MA 105 : Calculus
Division 1, Lecture 7

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Recap of the previous lecture

- Convexity and concavity, revisited
- Derivative Tests for Convexity and concavity
- Critical points and global extrema
- Local extrema and derivatives
- First Derivative Test and Second Derivative Test for Local Extrema
- Point of inflection. Derivative Tests
- Necessary condition for a point of inflection
- Sufficient condition for a point of inflection
- Summary of geometric properties of functions
- Brief discussion of asymptotes and "infinite limits"

'Infinite limits' and 'limits at infinity'

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) **tends to infinity** and write $a_n \to \infty$ if for every $\alpha > 0$, there is $n_0 \in \mathbb{N}$ such that $a_n > \alpha$ for all $n \geq n_0$.

We say that (a_n) **tends to minus infinity** and write $a_n \to -\infty$ if for every $\beta < 0$, there is $n_0 \in \mathbb{N}$ such that $a_n < \beta$ for all $n \geq n_0$.

Note: Since ∞ and $-\infty$ are not numbers, we avoid writing $\lim_{n\to\infty} a_n = \infty$ when $a_n \to \infty$. Likewise when $a_n \to -\infty$.

Examples: (i) Let $a_n := n^2$ for $n \in \mathbb{N}$. To see that $a_n \to \infty$, let $\alpha > 0$. Then $n^2 > \alpha$ if $n > \alpha^{1/2}$. So take $n_0 := [\alpha^{1/2}] + 1$.

(ii) Let $a_n := -n^{1/3}$ for $n \in \mathbb{N}$. To see that $a_n \to -\infty$, let $\beta < 0$. Then $-n^{1/3} < \beta$ if $n > -\beta^3$. So take $n_0 := [-\beta^3] + 1$.

Limits as $x \to \infty$ and as $x \to -\infty$

Definition

Suppose $D \subset \mathbb{R}$ is such that $(a, \infty) \subset D$ for some $a \in \mathbb{R}$. For a function $f : D \to \mathbb{R}$, we say that a **limit** of $f(x)$ exists as $x \to \infty$ if there is $\ell \in \mathbb{R}$ such that
(x_n) is a sequence in D and $x_n \to \infty \implies f(x_n) \to \ell$.

Notation: $f(x) \to \ell$ as $x \to \infty$ or $\lim_{x \to \infty} f(x) = \ell$.

Similarly, we define $f(x) \to \ell$ as $x \to -\infty$ or $\lim_{x \to -\infty} f(x) = \ell$.

Examples:

Let $f(x) := \dfrac{1}{x}$ for $x > 0$. Then $f(x) \to 0$ as $x \to \infty$.

Let $f(x) := \dfrac{x+1}{x-1}$ for $x < 1$. Then $f(x) \to 1$ as $x \to -\infty$.

$f(x) \to \infty$ and $f(x) \to -\infty$

Definition

Let $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that there is $r > 0$ with $(c - r, c) \cup (c, c + r) \subset D$. We say that $f(x)$ tends to ∞ as $x \to c$, and write $f(x) \to \infty$ as $x \to c$ if (x_n) is a sequence in D, $x_n \neq c$ and $x_n \to c \implies f(x_n) \to \infty$.

By restricting to sequences (x_n) in D with $x_n < c$ for all $n \in \mathbb{N}$, we can define $f(x) \to \infty$ as $x \to c^-$. Similarly, we can define $f(x) \to \infty$ as $x \to c^+$, and furthermore, we can also define $f(x) \to -\infty$ as $x \to c$ (or as $x \to c^-$ or as $x \to c^+$).

Note: As in the case of sequences, we shall not write $\lim_{x \to c} f(x) = \infty$ when $f(x) \to \infty$ as $x \to c$. Likewise, we shall not write $\lim_{x \to c} f(x) = -\infty$ when $f(x) \to -\infty$ as $x \to c$.

Example: Let $f(x) := 1/x$ for $x \in \mathbb{R} \setminus \{0\}$. Then $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^-$.

