MA 105 : Calculus
Division 3, Lecture 15

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Recap of the previous lecture

- Fundamental Theorem of Calculus: Part I and its proof
- Examples
- Notion of an antiderivative
- Fundamental Theorem of Calculus: Part II and its proof
- Consequences of the FTC
  - Integration by parts
  - Integration by substitution
  - Examples

The Fundamental Theorem of Calculus (FTC)

(The Fundamental Theorem of Calculus: Part I)

Let f be integrable on $[a, b]$. For $x \in [a, b]$, define

$$F(x) := \int_a^x f(t)\, dt.$$

Then F is continuous on $[a, b]$. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

(The Fundamental Theorem of Calculus: Part II)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a, b]$. Then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

What does the Fundamental Theorem of Calculus say?

(i) Part I: If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{for all } x \in [a, b].$$

In particular, continuous functions have a continuously differentiable antiderivative.

(ii) Part II: If f' exists and is integrable on $[a, b]$, then

$$\int_a^b \Big(\frac{d}{dx} f(x)\Big)\,dx = f(b) - f(a).$$

In particular, if an integrable function on $[a, b]$ has an antiderivative, then its definite integral is easily computed.

Moral: Thus the operations of differentiation and integration are 'inverse' to each other.

Side Remark: Some Pathologies

This optional slide (which was not shown during the lecture) elaborates on some remarks I made and also contains answers to some of the questions asked by the students during or after the class. The material here will not have bearing on any of the exams.

- An integrable function need not have an antiderivative. In fact, if $f : [a, b] \to \mathbb{R}$ has an antiderivative, then f has the IVP on $[a, b]$ (see, e.g, Theorem 4.16 in [GL-1]). In particular, $f : [-1, 1] \to \mathbb{R}$ given by $f(x) = [x]$ does not have an antiderivative, although f is clearly integrable.

- If $f : [a, b] \to \mathbb{R}$ has an antiderivative F, then f need not be integrable. In fact, f need not even be bounded. Example: $f = F'$ with $F : [-1, 1] \to \mathbb{R}$ given by $F(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $F(0) = 0$ [See also Remark 6.25 (iii) in [GL-1].]

- Composites $g \circ f$ of integrable functions $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ need not be integrable, unless g is continuous. See, e.g., Example 6.17 and Remark 6.18 in [GL-1].

Consequences of the FTC

(Integration by parts)

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable. Suppose $g : [a, b] \to \mathbb{R}$ is integrable and has an antiderivative G on $[a, b]$. Then

$$\begin{aligned}\int_a^b f(x)g(x)dx &= f(b)G(b) - f(a)G(a) - \int_a^b f'(x)G(x)dx \\ &= f(x)G(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)G(x)\, dx.\end{aligned}$$

Idea of Proof: Consider $H(x) := f(x)G(x)$ for $x \in [a, b]$, and use the product rule together with the FTC.

(Integration by substitution)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function such that $\phi([\alpha, \beta]) = [a, b]$. Then $(f \circ \phi)\phi' : [\alpha, \beta] \to \mathbb{R}$ is integrable, and

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f\big(\phi(t)\big)\phi'(t)dt.$$

In particular, if $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$, then

$$\int_{a}^{b} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))|\phi'(t)|dt.$$

Idea of Proof: Consider $F(x) := \int_a^x f(u)du$ for $x \in [a, b]$ and $H(t) := F(\phi(t))$ for $t \in [\alpha, \beta]$. Apply the Chain rule to H and then use the FTC.

Examples:

(i) To evaluate $\int_0^1 x\sqrt{1-x}\,dx$, let $f(x) := x$ and $g(x) := \sqrt{1-x}$ for $x \in [0,1]$. Then $f'(x) = 1$ for $x \in [0,1]$. Also, if $G(x) := -(2/3)(1-x)^{3/2}$, then $G'(x) = g(x)$ for $x \in [0,1]$, that is, $G' = g$. Integration by Parts yields

$$\int_0^1 x\sqrt{1-x}\,dx = 0 - 0 - \int_0^1 \left(-\frac{2}{3}\right)(1-x)^{3/2}dx = \frac{2}{3}\int_0^1 (1-x)^{3/2}dx.$$

If we let $F(x) := -(2/5)(1-x)^{5/2}$ for $x \in [0,1]$, then $F'(x) = (1-x)^{3/2}$ for $x \in [0,1]$. By the FTC, Part II, the integral equals $(2/3)(F(1) - F(0)) = (2/3)(2/5) = 4/15$.

