MA 105 : Calculus
Division 3, Lecture 11

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Recap of the previous lecture

- Infinite limits of sequences
- Limits of functions at infinity
- Infinite limits of functions
- Asymptotes: Horizontal, Vertical and Oblique
- Even, odd and periodic functions
- Curve Sketching: Tips and examples

Mean Value Theorem Revisited

Recall the most important result in differential calculus which we proved in Lecture 7:

Theorem (Mean Value Theorem: MVT)

Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

Let us now consider an extension of the above result.

Theorem (Extended Mean Value Theorem)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be such that f' exists on $[a, b]$. Suppose f' is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2.$$

Proof: For $x \in [a, b]$, let $P(x) := f(a) + f'(a)(x - a)$. Define

$$F(x) := f(x) - P(x) - \frac{f(b) - P(b)}{(b - a)^2}(x - a)^2 \quad \text{for } x \in [a, b].$$

Then $F(a) = 0 = F(b)$. By the Rolle theorem, there exists $c_1 \in (a, b)$ such that $F'(c_1) = 0$. Since $F'(a) = 0$ also, there is $c \in (a, c_1)$ such that $F''(c) = 0$. This gives the desired result.

More generally, we can obtain the following important result.

Theorem (Taylor Theorem)

Let $a < b$ and $n \in \mathbb{N}$. Let $f : [a, b] \to \mathbb{R}$ be such that $f', \ldots, f^{(n)}$ exist on $[a, b]$. Suppose $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) = f(a) + f'(a)(b - a) + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + R_n$ where $R_n := f^{(n+1)}(c)(b - a)^{n+1}/(n + 1)!$.

Proof (Sketch): For $x \in [a, b]$, let

$$P(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Consider $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) := f(x) - P(x) - \frac{f(b) - P(b)}{(b-a)^{n+1}}(x-a)^{n+1} \quad \text{for } x \in [a, b].$$

Then $F(a) = 0 = F(b)$. By the Rolle theorem, there exists $c_1 \in (a, b)$ such that $F'(c_1) = 0$. Further,

$$F'(a) = 0 = F'(c_1)$$

and so there is $c_2 \in (a, c_1)$ such that $F''(c_2) = 0$. Now

$$F''(a) = 0 = F''(c_2)$$

and so there is $c_3 \in (a, c_2)$ such that $F'''(c_3) = 0$. Continuing in this way, we see that there is $c = c_{n+1} \in (a, b)$ such that $F^{(n+1)}(c) = 0$, and this yields the desired result.

A result similar to Taylor's theorem holds with a and b interchanged. [This can be seen by applying Taylor's Theorem to $g : [a, b] \to \mathbb{R}$ defined by $g(x) := f(a + b - x)$.] As a consequence, we obtain the following result.

Taylor Formula: Let $n \in \mathbb{N}$. If I is an interval, $a \in I$, and $f : I \to \mathbb{R}$ is such that $f', f'', \ldots, f^{(n)}$ exist on I and $f^{(n+1)}$ exists at every interior point of I, then for $x \in I$, $x \neq a$, there is c_x between a and x such that $f(x) = P_n(x) + R_n(x)$, where

$$P_n(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$R_n(x) := \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-a)^{n+1}.$$

Here the polynomial $P_n(x)$ is of degree $\leq n$, and it can be considered as an 'approximation' of $f(x)$ of order n. It is called the nth **Taylor polynomial** of f around a, and $R_n(x)$ is called the nth **Lagrange remainder** of f around a.

Example:

Let $f(x) := \sin x$ for $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$, $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x, \ldots$, and in general,

$$f^{(k)}(x) = \begin{cases} (-1)^{k/2} \sin x & \text{if} \quad k \text{ is even,} \\ \\ (-1)^{(k-1)/2} \cos x & \text{if} \quad k \text{ is odd.} \end{cases}$$

Fix $n \in \mathbb{N}$. Then the nth Taylor polynomial of f around 0 is

$$\begin{aligned} P_n(x) &= f(0) + f'(0)(x-0) + \cdots + \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= \begin{cases} x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots (-1)^{(n-1)/2} \dfrac{x^n}{n!} & \text{if} \quad n \text{ is odd,} \\ \\ x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots (-1)^{(n-2)/2} \dfrac{x^{n-1}}{(n-1)!} & \text{if} \quad n \text{ is even} \end{cases} \end{aligned}$$

for $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$. Then the nth Lagrange remainder around 0 is

$$R_n(x) = \begin{cases} (-1)^{n/2} \dfrac{x^{n+1}}{(n+1)!} \cos c_x & \text{if} \quad n \text{ is even,} \\ \\ (-1)^{(n+1)/2} \dfrac{x^{n+1}}{(n+1)!} \sin c_x & \text{if} \quad n \text{ is odd,} \end{cases}$$

where c_x is between 0 and x. As $|\cos c_x| \leq 1$ & $|\sin c_x| \leq 1$,

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

Hence for $x \in \mathbb{R}$,

$$\sin x = \lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{(k-1)} \frac{x^{2k-1}}{(2k-1)!}, \text{ that is,}$$

$$\sin x = \sum_{k=1}^{\infty} (-1)^{(k-1)} \frac{x^{2k-1}}{(2k-1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

The RHS is the **Taylor series** of the function sin around 0.