Asymptotes

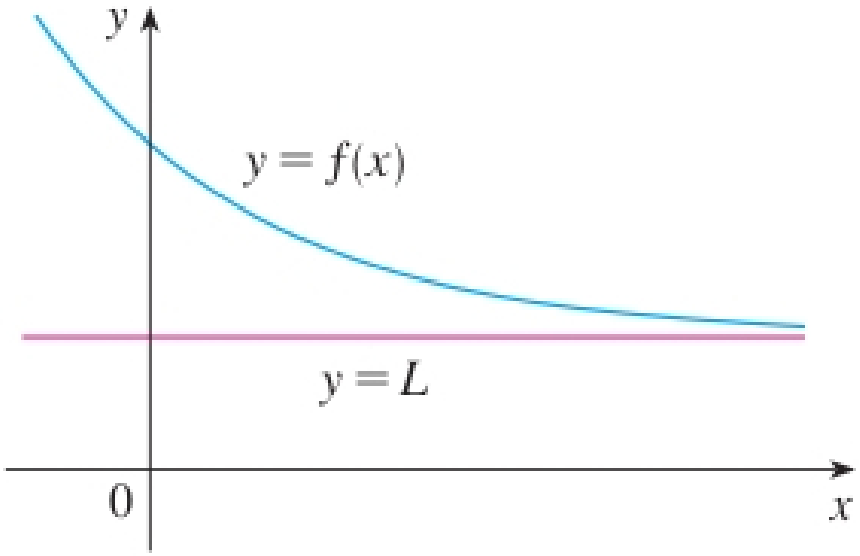
Let $D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$. There are three possible types of asymptotes: horizontal, oblique and vertical. They all imply nearness of the curve $y = f(x)$ to a straight line.

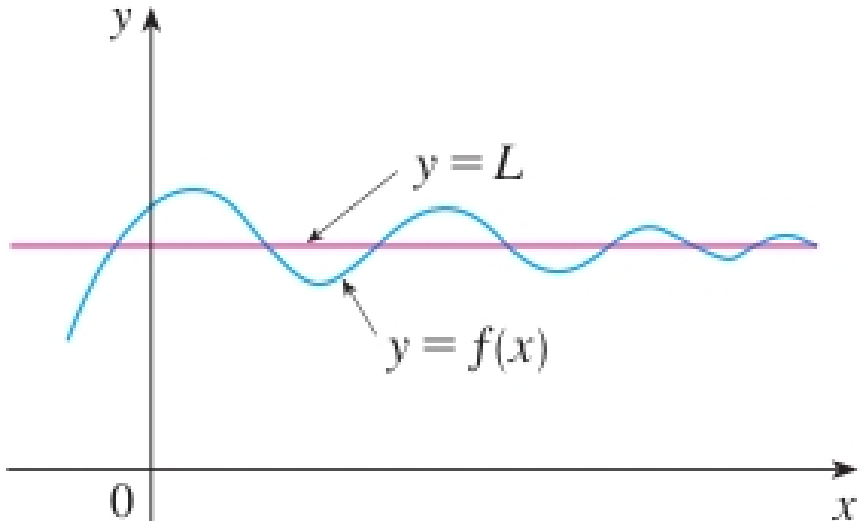
- A straight line $y = b$ is called a **horizontal asymptote** of the curve $y = f(x)$ if

