MA 105 : Calculus
Division 3, Lecture 10

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Recap of the previous lecture

- First Derivative Test for Local Extrema
- Second Derivative Test for Local Extrema
- Point of inflection.
- Derivative Tests for a point of inflection
- Necessary condition for a point of inflection
- Sufficient condition for a point of inflection
- Summary of geometric properties and analytic conditions
- Oral discussion of asymptotes and infinite limits

'Infinite limits' and 'limits at infinity'

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) **tends to infinity** and write $a_n \to \infty$ if for every $\alpha > 0$, there is $n_0 \in \mathbb{N}$ such that $a_n > \alpha$ for all $n \geq n_0$.

We say that (a_n) **tends to minus infinity** and write $a_n \to -\infty$ if for every $\beta < 0$, there is $n_0 \in \mathbb{N}$ such that $a_n < \beta$ for all $n \geq n_0$.

Note: Since ∞ and $-\infty$ are not numbers, we avoid writing $\lim_{n\to\infty} a_n = \infty$ when $a_n \to \infty$. Likewise when $a_n \to -\infty$.

Examples: (i) Let $a_n := n^2$ for $n \in \mathbb{N}$. To see that $a_n \to \infty$, let $\alpha > 0$. Then $n^2 > \alpha$ if $n > \alpha^{1/2}$. So take $n_0 := [\alpha^{1/2}] + 1$.

(ii) Let $a_n := -n^{1/3}$ for $n \in \mathbb{N}$. To see that $a_n \to -\infty$, let $\beta < 0$. Then $-n^{1/3} < \beta$ if $n > -\beta^3$. So take $n_0 := [-\beta^3] + 1$.

Limits as $x \to \infty$ and as $x \to -\infty$

Definition

Suppose $D \subset \mathbb{R}$ is such that $(a, \infty) \subset D$ for some $a \in \mathbb{R}$. For a function $f : D \to \mathbb{R}$, we say that a **limit** of $f(x)$ exists as $x \to \infty$ if there is $\ell \in \mathbb{R}$ such that
(x_n) is a sequence in D and $x_n \to \infty \implies f(x_n) \to \ell$.

Notation: $f(x) \to \ell$ as $x \to \infty$ or $\lim_{x \to \infty} f(x) = \ell$.

Similarly, we define $f(x) \to \ell$ as $x \to -\infty$ or $\lim_{x \to -\infty} f(x) = \ell$.

Examples:

Let $f(x) := \dfrac{1}{x}$ for $x > 0$. Then $f(x) \to 0$ as $x \to \infty$.

Let $f(x) := \dfrac{x+1}{x-1}$ for $x < 1$. Then $f(x) \to 1$ as $x \to -\infty$.

$f(x) \to \infty$ and $f(x) \to -\infty$

Definition

Let $f : D \to \mathbb{R}$ and $c \in \mathbb{R}$ be such that there is $r > 0$ with $(c - r, c) \cup (c, c + r) \subset D$. We say that $f(x)$ tends to ∞ as $x \to c$, and write $f(x) \to \infty$ as $x \to c$ if (x_n) is a sequence in D, $x_n \neq c$ and $x_n \to c \implies f(x_n) \to \infty$.

By restricting to sequences (x_n) in D with $x_n < c$ for all $n \in \mathbb{N}$, we can define $f(x) \to \infty$ as $x \to c^-$. Similarly, we can define $f(x) \to \infty$ as $x \to c^+$, and furthermore, we can also define $f(x) \to -\infty$ as $x \to c$ (or as $x \to c^-$ or as $x \to c^+$).

Note: As in the case of sequences, we shall not write $\lim_{x \to c} f(x) = \infty$ when $f(x) \to \infty$ as $x \to c$. Likewise, we shall not write $\lim_{x \to c} f(x) = -\infty$ when $f(x) \to -\infty$ as $x \to c$.

Example: Let $f(x) := 1/x$ for $x \in \mathbb{R} \setminus \{0\}$. Then $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^-$.

