MA 105 : Calculus
Division 3, Lecture 12

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Mean Value Theorem, revisited
- Extended Mean Value Theorem, and its proof
- Taylor theorem, and its proof
- Taylor formula, featuring Taylor polynomial and Lagrange (form of) remainder
- Example: f(x) = sin x. Taylor series for this function around 0.
- Brief discussion of area of planar regions and Riemann integration

Riemann Integration

Let $a, b \in \mathbb{R}$ with $a < b$, and let f be a bounded function defined on $[a, b]$. Let us first assume that $f \geq 0$ on $[a, b]$. We are interested in assigning a meaning to the concept of 'area' of the region that lies under the graph of f, between the lines $x = a$, $x = b$, and above the x-axis, that is, of the region

$$R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

A **partition** of $[a, b]$ is a finite ordered set $P := \{x_0, x_1, \ldots, x_n\}$ of points in $[a, b]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $n \in \mathbb{N}$.

The points in a partition divide the interval $[a, b]$ into nonoverlapping subintervals, $[x_0, x_1]$, $\ldots$, $[x_{n-1}, x_n]$. We call $[x_{i-1}, x_i]$ the ith **subinterval** of the partition P.

Example: For $n \in \mathbb{N}$, the set

$$\mathrm{P}_n := \left\{ a,\ a + \frac{(b-a)}{n}, \ldots, a + \frac{(n-1)(b-a)}{n},\ b \right\}$$

is a partition of $[a, b]$ into *n* equal parts, each of length $(b-a)/n$. If $n := 1$, then $\mathrm{P}_1 := \{a, b\}$ is the trivial partition.

Let $f : [a, b] \to \mathbb{R}$ be any bounded function, and define

$m(f) := \inf\{f(x) : x \in [a, b]\}$ and $M(f) := \sup\{f(x) : x \in [a, b]\}$.

Given a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, let us define

$$m_i(f) := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$M_i(f) := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$$

for $i = 1, \ldots, n$.

Lower sum and upper sum

Clearly, $m(f) \leq m_i(f) \leq M_i(f) \leq M(f)$ for $i = 1, \ldots, n$. Let

$$L(P, f) := \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}),$$

$$U(P, f) := \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

$L(P, f)$ is called the **lower sum** of f with respect to P and $U(P, f)$ is called the **upper sum** of f with respect to P.

Since $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, we obtain

$$m(f)(b - a) \leq L(P, f) \leq U(P, f) \leq M(f)(b - a).$$

Underlying assumption: The area of a rectangle $[x_{i-1}, x_i] \times [y_{i-1}, y_i]$ is equal to $(x_i - x_{i-1})(y_i - y_{i-1})$.