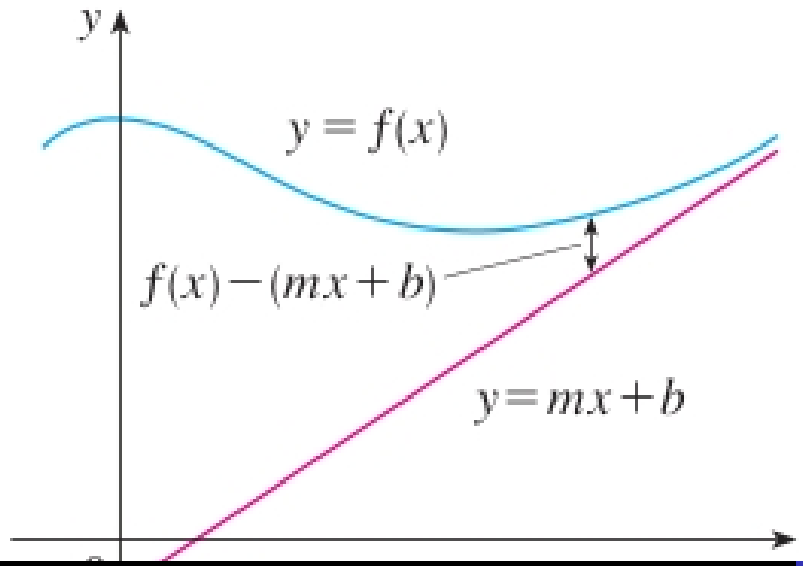
  $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.
- A straight line $y = ax + b$, where $a \neq 0$, is called an **oblique asymptote** of the curve $y = f(x)$ if

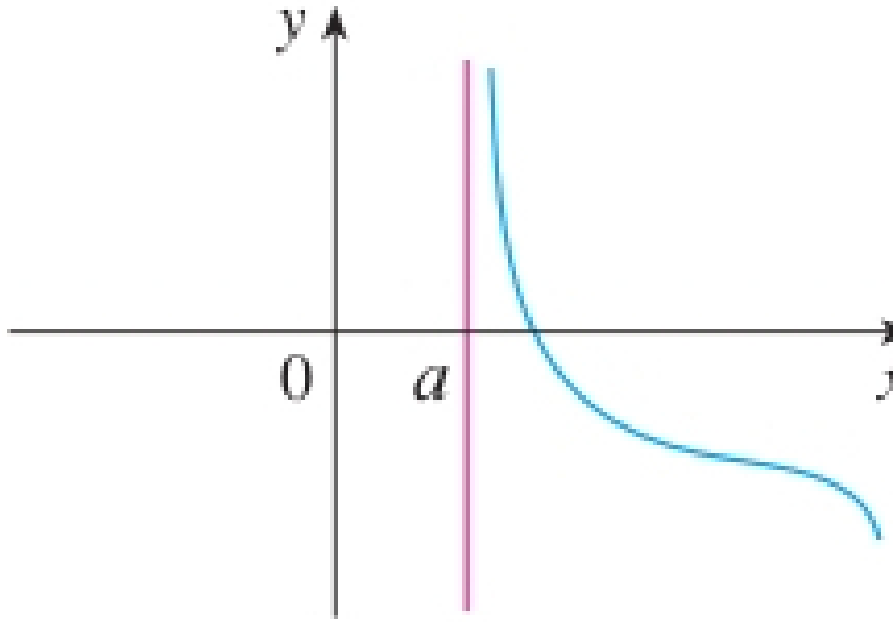
  $\lim_{x\to\infty} (f(x) - ax - b) = 0$ or $\lim_{x\to-\infty} (f(x) - ax - b) = 0$.
- A line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if
  $f(x) \to \infty$ as $x \to a^-$, or $f(x) \to -\infty$ as $x \to a^-$, or
  $f(x) \to \infty$ as $x \to a^+$, or $f(x) \to -\infty$ as $x \to a^+$.

Illustrations of asymptotes









Examples of asymptotes

Example: Consider $f : (-\infty, 0) \cup (1, \infty) \to \mathbb{R}$ defined by

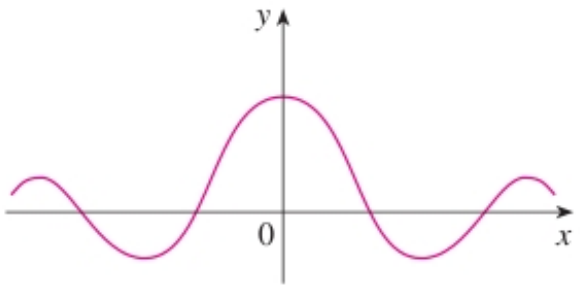
$$f(x) := \begin{cases} (3x^2 + 4x + 1)/x & \text{if } x < 0, \\ (2x - 1)/(x - 1) & \text{if } x > 1. \end{cases}$$

