MA 105 : Calculus
Division 1, Lecture 9

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Recap of the previous lecture

- Riemann integration
  - Partition of an interval, Subintervals of a partition
  - Upper sum and Lower sum of a function w.r.t a partition
  - Upper integral and Lower integral
  - Definition of integrable functions and of the Riemann integral
  - Example of integrable and non-integrable functions
- Results about Riemann integration
  - Criterion for integrability: Riemann condition. Corollary involving sequences of partitions
  - Domain additivity
  - (Classes of) integrable functions
  - Algebraic and Order Properties
  - Fundamental Theorem of Calculus: Statement of Part I

The Fundamental Theorem of Calculus (FTC)

(The Fundamental Theorem of Calculus: Part I)

Let f be integrable on $[a, b]$. For $x \in [a, b]$, define

$$F(x) := \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Proof: Let $\alpha \in \mathbb{R}$ satisfy $|f| \leq \alpha$ on $[a, b]$, and let $c \in [a, b]$. By the domain additivity of the Riemann integral,

$$|F(x) - F(c)| = \left| \int_c^x f(t)\, dt \right| \leq \alpha |x - c| \quad \text{for all } x \in [a, b].$$

If $x_n \to c$, then $F(x_n) \to F(c)$. Hence F is continuous at c.

Suppose f is continuous at $c \in [a,b]$. For $x \in [a,b]$, $x \neq c$, define $F_1(x) := \big(F(x) - F(c)\big)/(x-c)$, and $F_1(c) := f(c)$.

We show that the function F_1 is continuous at c. For $x \neq c$,

$$F_1(x) - f(c) = \left(\frac{1}{x-c}\int_c^x f(t)\,dt\right) - f(c) = \frac{1}{x-c}\int_c^x \big(f(t) - f(c)\big)\,dt$$

since $f(c) = \int_c^x f(c)\,dt/(x-c)$. Let $\epsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ with

$$t \in [a,b] \quad \text{and} \quad |t-c| < \delta \implies |f(t) - f(c)| < \epsilon.$$

Hence if $x \in [a,b]$ and $0 < |x-c| < \delta$, then

$$|F_1(x) - f(c)| = \frac{\left|\int_c^x \big(f(t) - f(c)\big)\,dt\right|}{|x-c|} \leq \frac{\epsilon\,|x-c|}{|x-c|} = \epsilon.$$

By the C-lemma, F' exists at c and $F'(c) = F_1(c) = f(c)$. □

Note: Suppose f is continuous on $[a, b]$. Then F is differentiable on $[a, b]$ and $F'(x) = f(x)$ for $x \in [a, b]$, that is,

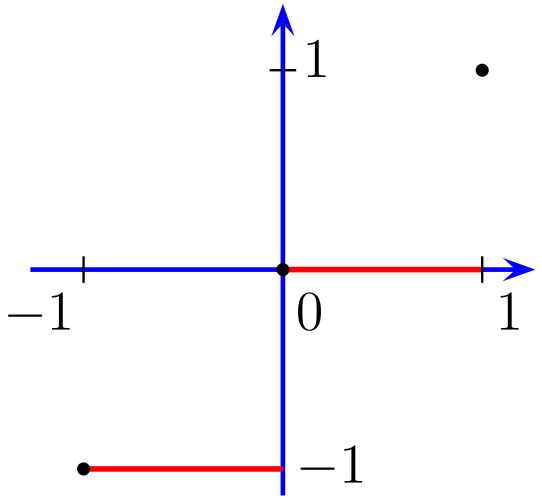
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x) \quad \text{for all } x \in [a, b].$$

Thus F is in fact **continuously differentiable** on $[a, b]$.

Examples:

(i) Let $f(x) := [x]$ for $x \in [-1, 1]$. Define $F(x) := \int_{-1}^x [t]dt$ for $x \in [-1, 1]$. Check that $F(x) = -1 - x$ if $x \in [-1, 0]$, and $F(x) = F(0) = -1$ if $x \in (0, 1]$. Note: f is not continuous on $[-1, 1]$, but F is continuous on $[-1, 1]$.

