MA 105 : Calculus
Division 1, Lecture 06

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Rolle's theorem and its proof (using Extreme Value Property)
- (Lagrange's) Mean Value Theorem and its proof
- Consequences of MVT
- Monotonicity. Relation with derivatives
- Convexity and Concavity.

Convexity and Concavity

Let I be an interval. A function $f : I \to \mathbb{R}$ is called **convex** on I if the line segment joining any two points on the graph of f lies on or above the graph of f. If $x_1, x_2 \in I$, then the equation of the line passing through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by $y = f(x_1) + \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$. Thus f is **convex** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \leq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Similarly, a function $f : I \to \mathbb{R}$ is called **concave** on I if

$$x_1, x_2,\ x \in I,\ x_1 < x < x_2 \Longrightarrow f(x) \geq f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1).$$

Remark: It is easy to see that f

$$f\left((1-\lambda)x_1 + \lambda x_2\right) \leq (1-\lambda)f(x_1) + \lambda f(x_2)\ \forall\, \lambda \in [0,1] \text{ and } x_1, x_2 \in I.$$

Likewise for concacity (with $\leq$ changed to $\geq$).

A function $f : I \to \mathbb{R}$ is called **strictly convex** on I if the inequality $\leq$ in the definition of a convex function can be replaced by the strict inequality $<$.

A function $f : I \to \mathbb{R}$ is called **strictly concave** on I if the inequality $\geq$ in the definition of a concave function can be replaced by the strict inequality $>$.

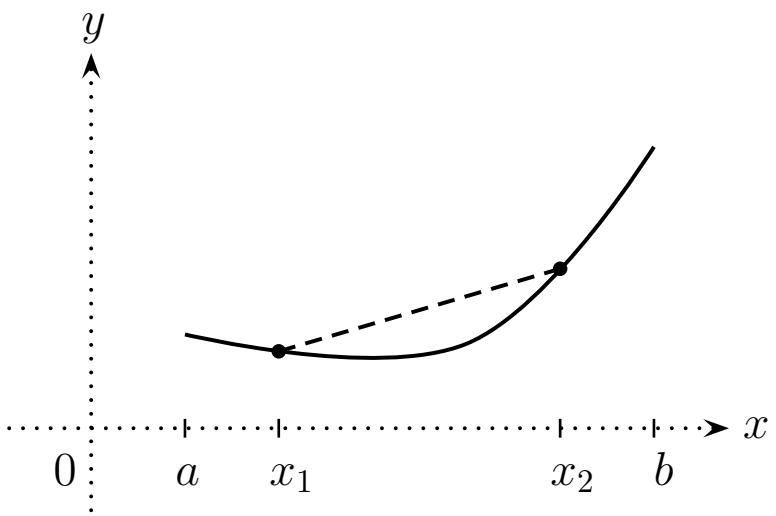
Examples:

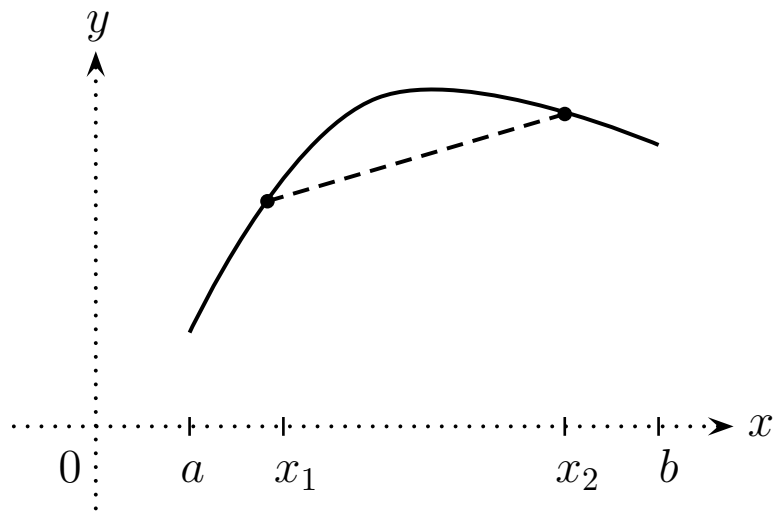
(i) Let $\alpha, \beta \in \mathbb{R}$, and let $f(x) := \alpha + \beta x$ for $x \in \mathbb{R}$. Then f is convex as well as concave on $\mathbb{R}$ (but not strictly).

