MA 105 : Calculus
Division 3, Lecture 16

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Recap of the previous lecture

- Review of the FTC and its consequences
- Examples illustrating the theorems on (i) Integration by parts, and (ii) Integration by substitution
- (Approximate) Riemann sum
  - Mesh of a partition
  - Convergence of Riemann sums of an integrable function
  - Examples
- Definition of natural logarithm

Logarithmic and Exponential Functions

Using Riemann integration, we can define 'new' functions. The FTC (Part I) allows us to investigate their properties. If $x \geq 1$, then the function $t \longmapsto 1/t$ is continuous on $[1, x]$, and if $0 < x < 1$, it is continuous on $[x, 1]$.

Definition

The **(natural) logarithmic function** defined by

$$\ln x := \int_1^x \frac{1}{t} dt \quad \text{for } x \in (0, \infty).$$

Remark: We have seen that for any $n \in \mathbb{Z}$ with $n \neq -1$, the function $f(t) = t^n$ can easily be integrated on any interval not containing zero. With the introduction of ln as above, every rational function $p(t)/q(t)$ where the polynomial $q(t)$ factors as a product of linear polynomials, can be integrated using the method of partial fractions.

Properties of (natural) logarithmic function $\ln : (0,\infty) \to \mathbb{R}$

(i) $\ln 1 = 0$, $\ln x < 0$ if $x \in (0,1)$, and $\ln x > 0$ if $x \in (1,\infty)$.
— Clear from the definition.

(ii) $\ln$ is differentiable and $(\ln)'(x) = 1/x$ for $x \in (0,\infty)$.
— Follows from the FTC

(iii) $\ln$ is strictly increasing and strictly concave on $(0,\infty)$.t
— $(\ln)'(x) > 0$ and $(\ln)''(x) = -1/x^2 < 0 \; \forall\, x \in (0,\infty)$.

(iv) $\ln x_1 + \ln x_2 = \ln x_1 x_2$ for all $x_1, x_2 \in (0,\infty)$.
— Let $x_1, x_2 \in (0,\infty)$. By substituting $t = x_1 s$, we obtain

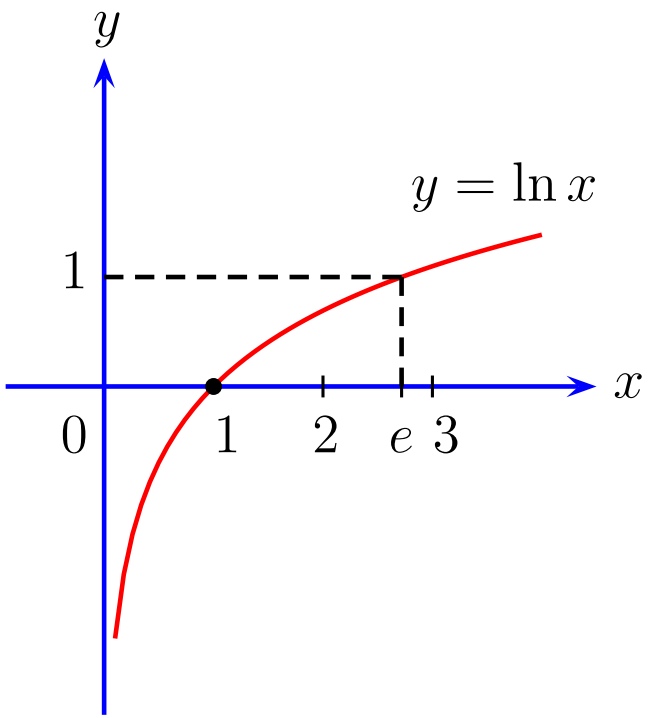
$$\ln x_1 x_2 = \int_1^{x_1 x_2} \frac{dt}{t} = \int_1^{x_1} \frac{dt}{t} + \int_{x_1}^{x_1 x_2} \frac{dt}{t} = \ln x_1 + \ln x_2.$$