- For $x > 1$, we have $f(x) = 2 + 1/(x - 1)$, and so $\lim_{x \to \infty} f(x) = 2$. Hence the straight line $y = 2$ is a horizontal asymptote of the curve $y = f(x)$.
- For $x < 0$, $f(x) = 3x + 4 + (1/x)$, and so $\lim_{x \to -\infty} (f(x) - (3x + 4)) = 0$. Hence the straight line $y = 3x + 4$ is an oblique asymptote of the curve $y = f(x)$.
- $f(x) \to \infty$ as $x \to 1^+$. Hence the straight line $x = 1$ is a vertical asymptote of the curve $y = f(x)$.
- $f(x) \to -\infty$ as $x \to 0^-$. Hence the straight line $x = 0$ is a vertical asymptote of the curve $y = f(x)$.

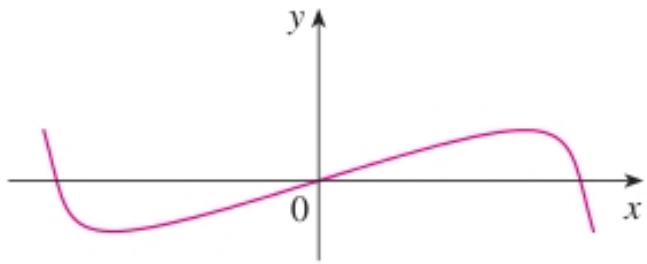
Exercise: Sketch the graph of f.

Even, odd and periodic functions

A function $f : (-a, a) \to \mathbb{R}$ is called **even** if $f(-x) = f(x)$, and it is called **odd** if $f(-x) = -f(x)$ for all $x \in (-a, a)$.

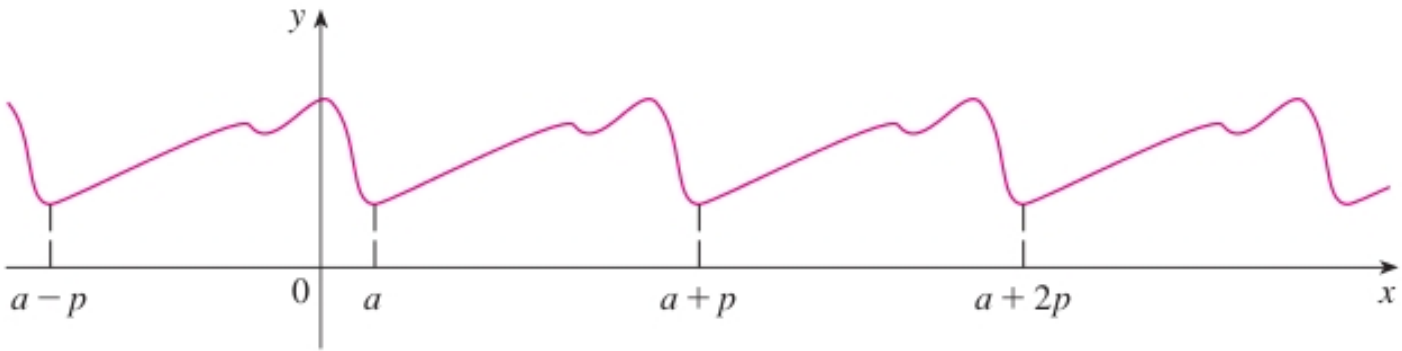


(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

If $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = f(x + p)$ for a fixed $p > 0$ and all $x \in \mathbb{R}$, then f is called **periodic** and p is called a **period** of f.



