MA 105 : Calculus
Division 3, Lecture 13

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Recap of the previous lecture

- Riemann integration:
  - Partition of an interval, Subintervals of a partition
  - Upper sum and Lower sum of a function w.r.t a partition
  - Upper integral and Lower integral
  - Definition of integrable functions and of the Riemann integral of a bounded function
  - Example of integrable and non-integrable functions

Riemann Integral

Let $a < b$, and $f : [a, b] \to \mathbb{R}$ be a bounded function. Define

$$\begin{aligned} L(f) &:= \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}, \\ U(f) &:= \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}. \end{aligned}$$

$L(f)$ is called the **lower Riemann integral** of f and $U(f)$ is called the **upper Riemann integral** of f. By (ii) above (and the definitions of sup and inf), we see that $L(f) \leq U(f)$.

Definition

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **(Riemann) integrable** on $[a, b]$ if $L(f) = U(f)$.

In this case, the **Riemann integral** of f on $[a, b]$ is the common value $U(f) = L(f)$, and it is denoted by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)dx.$$

A useful criterion for integrability

Recall that we discussed the following equivalent condition for integrability, which is often useful in practice.

Theorem (Riemann condition)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable on $[a, b]$ if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

This has the following consequence, which is also quite useful.

Corollary

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if there is a sequence (P_n) of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$. In this case,

$$L(P_n, f) \to \int_a^b f(x)dx \quad \text{and} \quad U(P_n, f) \to \int_a^b f(x)dx.$$

Example: Fix $m \in \mathbb{N}$, and let $f(x) := x^m$ for $x \in [a, b]$. For $n \in \mathbb{N}$, let P_n be the partition of $[a, b]$ in n equal parts. Then $x_i - x_{i-1} = (b-a)/n$ for $i = 1, \ldots, n$. If m is odd, or if m is even and $a \geq 0$, then

$$U(\mathrm{P}_n, f) - L(\mathrm{P}_n, f) = \sum_{i=1}^{n} (x_i^m - x_{i-1}^m) \frac{(b-a)}{n} = \frac{(b-a)}{n} (b^m - a^m) \to 0,$$

and so f is integrable. (What if m is even and $a < 0$?)

Now let us also try to determine $\int_a^b x^m \, dx$ using the second part of the Corollary. Thus, for the above partitions P_n, we try to determine the limit of $U(\mathrm{P}_n, f)$ as $n \to \infty$. As we shall see, this isn't always very easy. But smalll values of m are not difficult to handle.

In particular, if $m := 1$, then

$$\begin{aligned} U(\mathrm{P}_n, f) &= \frac{b-a}{n} \sum_{i=1}^{n} \left(a + i \, \frac{b-a}{n} \right) \\ &= a(b-a) + \frac{(b-a)^2}{n^2} \frac{n(n+1)}{2} \\ &\to a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2 - a^2}{2}, \end{aligned}$$

and so $\int_a^b x \, dx = (b^2 - a^2)/2$, as we have already seen.

Similarly, if $m := 2$ and $a \geq 0$, then

$$\begin{aligned} U(\mathrm{P}_n, f) &= \frac{b-a}{n} \sum_{i=1}^{n} \left(a + i \, \frac{b-a}{n} \right)^2 \\ &\to a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{3} = \frac{b^3 - a^3}{3}. \end{aligned}$$

and so $\int_a^b x^2 dx = (b^3 - a^3)/3$.

Results about Riemann integration

(Domain Additivity)

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on $[a, c]$ and $[c, b]$. In this case, $\int_a^b f = \int_a^c f + \int_c^b f$.

A proof can be given using the Riemann condition. We omit the details. See [GL-1] if interested.

In defining $\int_a^b f$, we have assumed $a < b$ so far.

Conventions: (i) Let $b = a$. Then every $f : [a, a] \to \mathbb{R}$ is integrable, and $\int_a^a f(x)\, dx := 0$.

(ii) Let $b < a$ and $f : [b, a] \to \mathbb{R}$ be integrable. Then $\int_a^b f(x)\, dx := -\int_b^a f(x)\, dx$.

It follows that $\int_c^d f = \int_a^d f - \int_a^c f$ for any $c, d \in [a, b]$.

Integrable functions

(i) If $f : [a, b] \to \mathbb{R}$ is monotonic, then f is integrable.

(ii) If $f : [a, b] \to \mathbb{R}$ is bounded and has at most a finite number of discontinuities in $[a, b]$, then f is integrable.