(ii) To evaluate $\int_0^1 t\sqrt{1-t^2}\,dt$, let $\phi(t) := 1 - t^2$ for $t \in [0,1]$ and $f(x) := \sqrt{x}$ for $x \in [0,1]$. Then $\phi(0) = 1$, $\phi(1) = 0$, and $\phi'(t) = -2t$ for $t \in [0,1]$. Let $F(x) := (2/3)x^{3/2}$, $x \in [0,1]$. Then $F'(x) = \sqrt{x}$, $x \in [0,1]$. Integration by Substitution yields

$$\int_0^1 t\sqrt{1-t^2}\,dt = \frac{1}{2}\int_0^1 f(\phi(t))|\phi'(t)|dt = \frac{1}{2}\int_0^1 \sqrt{x}\,dx = \frac{1}{3}.$$

(Approximate) Riemann Sum

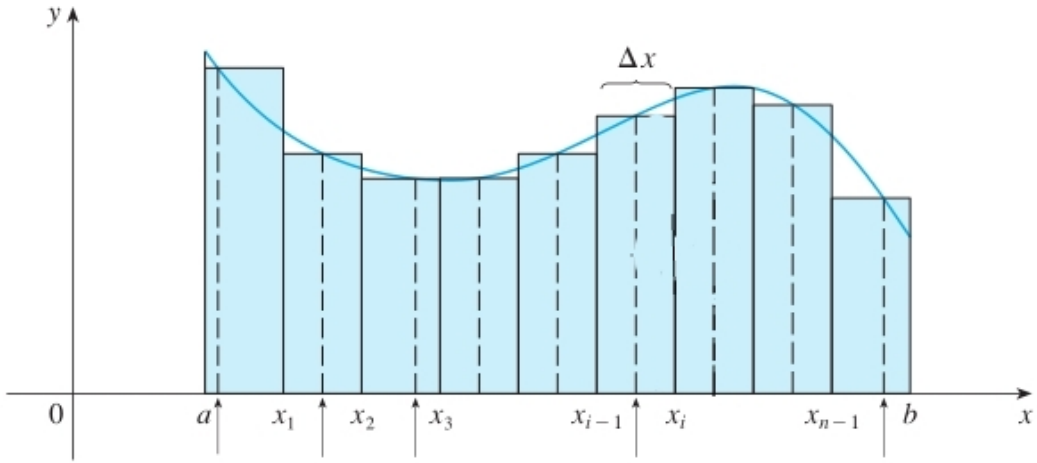
Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, the sum

$$S(P, f) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

is called a Riemann sum for f corresponding to P, where $t_i \in [x_{i-1}, x_i]$, $i = 1, \ldots, n$.

Since $m_i(f) \leq f(t_i) \leq M_i(f)$ for $i = 1, \ldots, n$, we note that

$$L(P, f) \leq S(P, f) \leq U(P, f) \quad \text{for any } t_i \in [x_{i-1}, x_i].$$



Approximation of a Riemann Integral

The **mesh** of a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ is defined by $\mu(P) := \max\{x_i - x_{i-1} : i = 1, \ldots, n\}$.

Theorem: Let f be integrable on $[a, b]$, and let $\epsilon > 0$ be given. Then there is $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition P satisfying $\mu(P) < \delta$.

This improves upon the Riemann condition. Proof is omitted.

Corollary. Let f be integrable on $[a, b]$, and let (P_n) be a sequence of partitions of $[a, b]$ such that $\mu(P_n) \to 0$. Then $U(P_n, f) - L(P_n, f) \to 0$. Further, if $S(P_n, f)$ is a Riemann sum corresponding to P_n and f, then $S(P_n, f) \to \int_a^b f(x)dx$.

Proof: Let $\epsilon > 0$ be given, and let $\delta > 0$ be as in the above theorem. For this δ, we find $n_0 \in \mathbb{N}$ such that $\mu(P_n) < \delta$ for all $n \geq n_0$. Then $U(P_n, f) - L(P_n, f) < \epsilon$ for all $n \geq n_0$. Thus $U(P_n, f) - L(P_n, f) \to 0$.

Since

$$\begin{aligned} L(P_n, f) &\leq S(P_n, f) \leq U(P_n, f) \quad \text{and} \\ L(P_n, f) &\leq L(f) = \int_a^b f(x)dx = U(f) \leq U(P_n, f), \end{aligned}$$

we see that

$$\left| S(P_n, f) - \int_a^b f(x)dx \right| \leq U(P_n, f) - L(P_n, f) \to 0. \quad \square$$

The above corollary is useful in two ways.