(ii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha |x|$ for $x \in \mathbb{R}$. Then f is convex on $\mathbb{R}$ if $\alpha > 0$, and f is concave on $\mathbb{R}$ if $\alpha < 0$ (but not strictly). (Prove this from the first principles.)

(iii) Let $\alpha \in \mathbb{R}$, and let $f(x) := \alpha x^2$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$ if $\alpha > 0$, and f is strictly concave on $\mathbb{R}$ if $\alpha < 0$. (Prove this from the first principles.)

Typical graphs of convex and concave functions on $I := [a, b]$ look as follows:





Let I be an interval, and let $f : I \to \mathbb{R}$ be a function. Suppose the curve $y = f(x)$ has a (unique) nonvertical tangent at each point on it.

It is apparent that if f is convex, then the slopes of the tangents increase as we move from left to right, and if f is concave, then then the slopes of the tangents decrease as we move from left to right. This is made precise in the following.

Proposition

Let I be an interval and $f : I \to \mathbb{R}$ be differentiable. Then

(i) f' is increasing on $I \iff f$ is convex on I.

(ii) f' is decreasing on $I \iff f$ is concave on I.

(iii) f' is strictly increasing on $I \iff f$ is strictly convex on I.

(iv) f' is strictly decreasing on $I \iff f$ is strictly concave on I.

We omit proofs of the above results as they are a bit involved.

Corollary

Let I be an interval and $f : I \to \mathbb{R}$ be twice differentiable.

(i) $f'' \geq 0$ on $I \iff f$ is convex on I.

(ii) $f'' \leq 0$ on $I \iff f$ is concave on I.

(iii) $f'' > 0$ on $I \implies f$ is strictly convex on I.

(iv) $f'' < 0$ on $I \implies f$ is strictly concave on I.

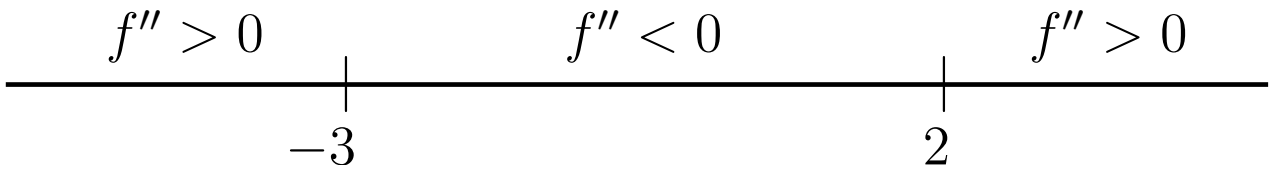
The converse of (iii) in the above corollary is false. To see this, consider $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is strictly convex on $\mathbb{R}$, but $f''(0) = 0$. Similarly, the converse of (iv) is false.

Example:

Let $f(x) := x^4 + 2x^3 - 36x^2 + 62x + 5$ for $x \in \mathbb{R}$. Let us find intervals on which f is (strictly) convex or (strictly) concave. For $x \in \mathbb{R}$,

$$\begin{aligned} f'(x) &= 4x^3 + 6x^2 - 72x + 62, \text{ and} \\ f''(x) &= 12x^2 + 12x - 72 = 12(x+3)(x-2). \end{aligned}$$

Note:



- f is strictly convex on $(-\infty, -3)$.
- f is strictly concave on $(-3, 2)$.
- f is strictly convex on $(2, \infty)$.

Critical Points and Global Extrema

Let $D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$. An interior point c of D is called a **critical point** of f if either f is not differentiable at c, or if f is differentiable at c and $f'(c) = 0$.

Proposition

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the global minimum $m := \min\{f(x) : x \in [a, b]\}$ as well as the global maximum $M := \max\{f(x) : x \in [a, b]\}$ of f on $[a, b]$ is attained either at a critical point of f or at an end-point of $[a, b]$.