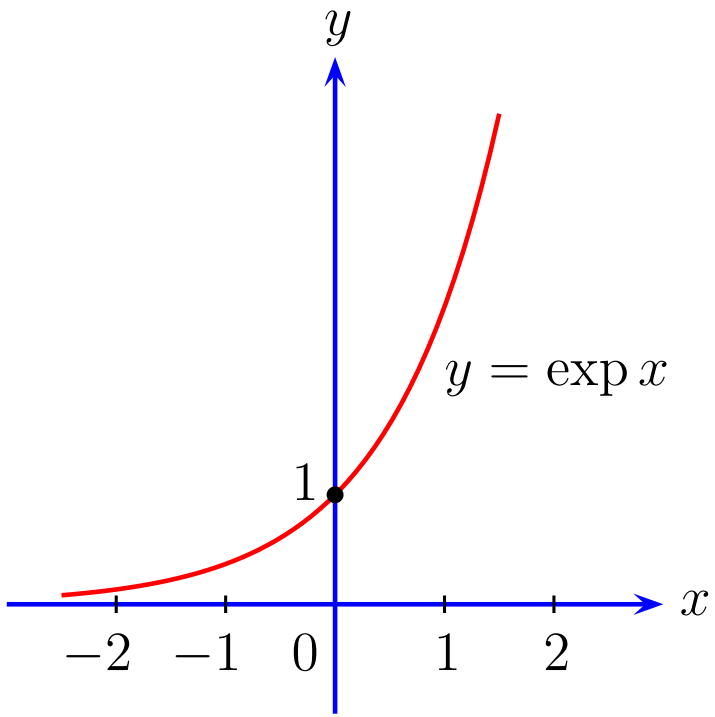
(v) $\ln x \to \infty$ as $x \to \infty$, and $\ln x \to -\infty$ as $x \to 0^+$.
— $\ln 2^n = n \ln 2$ for $n \in \mathbb{N}$ and $\ln(1/x) = -\ln x$.

(vi) $\ln : (0,\infty) \to \mathbb{R}$ is one-one and onto.
— Follows from (iii), (v) and the IVP of the function $\ln$.

The **exponential function** $\exp : \mathbb{R} \to (0, \infty)$ is the inverse of the logarithmic function $\ln : (0, \infty) \to \mathbb{R}$. Thus

$$\exp x = y \iff \ln y = x \quad \text{for } x \in \mathbb{R}.$$





Properties of the exponential function $\exp : \mathbb{R} \to \mathbb{R}$:

(i) $\exp 0 = 1$ and $\exp x > 0$ for every $x \in \mathbb{R}$.

(ii) $\exp$ is differentiable and $(\exp)'(x) = \exp x$ for $x \in \mathbb{R}$:
If $x = \ln y$, then $(\exp)'(x) = 1/(\ln)'(y) = y = \exp x$.

(iii) $\exp$ is strictly increasing and strictly convex on $\mathbb{R}$.
— $\exp' x > 0$ and $\exp'' x > 0$ for $x \in \mathbb{R}$.

(iv) $\exp(x_1 + x_2) = (\exp x_1)(\exp x_2)$ for all $x_1, x_2 \in \mathbb{R}$.
— If $\exp x_1 = y_1$ and $\exp x_2 = y_2$, then
$\ln(y_1 y_2) = \ln y_1 + \ln y_2 = x_1 + x_2$.

(v) $\exp : \mathbb{R} \to (0, \infty)$ is one-one and onto.

(vi) $\exp x \to \infty$ as $x \to \infty$, and $\exp x \to 0$ as $x \to -\infty$:

Let e denote the unique number in $(0, \infty)$ such that $\ln e = 1$.
Then $2 < e < 4$: $\ln 2 = \int_1^2 (1/t)dt < \int_1^2 1\,dt = 1 = \ln e$, and
$\ln 4 = \int_1^4 (1/t)dt = \int_1^2 (1/t)dt + \int_2^4 (1/t)dt > \frac{1}{2} + \frac{2}{4} = 1 = \ln e$.
Now $\exp n = e^n > 2^n \to \infty$ and $0 < \exp(-n) < 2^{-n} \to 0$ as $n \to \infty$. This together with (iii) yields (vi).

Real powers of a positive real number

Let $a > 0$. Observe that for any $r \in \mathbb{Q}$,

$$\ln a^r = r \ln a \quad \text{and hence} \quad a^r = \exp(r \ln a).$$

To see this, consider $f(x) = \ln x^r - r \ln x$ for $x \in (0, \infty)$.
Then f is differentiable and

$$f'(x) = \frac{1}{x^r} r x^{r-1} - r \frac{1}{x} = 0 \quad \text{for all } x \in (0, \infty),$$

and so $f(x) = f(1) = \ln 1^r - r \ln 1 = 0 - 0 = 0 \; \forall \, x \in (0, \infty)$.

Definition

Let $a > 0$. For $x \in \mathbb{R}$, define

$$a^x := \exp(x \ln a)$$

In particular, $e^x = \exp(x \ln e) = \exp x$ for all $x \in \mathbb{R}$.

We can use the above definition of real powers together with the properties of exponential functions to derive "standard properties" of the power function with a fixed base $a > 0$, i.e., the function

$$f_a : \mathbb{R} \to (0, \infty) \quad \text{defined by} \quad f_a(x) := a^x$$

and also of the power function with a fixed exponent $b \in \mathbb{R}$, i.e., the function

$$g_b : (0, \infty) \to \mathbb{R} \quad \text{defined by} \quad g_b(x) := x^b.$$

In particular, both f_a and g_b are differential functions and

$$f_a'(x) = (\ln a)a^x \quad \text{and} \quad g_b'(x) = bx^{b-1}.$$

Further, f_a is bijective if $a \neq 1$, whereas g_b is bijective if $b \neq 0$. For more details, see Section 7.1 of [GL-1].