(ii) Let $f(x) := |x|$ for $x \in [-1, 1]$. Define $F(x) := \int_{-1}^x |t|dt$ for $x \in [-1, 1]$. Check that $F(x) = \int_{-1}^x (-t)dt = (1 - x^2)/2$ if $x \in [-1, 0]$, and $F(x) = F(0) + \int_0^x t\, dt = (1 + x^2)/2$ if $x \in (0, 1]$. Note: f is not differentiable on $[-1, 1]$, but F is differentiable; in fact, $F'(x) = |x| = f(x)$ for $x \in [-1, 1]$.



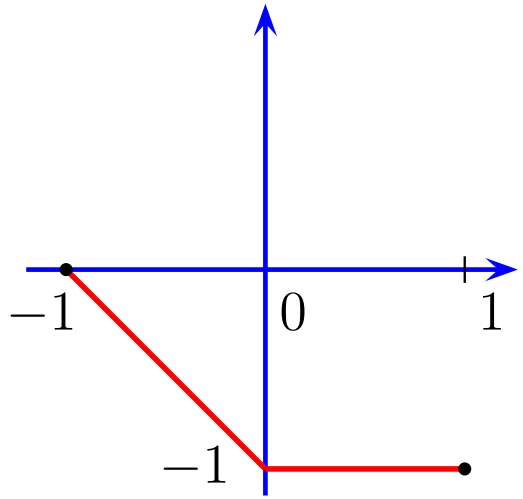
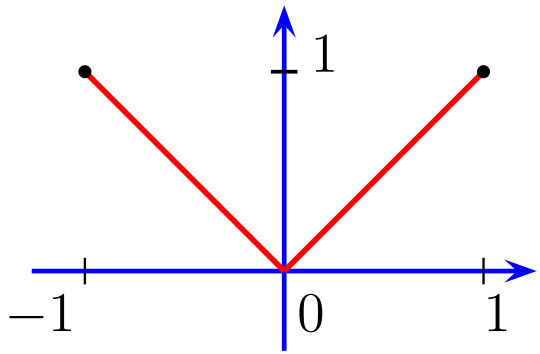


Figure: $y = [x]$ and $y = \int_{-1}^{x} [t]dt$



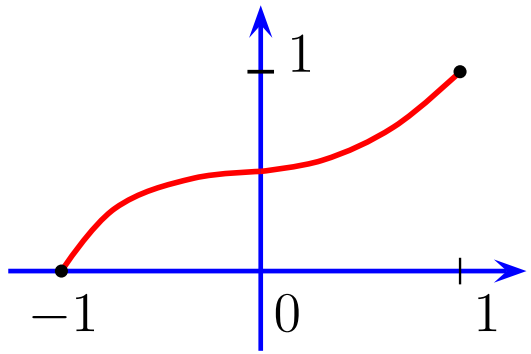


Figure: $y = |x|$ and $y = \int_{-1}^{x} |t|dt$

Antiderivative

Definition

Let I be an interval containing more than one point, and consider a function $f : I \to \mathbb{R}$. We say that f has an antiderivative on I if there is a differentiable function $F : I \to \mathbb{R}$ such that $F' = f$. Such a function F is called an antiderivative or a primitive of f, or an indefinite integral of f.

Lemma: If I is an interval and $f : I \to \mathbb{R}$ has an antiderivative F, then F is unique up to addition by a constant.

Proof: If $F' = f = G'$, then $(F - G)' = 0$ on the interval I, and hence $F - G$ is a constant function (by the MVT). □

As a consequence of the FTC (Part I), we see that if $f : [a, b] \to \mathbb{R}$ is continuous, then f has an antiderivative given by $F(x) := \int_a^x f(t)dt$ for $x \in [a, b]$. (Note: $F(a) = 0$.)

(The Fundamental Theorem of Calculus: Part II)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$. Then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof: Let $P := \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the MVT, there is $c_i \in (x_{i-1}, x_i)$ for $i = 1, \ldots, n$ such that

$$f(b) - f(a) = \sum_{i=1}^{n} \big(f(x_i) - f(x_{i-1})\big) = \sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}).$$

Hence $L(P, f') \le f(b) - f(a) \le U(P, f')$ for every partition P of $[a, b]$, and so $\int_a^b f' = L(f') \le f(b) - f(a) \le U(f') = \int_a^b f'$. Thus $\int_a^b f' = f(b) - f(a)$. □

Compare the conclusions of the MVT and the FTC, Part II:

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{and} \quad \frac{f(b)-f(a)}{b-a} = \frac{1}{b-a}\int_a^b f'(x)dx.$$

What does the Fundamental Theorem of Calculus say?