Proof. There is $c_1 \in [a, b]$ such that $f(c_1) = m$. If $c_1 = a$ or $c_1 = b$, then we are done. Next, let $c_1 \in (a, b)$. Then f has a local minimum at c_1. If f is not differentiable at c_1, then c_1 is a critical point of f. If f is differentiable at c_1, then $f'(c_1) = 0$ by an earlier result, and so c_1 is a critical point of f.
Similarly, we argue for the global maximum M of f. □

Finding Global Extrema

Example: Consider $f : [-1, 2] \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} -x & \text{if } -1 \le x \le 0, \\ 2x^3 - 4x^2 + 2x & \text{if } 0 < x \le 2. \end{cases}$$

f is continuous on $[-1, 2]$: $\lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x)$.
f is not differentiable at 0 : $f'_-(0) = -1$ and $f'_+(0) = 2$.
For $-1 < x < 0$, $f'(x) = -1 \neq 0$.
For $0 < x < 2$, $f'(x) = 6x^2 - 8x + 2 = 2(3x - 1)(x - 1)$.
Critical points of f: 0, $1/3$, 1. End-points of $[-1, 2]$: -1, 2.

x	-1	0	$1/3$	1	2
$f(x)$	1	0	$8/27$	0	4

f attains its global maximum 4 at $x = 2$, and its global minimum 0 at $x = 0$ as well as at $x = 1$.

Moral of the story

While finding global extrema of a continuous function f defined on $[a, b]$, it is of no use to know whether a local extremum of f is in fact a local maximum or a local minimum. Hence do not find the second derivative of f.

To determine the global extrema of f, find the critical points of f, that is, the points at which the derivative of f does not exist and the points in (a, b) at which the derivative of f exists and is equal to 0. Then find the values of f at the critical points of f, and at the end points a and b of $[a, b]$.

Compare these values of f. The smallest among them is the global minimum of f and the largest among them is the global maximum of f.

Local Extrema and Derivatives

Intuitively, it is clear that if the slopes of tangents to a curve $y = f(x)$ are positive to the left of $x = c$ and negative to the right of $x = c$, then f has a local maximum at c. A more precise and general formulation of this is the following.

Theorem (First derivative test for a local maximum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \geq 0$ on $(c - \delta, c)$ and $f' \leq 0$ on $(c, c + \delta)$.

Then f has a local maximum at c.

Proof: By (ii), f is increasing on $(c - \delta, c)$, and it is decreasing on $(c, c + \delta)$. Also, $\lim_{h \to 0} f(c + h) = f(c)$ by (i). Hence $x \in (c - \delta, c) \implies f(x) \leq \lim_{h \to 0^-} f(c + h) = f(c)$ and $x \in (c, c + \delta) \implies f(x) \leq \lim_{h \to 0^+} f(c + h) = f(c)$. Hence f has a local maximum at c. □

Theorem (First derivative test for a local minimum)

Let $D \subset \mathbb{R}$, c an interior point of D, and $f : D \to \mathbb{R}$. Suppose

(i) f is continuous at c, and

(ii) there is $\delta > 0$ such that $f' \leq 0$ on $(c - \delta, c)$ and $f' \geq 0$ on $(c, c + \delta)$.

Then f has a local minimum at c.

Proof: Similar to the proof of the previous result.

Examples:

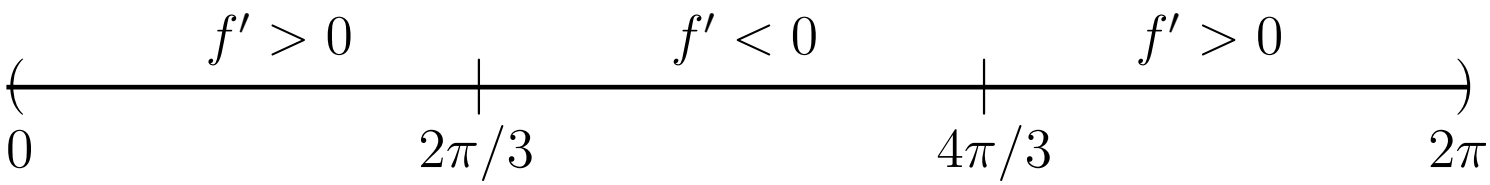
(i) Let $f(x) := |x|$ for $x \in \mathbb{R}$. Then f is continuous at 0, $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. Thus f has a local minimum at 0.

(ii) Let $f(x) := -x - 1$ for $x \in (-\infty, 0)$, $f(0) := 0$ and $f(x) := x + 1$ for $x \in (0, \infty)$. Then $f' = -1 \leq 0$ on $(-\infty, 0)$ and $f' = 1 \geq 0$ on $(0, \infty)$. But f does not have a local minimum at 0. Note: f is not continuous at 0.

(iii) Let $f(x) := x + 2\sin x$ for $x \in (0, 2\pi)$.

Then $f'(x) = 1 + 2\cos x = 0 \iff x \in \{2\pi/3, 4\pi/3\}$.

f is continuous at $2\pi/3$ and at $4\pi/3$. We note the following:



f has a local maximum at $2\pi/3$ and a local minimum at $4\pi/3$.