Curve Sketching: Tips

- Look for symmetries of the given function f, that is, find out if f is even/ odd/ periodic?
- Look for intercepts, that is, the intersections of the curve with co-ordinate axes.
- Look for points (if any) where f is not defined. If such a point exists, study the behaviour of f near that point.
- Locate critical points, local maxima, local minima of f.
- Find intervals on which f is increasing or decreasing.
- Find intervals on which f is convex or concave.
- Locate points of inflection of f.
- Determine the behaviour of $f(x)$ as $x \to \infty$ or $x \to -\infty$.
- Find the asymptotes, if any.

Curve Sketching: An Example

Example: Let $f(x) := (x^2-1)/(x+2)$ for $x \in \mathbb{R}$ with $x \neq -2$.

- f is not even, not odd, not periodic.
- Intercepts of f: $(0,-1/2), (1,0), (-1,0)$.
- $f'(x) = (x+2+\sqrt{3})(x+2-\sqrt{3})/(x+2)^2$ and $f''(x) = 6/(x+2)^3$ for $x \neq -2$.
- f is increasing on $(-\infty, -2-\sqrt{3})$ and on $(-2+\sqrt{3}, \infty)$. f is decreasing on $(-2-\sqrt{3}, -2)$ and on $(-2, -2+\sqrt{3})$.
- f is concave on $(-\infty, -2)$ and convex on $(-2, \infty)$.
- Since $f'(-2-\sqrt{3}) = 0 = f'(-2+\sqrt{3})$ and $f''(-2-\sqrt{3}) = -2/\sqrt{3} < 0$, $f''(-2+\sqrt{3}) = 2/\sqrt{3} > 0$, f has a local maximum at $-2-\sqrt{3}$ and f has a local minimum at $-2+\sqrt{3}$. Note: $f(-2 \pm \sqrt{3}) = -4 \pm 2\sqrt{3}$.
- There is no point of inflection for f.
- The straight line $x = -2$ is a vertical asymptote.
- Since $f(x) = x - 2 + 3/(x+2)$ for $x \neq -2$, $y = x - 2$ is an oblique asymptote.

Mean Value Theorem Revisited

Recall the most important result in differential calculus which we proved in Lecture 7:

Theorem (Mean Value Theorem: MVT)

Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

Let us now consider an extension of the above result.

Theorem (Extended Mean Value Theorem)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be such that f' exists on $[a, b]$. Suppose f' is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2.$$

Proof: For $x \in [a, b]$, let $P(x) := f(a) + f'(a)(x - a)$. Define

$$F(x) := f(x) - P(x) - \frac{f(b) - P(b)}{(b - a)^2}(x - a)^2 \quad \text{for } x \in [a, b].$$

Then $F(a) = 0 = F(b)$. By the Rolle theorem, there exists $c_1 \in (a, b)$ such that $F'(c_1) = 0$. Since $F'(a) = 0$ also, there is $c \in (a, c_1)$ such that $F''(c) = 0$. This gives the desired result.

More generally, we can obtain the following important result.

Theorem (Taylor Theorem)

Let $a < b$ and $n \in \mathbb{N}$. Let $f : [a, b] \to \mathbb{R}$ be such that $f', \ldots, f^{(n)}$ exist on $[a, b]$. Suppose $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + R_n$ where $R_n := f^{(n+1)}(c)(b - a)^{n+1}/(n + 1)!$.

Proof (Sketch): For $x \in [a, b]$, let

$$P(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Consider $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) := f(x) - P(x) - \frac{f(b) - P(b)}{(b-a)^{n+1}}(x-a)^{n+1} \quad \text{for } x \in [a, b].$$

Then $F(a) = 0 = F(b)$. By the Rolle theorem, there exists $c_1 \in (a, b)$ such that $F'(c_1) = 0$. Further,

$$F'(a) = 0 = F'(c_1)$$

and so there is $c_2 \in (a, c_1)$ such that $F''(c_2) = 0$. Now

$$F''(a) = 0 = F''(c_2)$$

and so there is $c_3 \in (a, c_2)$ such that $F'''(c_3) = 0$. Continuing in this way, we see that there is $c = c_{n+1} \in (a, b)$ such that $F^{(n+1)}(c) = 0$, and this yields the desired result.

