (strictly) concave. For $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= 4x^3 + 6x^2 - 72x + 62, \text{ and} \\ f''(x) &= 12x^2 + 12x - 72 = 12(x+3)(x-2). \end{aligned}$$

Note:



- f is strictly convex on $(-\infty, -3)$.
- f is strictly concave on $(-3, 2)$.
- f is strictly convex on $(2, \infty)$.