(i) If $\int_a^b f(x)dx$ is known, then we can find $\lim_{n\to\infty} a_n$, if $a_n := S(P_n, f)$ for $n \in \mathbb{N}$ and $\mu(P_n) \to 0$.

(ii) If $\int_a^b f(x)dx$ is not known, then $S(P_n, f)$ gives us an approximation of $\int_a^b f(x)dx$ if $\mu(P_n) \to 0$.

Suppose $f : [a, b] \to \mathbb{R}$ is known to be an integrable function. This is the case, for example, if f is monotonic on $[a, b]$, or if f is discontinuous at only a finite number of points in $[a, b]$.

How can we find the Riemann integral $\int_a^b f(x)dx$?

Try to find an antiderivative of f on $[a, b]$, that is, find a function F on $[a, b]$ such that $F' = f$ on $[a, b]$. In that case, $\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a)$ by the FTC (Part II).

For instance, let $r \in \mathbb{Q}$, $r \geq 0$, and $f(x) := x^r$ for $x \in [a, b]$. Let $F(x) := x^{r+1}/(r + 1)$ for $x \in [a, b]$. Then $F' = f$ on $[a, b]$. Hence $\int_a^b f(x)dx = F(b) - F(a) = (b^{r+1} - a^{r+1})/(r + 1)$.

Next, suppose $0 \notin [a, b]$. Then the above calculation holds if $r \in \mathbb{Q}$ and $r < 0$, provided $r \neq -1$. If $r := -1$, we do not yet know an antiderivative of the function f. Hence we look for an approximation of $\int_a^b f(x)dx$.

Examples:

(i) For $n \in \mathbb{N}$, let $a_n := \dfrac{1^r + 2^r + \cdots + n^r}{n^{r+1}}$, where $r \in \mathbb{Q}$ and $r \geq 0$. Let us find $\lim_{n\to\infty} a_n$. Note that

$$a_n = \sum_{i=1}^{n} \frac{i^r}{n^{r+1}} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^r = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^r \left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := x^r$.
Since f is increasing on $[0,1]$, it is integrable. For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $t_i := i/n$ for $i = 1, 2, \ldots, n$.
Then $a_n = S(P_n, f)$. Since $\mu(P_n) = 1/n \to 0$, it follows that

$$S(P_n, f) \to \int_0^1 f(x)dx, \text{ that is },$$

$$a_n = \sum_{i=1}^{n} \frac{i^r}{n^{r+1}} \to \int_0^1 x^r \, dx = \frac{1}{r+1}.$$

(ii) Let $f(x) := 1/x$ for $x \in [1, 2]$. (This is the case $r := -1$ mentioned earlier.)

Since f is continuous on $[1, 2]$, it is integrable. For $n \in \mathbb{N}$, let

$$\mathrm{P}_n := \left\{1,\, 1 + \frac{1}{n},\, 1 + \frac{2}{n}, \ldots, 1 + \frac{n-1}{n},\, 2\right\}$$

be the partition of $[1, 2]$ in n equal parts.

Let $t_{n,i} = 1 + (i-1)/n$ for $i = 1, \ldots, n$, so that

$$S(\mathrm{P}_n, f) = \sum_{i=1}^{n} \frac{1}{1 + (i-1)/n} \left(\frac{i}{n} - \frac{i-1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n+i-1}.$$

Since $\mu(\mathrm{P}_n) = 1/n \to 0$, we see that $S(\mathrm{P}_n, f) \to \int_a^b f(x)dx$, that is,

$$\sum_{i=1}^{n} \frac{1}{n+i-1} \to \int_1^2 \frac{1}{x}\, dx \text{ as } n \to \infty.$$

Logarithmic and Exponential Functions

Using Riemann integration, we can define 'new' functions. The FTC (Part I) allows us to investigate their properties. If $x \geq 1$, then the function $t \longmapsto 1/t$ is continuous on $[1, x]$, and if $0 < x < 1$, it is continuous on $[x, 1]$.

Definition

The **(natural) logarithmic function** defined by

$$\ln x := \int_1^x \frac{1}{t} dt \quad \text{for } x \in (0, \infty).$$

Remark: We have seen that for any $n \in \mathbb{Z}$ with $n \neq -1$, the function $f(t) = t^n$ can easily be integrated on any interval not containing zero. With the introduction of ln as above, every rational function $p(t)/q(t)$ where the polynomial $q(t)$ factors as a product of linear polynomials, can be integrated using the method of partial fractions.