A result similar to Taylor's theorem holds with a and b interchanged. [This can be seen by applying Taylor's Theorem to $g : [a, b] \to \mathbb{R}$ defined by $g(x) := f(a + b - x)$.] As a consequence, we obtain the following result.

Taylor Formula: Let $n \in \mathbb{N}$. If I is an interval, $a \in I$, and $f : I \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n)}$ exist on I and $f^{(n+1)}$ exists at every interior point of I, then for $x \in I$, $x \neq a$, there is c_x between a and x such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$R_n(x) := \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}.$$

Here the polynomial $P_n(x)$ is of degree $\leq n$, and it can be considered as an 'approximation' of $f(x)$ of order n. It is called the nth **Taylor polynomial** of f around a, and $R_n(x)$ is called the nth **Lagrange remainder** of f around a.

Example:

Let $f(x) := \sin x$ for $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \ldots$, and in general,

$$f^{(k)}(x) = \begin{cases} (-1)^{k/2} \sin x & \text{if } k \text{ is even,} \\ (-1)^{(k-1)/2} \cos x & \text{if } k \text{ is odd.} \end{cases}$$

Fix $n \in \mathbb{N}$. Then the nth Taylor polynomial of f around 0 is

$$\begin{aligned} P_n(x) &= f(0) + f'(0)(x-0) + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= \begin{cases} x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots (-1)^{(n-1)/2} \dfrac{x^n}{n!} & \text{if } n \text{ is odd,} \\ x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots (-1)^{(n-2)/2} \dfrac{x^{n-1}}{(n-1)!} & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

for $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. Then the nth Lagrange remainder around 0 is

$$R_n(x) = \begin{cases} (-1)^{n/2} \dfrac{x^{n+1}}{(n+1)!} \cos c_x & \text{if} \quad n \text{ is even,} \\ \\ (-1)^{(n+1)/2} \dfrac{x^{n+1}}{(n+1)!} \sin c_x & \text{if} \quad n \text{ is odd,} \end{cases}$$

where c_x is between 0 and x. As $|\cos c_x| \leq 1$ & $|\sin c_x| \leq 1$,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

Hence for $x \in \mathbb{R}$,

$$\sin x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{(k-1)} \frac{x^{2k-1}}{(2k-1)!}, \text{ that is,}$$

$$\sin x = \sum_{k=1}^{\infty} (-1)^{(k-1)} \frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

The RHS is the **Taylor series** of the function sin around 0.

Riemann Integration

Let $a, b \in \mathbb{R}$ with $a < b$, and let f be a bounded function defined on $[a, b]$. Let us first assume that $f \geq 0$ on $[a, b]$. We are interested in assigning a meaning to the concept of 'area' of the region that lies under the graph of f, between the lines $x = a$, $x = b$, and above the x-axis, that is, of the region

$$R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

A **partition** of $[a, b]$ is a finite ordered set $P := \{x_0, x_1, \ldots, x_n\}$ of points in $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $n \in \mathbb{N}$.

The points in a partition divide the interval $[a, b]$ into nonoverlapping subintervals, $[x_0, x_1]$, $\ldots$, $[x_{n-1}, x_n]$. We call $[x_{i-1}, x_i]$ the ith **subinterval** of the partition P.

Example: For $n \in \mathbb{N}$, the set

$$\mathrm{P}_n := \left\{ a,\ a + \frac{(b-a)}{n}, \ldots, a + \frac{(n-1)(b-a)}{n},\ b \right\}$$

is a partition of $[a, b]$ into *n* equal parts, each of length $(b-a)/n$. If $n := 1$, then $\mathrm{P}_1 := \{a, b\}$ is the trivial partition.

Let $f : [a, b] \to \mathbb{R}$ be any bounded function, and define

$m(f) := \inf\{f(x) : x \in [a, b]\}$ and $M(f) := \sup\{f(x) : x \in [a, b]\}$.