Asymptotes

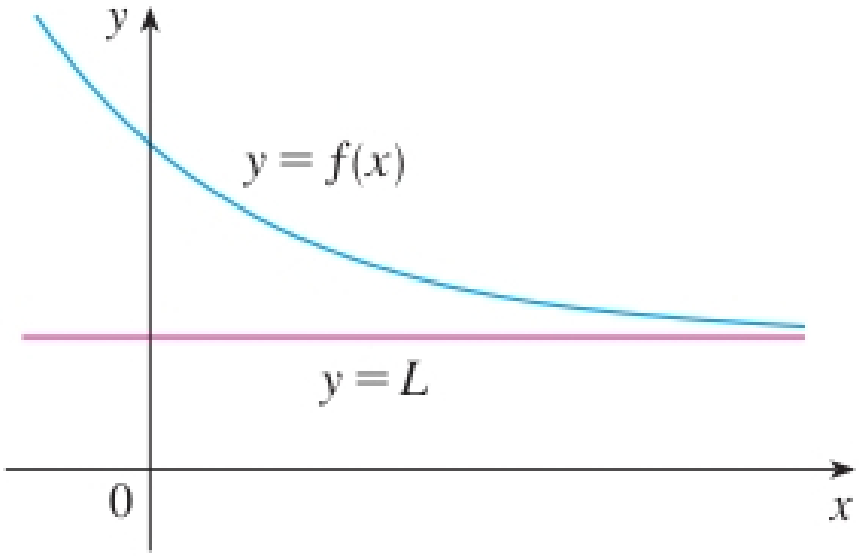
Let $D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$. There are three possible types of asymptotes: horizontal, oblique and vertical. They all imply nearness of the curve $y = f(x)$ to a straight line.

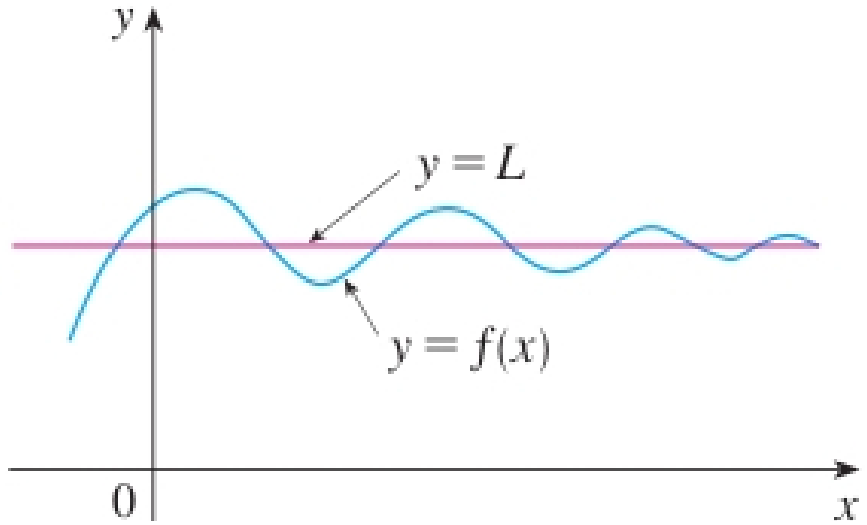
- A straight line $y = b$ is called a **horizontal asymptote** of the curve $y = f(x)$ if

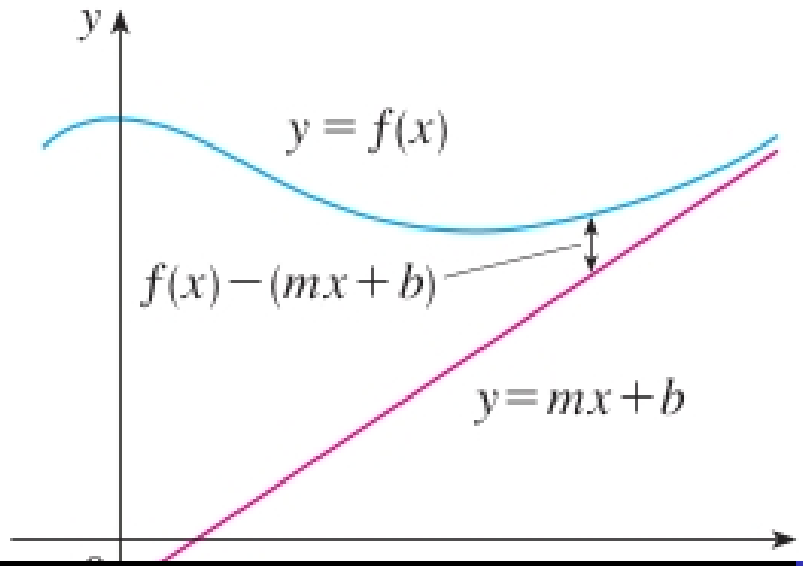
  $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.
- A straight line $y = ax + b$, where $a \neq 0$, is called an **oblique asymptote** of the curve $y = f(x)$ if

