# MA 105 : Calculus
# Division 3, Lecture 01

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# Generalities about the Course

- Instructor: Prof. Sudhir R. Ghorpade, 106B, Maths Dept.
- Lecture Hours:
  - Mon 8.30 -9.25 am
  - Tue 9.30 – 10.25 am
  - Thu 10.35 – 11.30 am

  (all in LA 001)
- Tutorial: Wed, 2 – 2.55 pm, in LT 201 – 206.
- Attendance: Compulsory! (Also, it will be good for you!)
- Office Hours: Mondays 10.30 – 11.30 am.
- Evaluation Plan: Short Quizzes in Tuts (10 %), Common quizzes ($10\% \times 2$), Mid-Sem (30 %), End-Sem (40%).
- More Information:
  - The Booklet
  - Moodle page of the course
  - Instructor's web page, and especially, the course page: http://www.math.iitb.ac.in/∼srg/autumn2019.html

# Text, References, and Acknowledgements

The treatment of calculus in these lectures will be based on the following two books by S. R. Ghorpade and B. V. Limaye, which are published by Springer, New York.

[GL-1] A Course in Calculus and Real Analysis, 2nd Ed., 2018.

[GL-2] A Course in Multivariable Calculus and Analysis, 2010.

Besides these, the other references listed in the booklet, especially the book of Thomas and Finney, may be consulted. For later parts of the course, it is also useful to see the book Basic Multivariable Calculus by J. E. Marsden, A. J. Tromba and A. Weinstein (Springer, New York, 1993).

Acknowledgement: I shall mainly use the slides of Calculus lectures prepared recently by Prof. B. V. Limaye. These slides acknowledged the use of the lecture notes of similar courses given by myself and by Prof. Prachi Mahajan in the past.

# Notation

- $\mathbb{N} := \{1, 2, 3, \ldots\}$
- $\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Q} :=$ the set of all rational numbers
  $= \{m/n : m, n \in \mathbb{Z},\ n \neq 0\}$

**There is no rational number whose square is 2.**

Proof: Suppose not! Then $(p/q)^2 = 2$, that is, $p^2 = 2q^2$ for some $p, q \in \mathbb{Z}$ such that $q \neq 0$, and $p$ and $q$ have no common factor. Now $p^2$ is even, and so $p$ is even. Hence there is an integer $r$ such that $p = 2r$. Then $2q^2 = p^2 = (2r)^2 = 4r^2$, and so $q^2 = 2r^2$. Thus $q^2$ is even, and so $q$ is also even. Thus 2 is a common factor of $p$ and $q$, which is a contradiction.

Let $\mathbb{R}$ denote the set of all real numbers. We will assume the following things about the set $\mathbb{R}$.

The set $\mathbb{Q}$ of all rational numbers is contained in $\mathbb{R}$, and the set $\mathbb{R}$ of all real numbers satisfies

- Algebraic Properties regarding addition and multiplication.
- Order Properties: There is a subset $\mathbb{R}^+$ of $\mathbb{R}$ such that
  (i) Given $a \in \mathbb{R}$, exactly one of the following holds:

  $a \in \mathbb{R}^+$ or $a = 0$ or $-a \in \mathbb{R}^+$

  (ii) $a, b \in \mathbb{R}^+ \Longrightarrow a + b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$.

  Define $a < b$ if $b - a \in \mathbb{R}^+$. Thus $\mathbb{R}^+ = \{x \in \mathbb{R} : 0 < x\}$.
- Completeness Property, which we shall state later.

Elements of the set $\mathbb{R} \setminus \mathbb{Q}$, that is, those real numbers which are not rational numbers, are called **irrational numbers**.

Notation: We write $a \leq b$ if $a < b$ or $a = b$.
Also, we write $a > b$ if $b < a$, and $a \geq b$ if $a > b$ or $a = b$.

## Boundedness of a subset of $\mathbb{R}$

Let $E$ be a subset of $\mathbb{R}$, that is, $E \subset \mathbb{R}$.

- $E$ is called **bounded above** if there is $\alpha \in \mathbb{R}$ such that $x \leq \alpha$ for all $x \in E$.
  Any such $\alpha$ is an **upper bound** of $E$.
- $E$ is called **bounded below** if there is $\beta \in \mathbb{R}$ such that $x \geq \beta$ for all $x \in E$.
  Any such $\beta$ is a **lower bound** of $E$.
- $E$ is **bounded** if it is bounded above and bounded below.