Given a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, let us define

$$m_i(f) := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$M_i(f) := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, \ldots, n$.

Lower sum and upper sum

Clearly, $m(f) \leq m_i(f) \leq M_i(f) \leq M(f)$ for $i = 1, \ldots, n$. Let

$$L(P, f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}),$$

$$U(P, f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

$L(P, f)$ is called the **lower sum** of f with respect to P and $U(P, f)$ is called the **upper sum** of f with respect to P.

Since $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, we obtain

$$m(f)(b - a) \leq L(P, f) \leq U(P, f) \leq M(f)(b - a).$$

Underlying assumption: The area of a rectangle $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$ is equal to $(x_i - x_{i-1})(y_i - y_{i-1})$.

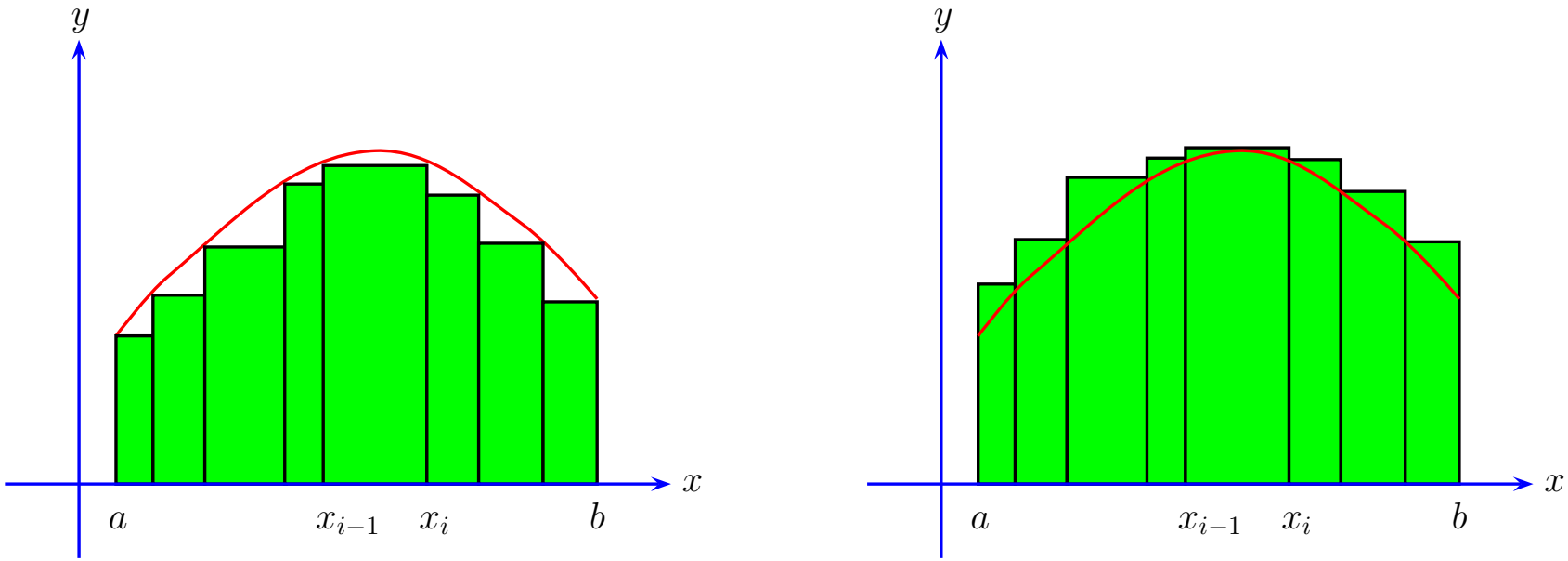


Figure: Approximating the 'area' under a curve by a lower sum which is sum of the areas of the inscribed rectangles, and by an upper sum which is the sum of the areas of the circumscribing rectangles.