In general, the first derivative test for a local extremum can be informally summarized by the following Thumb Rule.

Suppose f is continuous at c.

f' changes from $+$ to $-$ at $c \implies$ f has a local maximum at c,

f' changes from $-$ to $+$ at $c \implies$ f has a local minimum at c.

Theorem (Second derivative test for a local extremum)

Let $D \subset \mathbb{R}$, and let c be an interior point of D. Suppose $f : D \to \mathbb{R}$ is twice differentiable at c, and $f'(c) = 0$.

(i) If $f''(c) < 0$, then f has a local maximum at c.

(ii) If $f''(c) > 0$, then f has a local minimum at c.

Proof: (i) Let $f''(c) < 0$. Since $f''(c)$ exists, there is $r > 0$ such that f' exits on $(c - r, c + r)$. Now

$$\lim_{x \to c} \frac{f'(x)}{x - c} = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c) < 0.$$

Hence there is $\delta > 0$ such that $f'(x)/(x - c) < 0$ for all $x \in (c - \delta, c) \cup (c, c + \delta)$. [This follows from the ϵ-δ characterization of limits. Verify!] This implies that $f'(x) > 0$ when $x \in (c - \delta, c)$ and $f'(x) < 0$ when $x \in (c, c + \delta)$. Also f is continuous at c (being differentiable). Hence by the first derivative test, f has a local maximum at c.

A similar argument works for (ii). □

Examples:

(i) Let $f(x) := x^4 - 2x^2$ for $x \in \mathbb{R}$. Then

$f'(x) = 4x^3 - 4x = 4(x+1)x(x-1) = 0 \iff x \in \{-1, 0, 1\}$.

f is twice differentiable, and $f''(x) = 4(3x^2 - 1)$ for $x \in \mathbb{R}$.

Since $f''(-1) = 8 > 0$, $f''(0) = -4 < 0$ and $f''(1) = 8 > 0$,
f has a local maximum at 0, and has a local minimum at ± 1.

(ii) Let $f(x) := x^4$ for $x \in (-1, 1)$.
Then $f'(x) = 4x^3$ and $f''(x) = 12x^2$ for $x \in (-1, 1)$,
and so $f(0) = f'(0) = f''(0) = 0$.

Hence the second derivative test is not applicable at 0. But since $f' < 0$ on $(-1, 0)$ and $f' > 0$ on $(0, 1)$, the first derivative test shows that f has a local minimum at 0.

Note: The first derivative test is more general than the second.

Point of Inflection

Let I be an interval, and let $f : I \to \mathbb{R}$. An interior point of I at which convexity of f changes to concavity, or the other way round, is called a point of inflection for f. More precisely,

Definition

An interior point c of I is called a **point of inflection** for f if there is $\delta > 0$ such that
either f is convex on $(c - \delta, c)$ and concave on $(c, c + \delta)$,
or f is concave on $(c - \delta, c)$ and convex on $(c, c + \delta)$.

Examples:

Let $f(x) := x^3$ and $g(x) := x^{1/3}$ for $x \in \mathbb{R}$. Then 0 is a point of inflection for f as well as for g.

We give characterizations of a point of inflection of a differentiable and of a twice differentiable function.

Point of Inflection

Let I be an interval, and let $f : I \to \mathbb{R}$. An interior point of I at which convexity of f changes to concavity, or vice-versa, is called a point of inflection for f. More precisely,

Definition

An interior point c of I is called a **point of inflection** for f if there is $\delta > 0$ such that
either f is convex on $(c - \delta, c)$ and concave on $(c, c + \delta)$,
or f is concave on $(c - \delta, c)$ and convex on $(c, c + \delta)$.

Examples:

Let $f(x) := x^3$ and $g(x) := x^{1/3}$ for $x \in \mathbb{R}$. Then 0 is a point of inflection for f as well as for g.

We give characterizations of a point of inflection of a differentiable and of a twice differentiable function.

Point of Inflection and Derivatives

Theorem (Derivative tests for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$.

(i) (First derivative test) Suppose f is differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$. Then c is point of inflection for $f \iff$ there is $\delta > 0$ with $\delta < r$ such that f' is increasing on $(c-\delta, c)$ and f' is decreasing on $(c, c+\delta)$, or vice-versa.