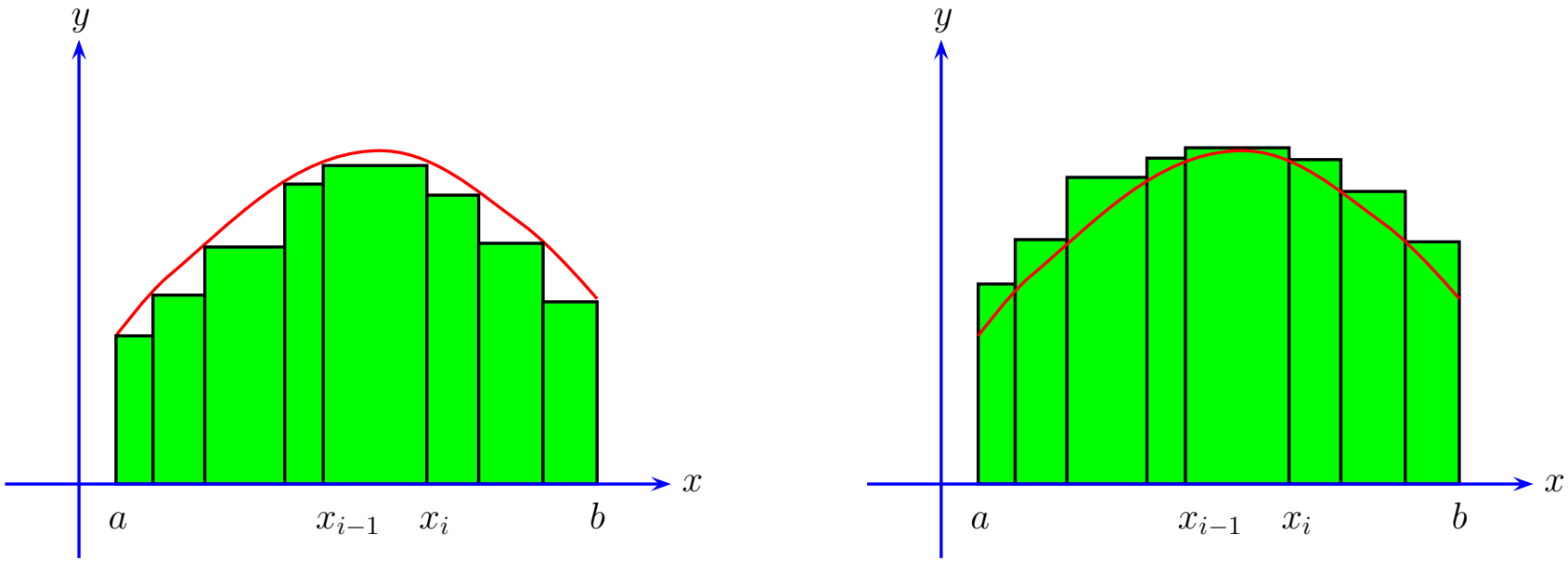


Figure: Approximating the 'area' under a curve by a lower sum which is sum of the areas of the inscribed rectangles, and by an upper sum which is the sum of the areas of the circumscribing rectangles.

Refinement of a partition

Facts:

(i) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If P is a partition of $[a, b]$ and P^* is a **refinement** of P, that is, if every point of P is also a point of P^*, then

$$L(P, f) \leq L(P^*, f) \quad \text{and} \quad U(P^*, f) \leq U(P, f).$$

Thus the lower sum increases and the upper sum decreases when a partition is refined. (See the figures on the next page. A formal proof is left as an exercise)

(ii) Let P_1, P_2 be any partitions of $[a, b]$, and let $P^* = P_1 \cup P_2$ be the **common refinement** of P_1 and P_2. Then by (i),

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

Thus any lower sum is less than or equal to any upper sum.

Effect of Refinement on Lower sum and Upper sum

To illustrate the effect of a refinement of the partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ on the corresponding lower sum and the upper sum of $f : [a, b] \to \mathbb{R}$, let us add an extra point x^* in the interval (x_{i-1}, x_i). We see that $L(P, f)$ can become bigger, while $U(P, f)$ may become smaller.