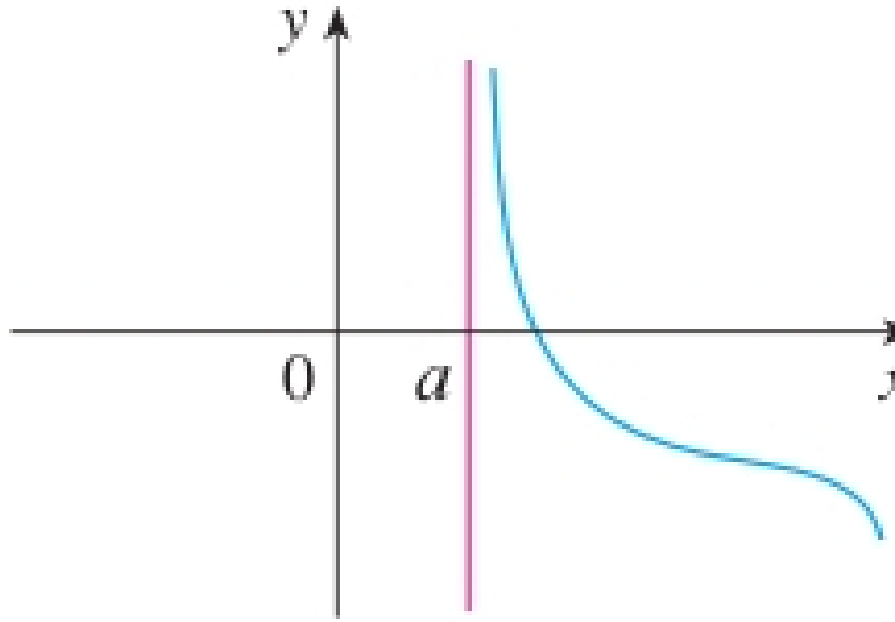
  $\lim_{x\to\infty} (f(x) - ax - b) = 0$ or $\lim_{x\to-\infty} (f(x) - ax - b) = 0$.
- A line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if
  $f(x) \to \infty$ as $x \to a^-$, or $f(x) \to -\infty$ as $x \to a^-$, or
  $f(x) \to \infty$ as $x \to a^+$, or $f(x) \to -\infty$ as $x \to a^+$.

Illustrations of asymptotes









Examples of asymptotes

Example: Consider $f : (-\infty, 0) \cup (1, \infty) \to \mathbb{R}$ defined by

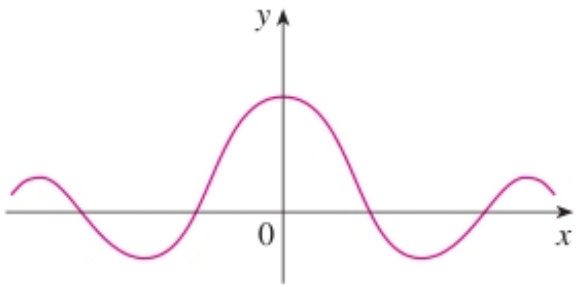
$$f(x) := \begin{cases} (3x^2 + 4x + 1)/x & \text{if } x < 0, \\ (2x - 1)/(x - 1) & \text{if } x > 1. \end{cases}$$

- For $x > 1$, we have $f(x) = 2 + 1/(x - 1)$, and so $\lim_{x \to \infty} f(x) = 2$. Hence the straight line $y = 2$ is a horizontal asymptote of the curve $y = f(x)$.
- For $x < 0$, $f(x) = 3x + 4 + (1/x)$, and so $\lim_{x \to -\infty} (f(x) - (3x + 4)) = 0$. Hence the straight line $y = 3x + 4$ is an oblique asymptote of the curve $y = f(x)$.
- $f(x) \to \infty$ as $x \to 1^+$. Hence the straight line $x = 1$ is a vertical asymptote of the curve $y = f(x)$.
- $f(x) \to -\infty$ as $x \to 0^-$. Hence the straight line $x = 0$ is a vertical asymptote of the curve $y = f(x)$.

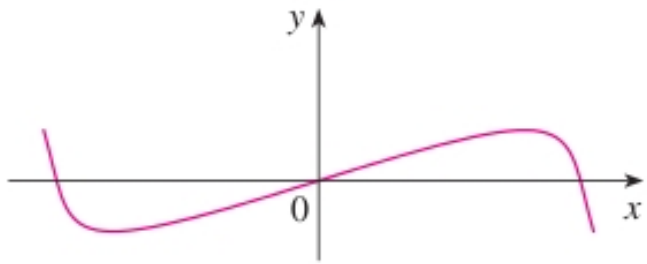
Exercise: Sketch the graph of f.