(ii) (Second derivative test) Suppose f is twice differentiable on $(c-r, c) \cup (c, c+r)$ for some $r > 0$.. Then c is point of inflection for $f \iff$ there is $\delta > 0$ such that $f'' \geq 0$ on $(c-\delta, c)$ and $f'' \leq 0$ on $(c, c+\delta)$, or vice-versa.

Proof: Use the characterizations of convexity and concavity. □

Thumb Rule:

f'' changes sign at $c \iff c$ is a point of inflection for f.

(Necessary condition for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$. Suppose f is twice differentiable at c. If c is point of inflection for f, then $f''(c) = 0$.

Proof: Since $f''(c)$ exists, there is $r > 0$ such that f is differentiable on $(c - r, c + r)$. Suppose c is point of inflection for f. By the first derivative test for a point of inflection, there is $\delta > 0$ with $\delta < r$ such that (i) f' is increasing on $(c - \delta, c)$ and f' is decreasing on $(c, c + \delta)$, or (ii) vice-versa. Suppose we are in Case (i). Then for all $x, y \in (c - \delta, c)$ with $x < y$ and for all $z, w \in (c, c + \delta)$ with $z < w$, we must have

$$f'(x) \leq f'(y) \quad \text{and} \quad f'(z) \geq f'(w).$$

Since $f''(c)$ exists, f' is continuous at c. Hence taking limit as $y \to c^-$ and as $z \to c^+$, we see that $f'(x) \leq f'(c)$ for all $x \in (c - \delta, c + \delta)$. Thus f' has a local maximum at c. Consequently, $f''(c) = 0$. The proof in Case (ii) is similar. □

The condition '$f''(c) = 0$' is not sufficient to conclude that c is a point of inflection for f. Example: Let $f(x) := x^4$ for $x \in \mathbb{R}$. Then f is twice differentiable at 0 and $f''(0) = 0$. But 0 is not a point of inflection for f.

(Sufficient condition for a point of inflection)

Let c be an interior point of I, and let $f : I \to \mathbb{R}$. Suppose f is thrice differentiable at c. If $f''(c) = 0$ and $f'''(c) \neq 0$, then c is point of inflection for f.

Proof: Without loss of generality, suppose $f'''(c) < 0$. Then

$$\lim_{x \to c} \frac{f''(x)}{x - c} = \lim_{x \to c} \frac{f''(x) - f''(c)}{x - c} = f'''(c) < 0.$$

Hence there is $\delta > 0$ such that $f''(x)/(x - c) < 0$ for all $x \in (c - \delta, c) \cup (c, c + \delta)$. Now $f'' > 0$ on $(c - \delta, c)$, and $f'' < 0$ on $(c, c + \delta)$. By the second derivative test for a point of inflection, c is point of inflection for f. □

The condition '$f'''(c) \neq 0$' is not necessary for c to be a point of inflection of f. Example: Let $f(x) := x^5$ for $x \in \mathbb{R}$. Then 0 is a point of inflection for f, but $f'''(0) = 0$.

To conclude the topic of local extrema and points of inflection, let us consider the following example.

Let $f(x) := x^4 - 4x^3$ for $x \in \mathbb{R}$. Then

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \iff x \in \{0, 3\},$$

$$f''(x) = 12x^2 - 24x = 12x(x-2) = 0 \iff x \in \{0, 2\}.$$

Also, $f'''(x) = 24(x-1)$ for $x \in \mathbb{R}$.

Since $f''(0) = 0 = f''(2)$ and $f'''(0) = -f'''(2) = -24 \neq 0$, we see that 0 and 2 are points of inflection for f.

Since $f' \leq 0$ on $(-\infty, 3)$ and $f' \neq 0$ on $(-\infty, 0) \cup (0, 3)$, the function f is strictly decreasing on $(-3, 3)$. Hence f does not have a local extremum at 0. Further, since $f'(3) = 0$ and $f''(3) = 36 > 0$, the function f has a local minimum at 3.

Geometric properties of a function

Let I be an interval. We have so far considered the following geometric properties of a function from I to $\mathbb{R}$.

- Boundedness, and the bounds being attained
- Intermediate value property
- Monotonicity
- Convexity/Concavity
- Having global extrema, having local extrema
- Having points of inflection

Further, we have given analytical conditions on the function (such as continuity, nonnegativity of the derivatives, the vanishing of the derivatives etc.) which imply the above-mentioned geometric properties.
Next, we shall now prepare the groundwork to be able to talk about asymptotes.