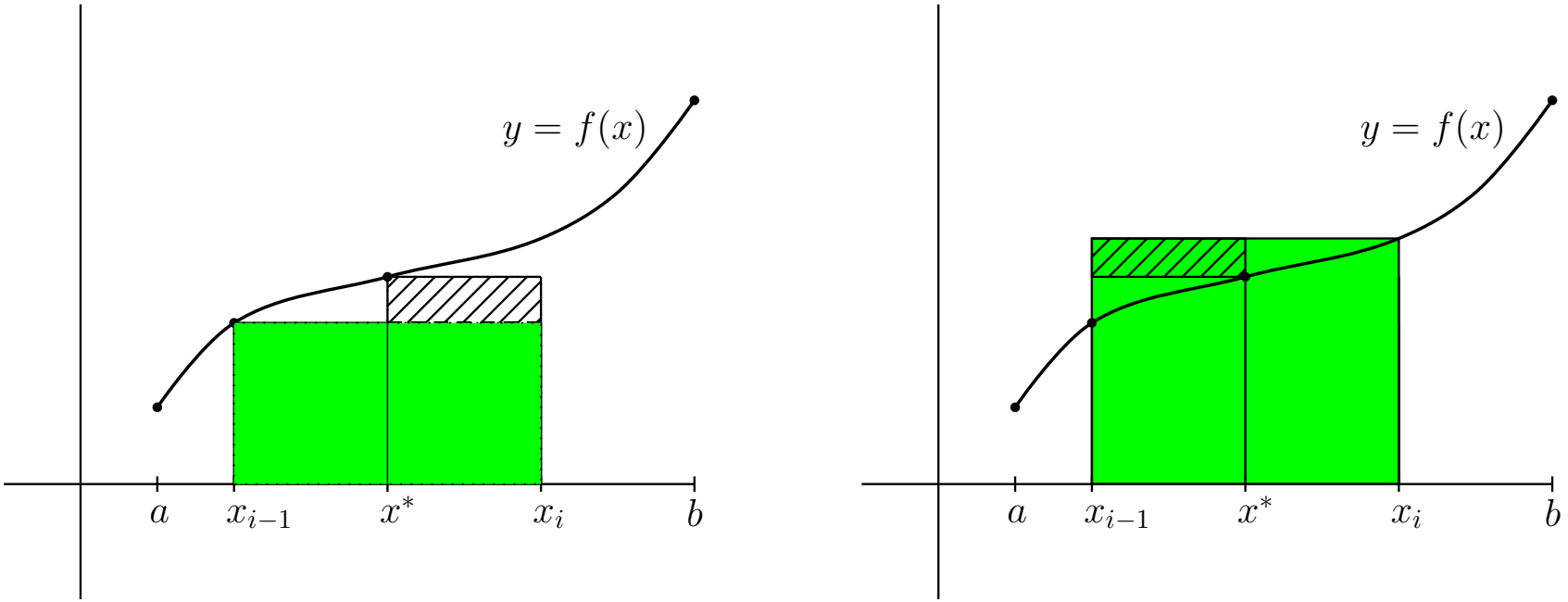


Figure: On the left, the shaded area is added to the lower sum. On the right, the shaded are is removed from the upper sum.

Riemann Integral

Let $a < b$, and $f : [a, b] \to \mathbb{R}$ be a bounded function. Define

$$\begin{aligned} L(f) &:= \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}, \\ U(f) &:= \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}. \end{aligned}$$

$L(f)$ is called the **lower Riemann integral** of f and $U(f)$ is called the **upper Riemann integral** of f. By (ii) above (and the definitions of sup and inf), we see that $L(f) \leq U(f)$.

Definition

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **(Riemann) integrable** on $[a, b]$ if $L(f) = U(f)$.

In this case, the **Riemann integral** of f on $[a, b]$ is the common value $U(f) = L(f)$, and it is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

Example 1: Let $c \in \mathbb{R}$, and let $f(x) := c$ for all $x \in [a, b]$.

For a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and $i = 1, \ldots, n$,

$$m_i(f) := \inf_{x \in [x_{i-1}, x_i]} f(x) = c \quad \text{and} \quad M_i(f) := \sup_{x \in [x_{i-1}, x_i]} f(x) = c,$$

and so

$$L(P, f) = U(P, f) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b - a).$$

Hence $L(f) = U(f) = c(b - a)$.

Thus f is integrable, and $\int_a^b f(x)\, dx = c(b - a)$.

Example 2: Define the **Dirichlet function** $f : [a, b] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1 & \text{if } x \text{ is rational,} \\ \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is a bounded function on $[a, b]$.
Recall: Between any two real numbers, there is a rational number, and also an irrational number.

For a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and $i = 1, \ldots, n$,

$$m_i(f) := \inf_{x \in [x_{i-1}, x_i]} f(x) = 0 \quad \text{and} \quad M_i(f) := \sup_{x \in [x_{i-1}, x_i]} f(x) = 1,$$

and so $L(P, f) = 0$ and $U(P, f) = b - a$. Hence $L(f) = 0$ and $U(f) = b - a$. Since $L(f) \neq U(f)$, f is not integrable.

Example 3: Let $f(x) := x$ for $x \in [a, b]$.

For a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ and $i = 1, \ldots, n$,

$$m_i(f) := \inf_{x \in [x_{i-1}, x_i]} f(x) = x_{i-1} \quad \text{and} \quad M_i(f) := \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i,$$

and so

$$L(P, f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1}) \quad \text{and} \quad U(P, f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}).$$

Now $U(P, f) - L(P, f) = \sum_{i=1}^{n} (x_i - x_{i-1})^2$ and

$U(P, f) + L(P, f) = \sum_{i=1}^{n} (x_i^2 - x_{i-1}^2) = x_n^2 - x_0^2 = b^2 - a^2.$