Proofs are a bit involved, especially for (ii), and thus skipped. See [GL-1] is interested.

Examples:

(i) Let $f(x) := [x]$ for $x \in [a, b]$. Since f is increasing on $[a, b]$, f is integrable on $[a, b]$.

(ii) Define $f : [0, 1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := 1/n$ if $1/(n+1) < x \leq 1/n$ for $n \in \mathbb{N}$. Since f is increasing on $[0, 1]$, f is integrable on $[a, b]$.

(iii) A polynomial function is integrable on $[a, b]$.

(iv) Define $f : [0, 1] \to \mathbb{R}$ by $f(0) := 0$ and $f(x) := \sin(1/x)$ for $x \in (0, 1]$. Since f is bounded, and it is discontinuous only at 0, it follows that f is integrable on $[a, b]$.

Algebraic and Order Properties

Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Then

(i) $f + g$ is integrable, and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(ii) $c\,f$ is integrable, and $\int_a^b c\,f = c \int_a^b f$ for $c \in \mathbb{R}$.

(iii) $f \cdot g$ is integrable.

(iv) If there is $\delta > 0$ such that $|f(x)| \geq \delta$ for all $x \in [a, b]$ (so that $1/f$ is bounded), then $1/f$ is integrable.

(v) If $f \leq g$, then $\int_a^b f \leq \int_a^b g$.

(vi) $|f|$ is integrable, and $|\int_a^b f| \leq \int_a^b |f|$.

Proving (ii) is easy. Proofs of (i), (iii) and (iv) are omitted. For (v), note that $U(P, f) \leq U(P, g)$ for any partition P of $[a, b]$, and so $\int_a^b f = U(f) \leq U(g) = \int_a^b g$. Finally, (vi) follows since $U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f)$ for each P, and also since $-|f| \leq f \leq |f| \implies -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$.

Area of planar regions

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. If $f \geq 0$, i.e., if $f(x) \geq 0$ for all $x \in [a, b]$, then we say that the planar region

$$R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

has an area if f is integrable on $[a, b]$, and in this case, define

$$\text{Area}(R_f) = \int_a^b f(x)dx.$$

Note that this is a nonnegative real number, thanks to (v).
Note also that if $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$, then so are $\max(f, g)$ and $\min(f, g)$. To see this, observe that

$$\max(f, g) = \frac{f + g + |f - g|}{2} \text{ and } \min(f, g) = \frac{f + g - |f - g|}{2}.$$

It follows that in general, if $f : [a, b] \to \mathbb{R}$ is integrable, then $\int_a^b f(x)dx = \text{Area}(R_{f^+}) - \text{Area}(R_{f^-})$, where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Thus $\int_a^b f(x)dx$ is like "**signed area**".

The Fundamental Theorem of Calculus (FTC)

(The Fundamental Theorem of Calculus: Part I)

Let f be integrable on $[a, b]$. For $x \in [a, b]$, define

$$F(x) := \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Proof: Let $\alpha \in \mathbb{R}$ satisfy $|f| \leq \alpha$ on $[a, b]$, and let $c \in [a, b]$. By the domain additivity of the Riemann integral,

$$|F(x) - F(c)| = \left| \int_c^x f(t)\, dt \right| \leq \alpha |x - c| \quad \text{for all } x \in [a, b].$$

If $x_n \to c$, then $F(x_n) \to F(c)$. Hence F is continuous at c.

Suppose f is continuous at $c \in [a,b]$. For $x \in [a,b]$, $x \neq c$, define $F_1(x) := \big(F(x) - F(c)\big)/(x-c)$, and $F_1(c) := f(c)$.

We show that the function F_1 is continuous at c. For $x \neq c$,

$$F_1(x) - f(c) = \left(\frac{1}{x-c}\int_c^x f(t)dt\right) - f(c) = \frac{1}{x-c}\int_c^x \big(f(t) - f(c)\big)\,dt$$

since $f(c) = \int_c^x f(c)dt/(x-c)$. Let $\epsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ with

$$t \in [a,b] \quad \text{and} \quad |t-c| < \delta \implies |f(t) - f(c)| < \epsilon.$$

Hence if $x \in [a,b]$ and $0 < |x-c| < \delta$, then

$$|F_1(x) - f(c)| = \frac{\left|\int_c^x \big(f(t) - f(c)\big)\,dt\right|}{|x-c|} \leq \frac{\epsilon\,|x-c|}{|x-c|} = \epsilon.$$

By the C-lemma, F' exists at c and $F'(c) = F_1(c) = f(c)$. □