(i) Part I: If f is continuous on $[a,b]$, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{for all } x \in [a,b].$$

(ii) Part II: If f' exists and is integrable on $[a,b]$, then

$$\int_a^b \left(\frac{d}{dx}f(x)\right)dx = f(b)-f(a).$$

Thus the operations of differentiation and integration are 'inverse' to each other.

Side Remark: Some Pathologies

This optional slide (which was not shown during the lecture) elaborates on some remarks I made and also contains answers to some of the questions asked by the students during or after the class. The material here will not have bearing on any of the exams.

- An integrable function need not have an antiderivative. In fact, if $f : [a, b] \to \mathbb{R}$ has an antiderivative, then f has the IVP on $[a, b]$ (see, e.g, Theorem 4.16 in [GL-1]). In particular, $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = [x]$ does not have an antiderivative, although f is clearly integrable.

- If $f : [a, b] \to \mathbb{R}$ has an antiderivative F, then f need not be integrable. In fact, f need not even be bounded. Example: $f = F'$ with $F : [-1, 1] \to \mathbb{R}$ given by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$ [See also Remark 6.25 (iii) in [GL-1].]

- Composites $g \circ f$ of integrable functions $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ need not be integrable, unless g is continuous. See, e.g., Example 6.17 and Remark 6.18 in [GL-1].

Consequences of the FTC

(Integration by parts)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable. Suppose $g : [a, b] \to \mathbb{R}$ is integrable and has an antiderivative G on $[a, b]$. Then

$$\begin{aligned}\int_a^b f(x)g(x)dx &= f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx \\ &= f(x)G(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)G(x)\, dx.\end{aligned}$$

Proof: For $x \in [a, b]$, define $H(x) := f(x)G(x)$, so that $H'(x) = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x)$. Then H' is integrable on $[a, b]$. Hence by the FTC (Part II),

$$\int_a^b H'(x)dx = H(b) - H(a) = f(b)G(b) - f(a)G(a). \quad \square$$

(Integration by substitution)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function such that $\phi([\alpha, \beta]) = [a, b]$. Then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable, and

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(\phi(t)\big)\phi'(t)dt.$$

In particular, if $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$, then

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))|\phi'(t)|dt.$$

Proof: Define $F : [a, b] \to \mathbb{R}$ by $F(x) := \int_a^x f(u)du$ for $x \in [a, b]$, and $H : [\alpha, \beta] \to \mathbb{R}$ by $H(t) := F(\phi(t))$ for $t \in [\alpha, \beta]$.

By the chain rule and by the FTC (Part I),

$$H'(t) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t) \quad \text{for } t \in [\alpha, \beta].$$

Then H' is integrable on $[\alpha, \beta]$. Hence by the FTC (Part II),

$$\int_\alpha^\beta H'(t)dt = H(\beta) - H(\alpha) = F(\phi(\beta)) - F(\phi(\alpha)) = \int_{\phi(\alpha)}^{\phi(\beta)} f(u)du,$$

that is,

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_\alpha^\beta f\big(\phi(t)\big)\phi'(t)dt.$$

Now suppose $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$. By the Intermediate Value Property of ϕ', either $\phi'(t) > 0$ for all $t \in (\alpha, \beta)$, or $\phi'(t) < 0$ for all $t \in (\alpha, \beta)$. In the former case, ϕ is strictly increasing on $[\alpha, \beta]$, $\phi(\alpha) = a$, $\phi(\beta) = b$ and $|\phi'| = \phi'$. In the latter case, ϕ is strictly decreasing on $[\alpha, \beta]$, $\phi(\alpha) = b$, $\phi(\beta) = a$ and $|\phi'| = -\phi'$. Hence the desired result follows. □

Examples:

(i) To evaluate $\int_0^1 x\sqrt{1-x}\,dx$, let $f(x) := x$ and $g(x) := \sqrt{1-x}$ for $x \in [0,1]$. Then $f'(x) = 1$ for $x \in [0,1]$. Also, if $G(x) := -(2/3)(1-x)^{3/2}$, then $G'(x) = g(x)$ for $x \in [0,1]$, that is, $G' = g$. Integration by Parts yields

$$\int_0^1 x\sqrt{1-x}\,dx = 0 - 0 - \int_0^1 \left(-\frac{2}{3}\right)(1-x)^{3/2}dx = \frac{2}{3}\int_0^1 (1-x)^{3/2}dx.$$

If we let $F(x) := -(2/5)(1-x)^{5/2}$ for $x \in [0,1]$, then $F'(x) = (1-x)^{3/2}$ for $x \in [0,1]$. By the FTC, Part II, the integral equals $(2/3)(F(1) - F(0)) = (2/3)(2/5) = 4/15$.

(ii) To evaluate $\int_0^1 t\sqrt{1-t^2}\,dt$, let $\phi(t) := 1 - t^2$ for $t \in [0,1]$ and $f(x) := \sqrt{x}$ for $x \in [0,1]$. Then $\phi(0) = 1$, $\phi(1) = 0$, and $\phi'(t) = -2t$ for $t \in [0,1]$. Let $F(x) := (2/3)x^{3/2}$, $x \in [0,1]$. Then $F'(x) = \sqrt{x}$, $x \in [0,1]$. Integration by Substitution yields

$$\int_0^1 t\sqrt{1-t^2}\,dt = \frac{1}{2}\int_0^1 f(\phi(t))|\phi'(t)|dt = \frac{1}{2}\int_0^1 \sqrt{x}\,dx = \frac{1}{3}.$$

(Approximate) Riemann Sum

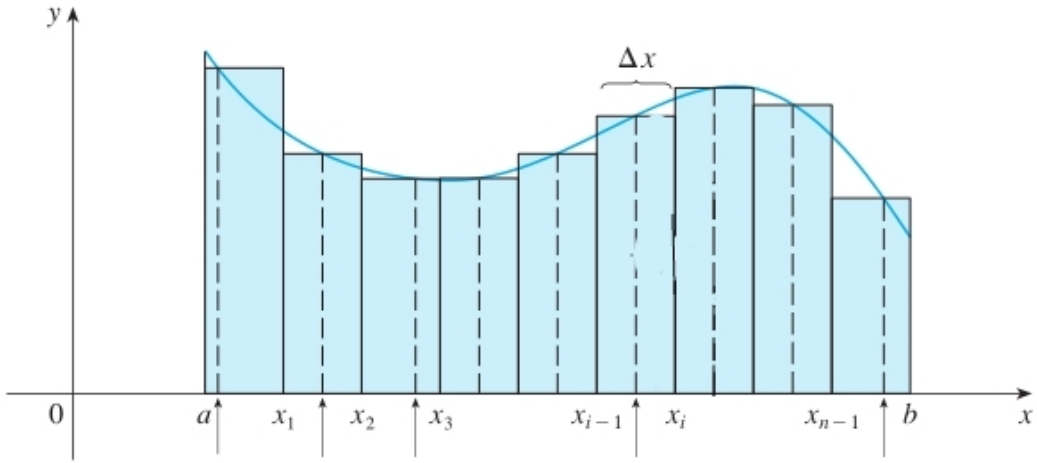
Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, the sum

$$S(P, f) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

is called a Riemann sum for f corresponding to P, where $t_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Since $m_i(f) \leq f(t_i) \leq M_i(f)$ for $i = 1, \ldots, n$, we note that

$$L(P, f) \leq S(P, f) \leq U(P, f) \quad \text{for any } t_i \in [x_{i-1}, x_i].$$



Approximation of a Riemann Integral

The **mesh** of a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ is defined by $\mu(P) := \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$.

Theorem: Let f be integrable on $[a, b]$, and let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition P satisfying $\mu(P) < \delta$.

This improves upon the Riemann condition. Proof is omitted.

Corollary. Let f be integrable on $[a, b]$, and let (P_n) be a sequence of partitions of $[a, b]$ such that $\mu(P_n) \to 0$. Then $U(P_n, f) - L(P_n, f) \to 0$. Further, if $S(P_n, f)$ is a Riemann sum corresponding to P_n and f, then $S(P_n, f) \to \int_a^b f(x)dx$.