Hence

$U(P, f) = (b^2 - a^2)/2 + \sum_{i=1}^{n} (x_i - x_{i-1})^2/2$, and

$L(P, f) = (b^2 - a^2)/2 - \sum_{i=1}^{n} (x_i - x_{i-1})^2/2.$

Let $n \in \mathbb{N}$, and let P_n denote the partition of $[a, b]$ in n equal parts. Then $x_i - x_{i-1} = (b-a)/n$ for $i = 1, \ldots, n$, and so

$$U(\mathrm{P}_n, f) = \frac{b^2 - a^2}{2} + \frac{1}{2} \sum_{i=1}^{n} \frac{(b-a)^2}{n^2} = \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n},$$

$$L(\mathrm{P}_n, f) = \frac{b^2 - a^2}{2} - \frac{1}{2} \sum_{i=1}^{n} \frac{(b-a)^2}{n^2} = \frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n}.$$

Since $L(\mathrm{P}_n, f) \leq L(f)$ and $U(f) \leq U(\mathrm{P}_n, f)$, we obtain

$$\frac{b^2 - a^2}{2} - \frac{(b-a)^2}{2n} \leq L(f) \leq U(f) \leq \frac{b^2 - a^2}{2} + \frac{(b-a)^2}{2n}.$$

Letting $n \to \infty$, we see that $L(f) = U(f) = (b^2 - a^2)/2$.

Thus f is integrable, and $\int_a^b f(x)\, dx = (b^2 - a^2)/2$.

A useful criterion for integrability

The above example illustrates the difficulty in proving the integrability of a bounded function f on $[a, b]$ by showing $U(f) = L(f)$. We now give a necessary and sufficient condition for the integrability of such a function, which is much easier to verify.

Theorem (Riemann condition)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

Proof: ($\Longrightarrow$) Let $\epsilon > 0$. There are partitions P_1, P_2 such that

$$U(P_1, f) < U(f) + \frac{\epsilon}{2} \quad \text{and} \quad L(f) - \frac{\epsilon}{2} < L(P_2, f).$$

Let P_ϵ denote the common refinement of P_1 and P_2. Then

$$U(P_\epsilon, f) - L(P_\epsilon, f) \leq U(P_1, f) - L(P_2, f) < U(f) - L(f) + \epsilon = \epsilon.$$

($\Longleftarrow$) Given any $\epsilon > 0$, by the Riemann condition, we obtain

$$0 \leq U(f) - L(f) \leq U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Letting $\epsilon \to 0$, we see that $U(f) = L(f)$. □

Corollary

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there is a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$. In this case,

$$L(P_n, f) \to \int_a^b f(x)dx \quad \text{and} \quad U(P_n, f) \to \int_a^b f(x)dx.$$

Proof: ($\Longrightarrow$) For $n \in \mathbb{N}$, let $\epsilon = 1/n$, and use the Riemann condition to find a partition P_n of $[a, b]$ such that

$$U(P_n, f) - L(P_n, f) < 1/n.$$

($\Longleftarrow$) Given $\epsilon > 0$, find $n_0 \in \mathbb{N}$ such that

$$U(P_n, f) - L(P_n, f) < \epsilon \quad \text{for all } n \geq n_0.$$

Then the Riemann condition is satisfied with $P_\epsilon := P_{n_0}$.

Next, let (P_n) be a sequence of partitions such that $U(P_n, f) - L(P_n, f) \to 0$. Then for $n \in \mathbb{N}$,

$$0 \le L(f) - L(P_n, f) \le U(f) - L(P_n, f) \le U(P_n, f) - L(P_n, f),$$

and hence $L(P_n, f) \to L(f) = \int_a^b f(x)dx$. Similarly, for $n \in \mathbb{N}$,

$$0 \le U(P_n, f) - U(f) \le U(P_n, f) - L(f) \le U(P_n, f) - L(P_n, f),$$

and hence $U(P_n, f) \to U(f) = \int_a^b f(x)dx$. □

Example: Fix $m \in \mathbb{N}$, and let $f(x) := x^m$ for $x \in [a, b]$. For $n \in \mathbb{N}$, let P_n be the partition of $[a, b]$ in n equal parts. Then $x_i - x_{i-1} = (b - a)/n$ for $i = 1, \ldots, n$. If m is odd, or if m is even and $a \ge 0$, then

$$U(\mathrm{P}_n, f) - L(\mathrm{P}_n, f) = \sum_{i=1}^{n} (x_i^m - x_{i-1}^m) \frac{(b-a)}{n} = \frac{(b-a)}{n}(b^m - a^m) \to 0,$$

and so f is integrable. (What if m is even and $a < 0$?)