Even, odd and periodic functions

A function $f : (-a, a) \to \mathbb{R}$ is called **even** if $f(-x) = f(x)$, and it is called **odd** if $f(-x) = -f(x)$ for all $x \in (-a, a)$.

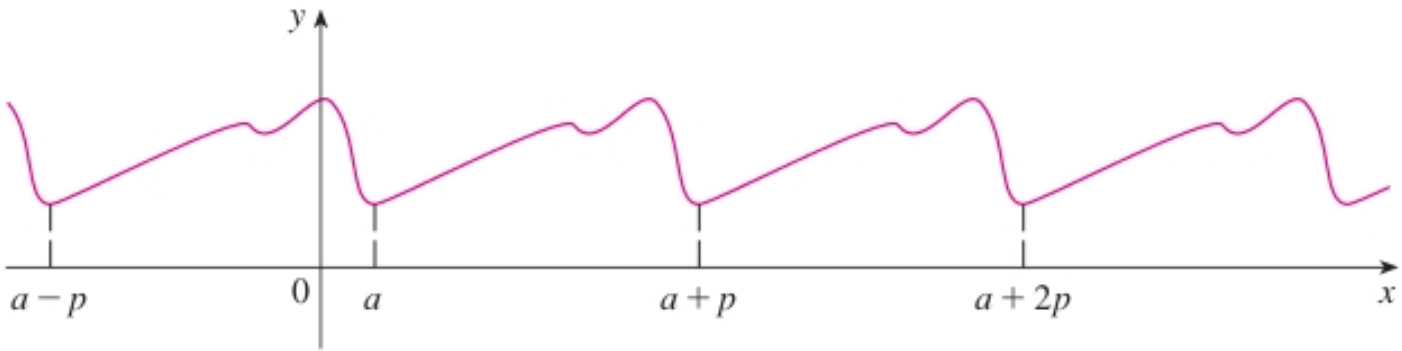


(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

If $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = f(x + p)$ for a fixed $p > 0$ and all $x \in \mathbb{R}$, then f is called **periodic** and p is called a **period** of f.



Curve Sketching: Tips

- Look for symmetries of the given function f, that is, find out if f is even/ odd/ periodic?
- Look for intercepts, that is, the intersections of the curve with co-ordinate axes.
- Look for points (if any) where f is not defined. If such a point exists, study the behaviour of f near that point.
- Locate critical points, local maxima, local minima of f.
- Find intervals on which f is increasing or decreasing.
- Find intervals on which f is convex or concave.
- Locate points of inflection of f.
- Determine the behaviour of $f(x)$ as $x \to \infty$ or $x \to -\infty$.
- Find the asymptotes, if any.

Curve Sketching: An Example

Example: Let $f(x) := (x^2-1)/(x+2)$ for $x \in \mathbb{R}$ with $x \neq -2$.

- f is not even, not odd, not periodic.
- Intercepts of f: $(0,-1/2), (1,0), (-1,0)$.
- $f'(x) = (x+2+\sqrt{3})(x+2-\sqrt{3})/(x+2)^2$ and $f''(x) = 6/(x+2)^3$ for $x \neq -2$.
- f is increasing on $(-\infty, -2-\sqrt{3})$ and on $(-2+\sqrt{3}, \infty)$. f is decreasing on $(-2-\sqrt{3}, -2)$ and on $(-2, -2+\sqrt{3})$.
- f is concave on $(-\infty, -2)$ and convex on $(-2, \infty)$.
- Since $f'(-2-\sqrt{3}) = 0 = f'(-2+\sqrt{3})$ and $f''(-2-\sqrt{3}) = -2/\sqrt{3} < 0$, $f''(-2+\sqrt{3}) = 2/\sqrt{3} > 0$, f has a local maximum at $-2-\sqrt{3}$ and f has a local minimum at $-2+\sqrt{3}$. Note: $f(-2 \pm \sqrt{3}) = -4 \pm 2\sqrt{3}$.
- There is no point of inflection for f.
- The straight line $x = -2$ is a vertical asymptote.
- Since $f(x) = x - 2 + 3/(x+2)$ for $x \neq -2$, $y = x - 2$ is an oblique asymptote.