We say that $M$ is the **maximum** of $E$ if $M$ is an upper bound of $E$ and $M \in E$, and we say that $m$ is the **minimum** of $E$ if $m$ is a lower bound of $E$ and $m \in E$.

# Supremum (sup or lub) and Infimum (inf or glb)

Let $E \subset \mathbb{R}$.

- A real number $\alpha$ is called a **supremum** or a **least upper bound** of $E$ if $\alpha$ is an upper bound of $E$ (that is, $x \leq \alpha$ for all $x \in E$), and $\alpha \leq u$ for every upper bound $u$ of $E$.

- A real number $\beta$ is called an **infimum** or a **greatest lower bound** of $E$ if $\beta$ is a lower bound of $E$ (that is, $\beta \leq x$ for all $x \in E$), and $v \leq \beta$ for every lower bound $v$ of $E$.

- If $E$ has a supremum, then it is unique, and it is denoted by sup $E$ or lub $E$. Similarily, if $E$ has an infimum, then it is unique and is denoted by inf $E$ or glb $E$.

Example: Let $E := \{x \in \mathbb{R} : 0 < x \leq 1\}$. Then $\sup E = 1$ and $\inf E = 0$. Also, $\max E = 1$, but $E$ has no minimum.

## Completeness Property of $\mathbb{R}$:

A nonempty subset of $\mathbb{R}$ that is bounded above has a supremum, that is, a least upper bound.

## Consequences of the Completeness Property:

- A nonempty subset $E$ of $\mathbb{R}$ that is bounded below has an infimum, that is, a greatest lower bound.
  In fact, the set $F := \{-x : x \in E\}$ is bounded above, and if $\alpha$ is the lub of $F$, then $\beta := -\alpha$ is the glb of $E$.
- Archimedean Property: Given $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$.
  Proof: Suppose not! Then $n \leq x$ for all $n \in \mathbb{N}$, that is, $x$ is an upper bound of the set $\mathbb{N}$. Let $\alpha := \sup \mathbb{N}$. Then $\alpha - 1$ is not an upper bound of $\mathbb{N}$, that is, there is $n_0 \in \mathbb{N}$ such that $\alpha - 1 < n_0$. But then $\alpha < n_0 + 1 \leq \alpha$, since $(n_0 + 1) \in \mathbb{N}$ and $\alpha$ is an upper bound of $\mathbb{N}$. Thus we obtain $\alpha < \alpha$, which is a contradiction.

- Let $x \in \mathbb{R}$. Applying the Archimedean property to $x$ and $-x$, we see that there are $\ell, n \in \mathbb{N}$ such that $-\ell < x < n$. The largest among finitely many integers $k$ satisfying $-\ell \leq k \leq n$ and also $k \leq x$ is called the **integer part** of $x$, and is denoted by $[x]$ or by $\lfloor x \rfloor$. Note that $[x]$ is the largest integer $\leq x$ and it is characterized by the following two properties: (i) $[x] \in \mathbb{Z}$ and (ii) $x - 1 \leq [x] \leq x$.

- Let $a \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Then there is a unique $b \in \mathbb{R}^+$ such that $b^n = a$. This real number $b$ is called the **positive $n$th root of $a$**, and we denote it by $a^{1/n}$.

  **Example (the positive square root of 2)**:
  Let $S := \{x \in \mathbb{R} : x^2 \leq 2\}$. Then $S$ is nonempty since $1 \in S$ and $S$ is bounded above by 2. By the completeness property of $\mathbb{R}$, let $b := \sup S$. Then $b \geq 1 > 0$. Also, we obtain $b^2 = 2$ by showing that both $b^2 < 2$ and $b^2 > 2$ lead to contradictions. (Verify!) Thus $b := \sqrt{2}$. Since $b \in S$, we see that $b = \max S$.

- Let $a < b$ in $\mathbb{R}$. Then there is a rational number $r$ such that $a < r < b$. In fact, we can consider $r := m/n$, where $n > 1/(b-a)$ and $m := [na] + 1$.
- Let $a < b$ in $\mathbb{R}$. Then there is an irrational number $s$ such that $a < s < b$. In fact, since $a + \sqrt{2} < b + \sqrt{2}$, let $r \in \mathbb{Q}$ be such that $a + \sqrt{2} < r < b + \sqrt{2}$. Then $a < r - \sqrt{2} < b$, where $s := r - \sqrt{2}$ is an irrational number.

Thus we obtain the following important result.

Between any two real numbers, there is a rational number as well as an irrational number.

Optional Exercise: Write down more detailed versions of the "proofs" sketched above. Consult [GL-1], if desired.