Proof: Let $\epsilon > 0$ be given, and let $\delta > 0$ be as in the above theorem. For this δ, we find $n_0 \in \mathbb{N}$ such that $\mu(P_n) < \delta$ for all $n \geq n_0$. Then $U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq n_0$. Thus $U(P_n, f) - L(P_n, f) \to 0$.

Since

$$\begin{aligned} L(P_n, f) &\leq S(P_n, f) \leq U(P_n, f) \quad \text{and} \\ L(P_n, f) &\leq L(f) = \int_a^b f(x)dx = U(f) \leq U(P_n, f), \end{aligned}$$

we see that

$$\left| S(P_n, f) - \int_a^b f(x)dx \right| \leq U(P_n, f) - L(P_n, f) \to 0. \quad \square$$

The above corollary is useful in two ways.

(i) If $\int_a^b f(x)dx$ is known, then we can find $\lim_{n\to\infty} a_n$, if $a_n := S(P_n, f)$ for $n \in \mathbb{N}$ and $\mu(P_n) \to 0$.

(ii) If $\int_a^b f(x)dx$ is not known, then $S(P_n, f)$ gives us an approximation of $\int_a^b f(x)dx$ if $\mu(P_n) \to 0$.

Suppose $f : [a, b] \to \mathbb{R}$ is known to be an integrable function. This is the case, for example, if f is monotonic on $[a, b]$, or if f is discontinuous at only a finite number of points in $[a, b]$.

How can we find the Riemann integral $\int_a^b f(x)dx$?

Try to find an antiderivative of f on $[a, b]$, that is, find a function F on $[a, b]$ such that $F' = f$ on $[a, b]$. In that case, $\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$ by the FTC (Part II).

For instance, let $r \in \mathbb{Q}$, $r \geq 0$, and $f(x) := x^r$ for $x \in [a, b]$. Let $F(x) := x^{r+1}/(r + 1)$ for $x \in [a, b]$. Then $F' = f$ on $[a, b]$. Hence $\int_a^b f(x)dx = F(b) - F(a) = (b^{r+1} - a^{r+1})/(r + 1)$.

Next, suppose $0 \notin [a, b]$. Then the above calculation holds if $r \in \mathbb{Q}$ and $r < 0$, provided $r \neq -1$. If $r := -1$, we do not yet know an antiderivative of the function f. Hence we look for an approximation of $\int_a^b f(x)dx$.

Examples:

(i) For $n \in \mathbb{N}$, let $a_n := \dfrac{1^r + 2^r + \cdots + n^r}{n^{r+1}}$, where $r \in \mathbb{Q}$ and $r \geq 0$. Let us find $\lim_{n\to\infty} a_n$. Note that

$$a_n = \sum_{i=1}^{n} \frac{i^r}{n^{r+1}} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^r = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^r \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := x^r$.
Since f is increasing on $[0,1]$, it is integrable. For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then $a_n = S(P_n, f)$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f) \to \int_0^1 f(x)dx, \text{ that is },$$

$$a_n = \sum_{i=1}^{n} \frac{i^r}{n^{r+1}} \to \int_0^1 x^r \, dx = \frac{1}{r+1}.$$

(ii) Let $f(x) := 1/x$ for $x \in [1, 2]$. (This is the case $r := -1$ mentioned earlier.)

Since f is continuous on $[1, 2]$, it is integrable. For $n \in \mathbb{N}$, let

$$\mathrm{P}_n := \left\{1,\, 1 + \frac{1}{n},\, 1 + \frac{2}{n}, \ldots, 1 + \frac{n-1}{n},\, 2\right\}$$

be the partition of $[1, 2]$ in n equal parts.

Let $t_{n,i} = 1 + (i-1)/n$ for $i = 1, \ldots, n$, so that

$$S(\mathrm{P}_n, f) = \sum_{i=1}^{n} \frac{1}{1 + (i-1)/n} \left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n+i-1}.$$

Since $\mu(\mathrm{P}_n) = 1/n \to 0$, we see that $S(\mathrm{P}_n, f) \to \int_a^b f(x)dx$, that is,

$$\sum_{i=1}^{n} \frac{1}{n+i-1} \to \int_1^2 \frac{1}{x}\, dx \text{ as } n \to \infty.$$