Inverse Trigonometric and Trigonometric Functions

In our development of calculus thus far, we have freely used the trigonometric functions such as $\sin, \cos, \tan$, and their basic properties without giving a precise definition of these real-valued functions of a real variable. A self-contained and rigorous definition can be given along the similar lines as that of the logarithmic and exponential functions. This is also partly motivated by algebraic considerations.

Noting that the function given by $t \longmapsto 1/(t^2+1) \in \mathbb{R}$ is continuous on $\mathbb{R}$, we make the following

Definition

The **arctangent function** $\arctan : \mathbb{R} \to \mathbb{R}$ is defined by

$$\arctan x := \int_0^x \frac{1}{1+t^2}\,dt \quad \text{for } x \in \mathbb{R}.$$

Properties of the Arctangent Function

From the definition of arctan : $\mathbb{R} \to \mathbb{R}$, the following properties are easily derived.

- $\arctan 0 = 0$,
- $\arctan x \geq 0$ if $x > 0$, while $\arctan x \leq 0$ if $x < 0$.
- arctan is a differentiable function on $\mathbb{R}$ and
$$(\arctan)'x = \frac{1}{1+x^2} \quad \text{for every } x \in \mathbb{R}.$$
- arctan is strictly increasing on $\mathbb{R}$, strictly convex on $(-\infty, 0)$, strictly concave on $(0, \infty)$, and 0 is a point of inflection for arctan.
- arctan is an odd function.

Theorem

arctan : $\mathbb{R} \to \mathbb{R}$ *is a bounded function. In fact,*

$$-2 < \arctan x < 2 \quad \textit{for all } x \in \mathbb{R}.$$

Boundedness of the Arctangent Function

For $x \in (1, \infty)$,

$$\arctan x = \int_0^1 \frac{1}{1+t^2}\,dt + \int_1^x \frac{1}{1+t^2}\,dt.$$

Since $1 \geq t^2$ for $t \in [0,1]$, while $t^2 \geq 1$ for $t \in [1,x]$,

$$\int_0^1 \frac{1}{1+1}\,dt + \int_1^x \frac{1}{t^2+t^2}\,dt \leq \arctan x \leq \int_0^1 \frac{1}{1}\,dt + \int_1^x \frac{1}{t^2}\,dt.$$

The definite integrals above are easy to evaluate, and thus

$$1 - \frac{1}{2x} \leq \arctan x \leq 2 - \frac{1}{x} \quad \text{for all } x \in (1, \infty).$$

Since the function arctan is strictly increasing and odd, it follows that $-2 < \arctan x < 2$ for all $x \in \mathbb{R}$. $\square$

Definition of π

We can now give a self-contained definition of π without getting into vicious circles! By the above results, there is a well-defined real number $\pi \leq 4$ such that

$$\pi = 2\sup\{\arctan x : x \in (0, \infty)\}.$$

Since arctan is increasing and since $\arctan x \geq 1 - (1/2x)$ for all $x \in (1, \infty)$, we see that

$$\frac{\pi}{2} = \sup\{\arctan x : x \in (1, \infty)\} \geq \sup\left\{1 - \frac{1}{2x} : x \in (0, \infty)\right\} = 1.$$

Thus the real number π satisfies $2 \leq \pi \leq 4$.

With this definition, it can be seen that arctan maps $\mathbb{R}$ bijectively onto $(-\pi/2, \pi/2)$ and $\arctan 1 = \pi/4$. Thus, the inverse function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is well-defined and it can easily be shown that this has the "standard properties".

Sine and Cosine Functions

To begin with, we define the sine function and the cosine function on the interval $(-\pi/2, \pi/2)$ by

$$\sin x := \frac{\tan x}{\sqrt{1+\tan^2 x}} \quad \text{and} \quad \cos x := \frac{1}{\sqrt{1+\tan^2 x}}$$

It is clear from the definition that

$$\tan x = \frac{\sin x}{\cos x} \quad \text{for } x \in (-\pi/2, \pi/2).$$

Further, the "standard properties" of the tangent function imply the "standard properties" of the sine and cosine functions. Next, define

$$\sin(\pi/2) := 1 \quad \text{and} \quad \cos(\pi/2) := 0.$$

Finally, extend sin and cos to $\mathbb{R}$ by requiring

$$\sin(x+\pi) := -\sin x \quad \text{and} \quad \cos(x+\pi) := -\cos x \quad \text{for } x \in \mathbb{R}.$$

For more on this approach, see Section 7.2 of [GL-1].