MA 105 : Calculus
Division 3, Lecture 17

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Recap of the previous lecture

- The natural logarithm function and its properties
- The exponential function and its properties
- Real powers of a positive real number
- Definition of the arctangent function via integration, and its properties
- A self-contained definition of using the arctangent function
- An outline of the development of trigonometric functions beginning with arctan

Logarithmic and Arctangent Functions

Definition

The **(natural) logarithmic function** defined by

$$\ln x := \int_1^x \frac{1}{t}\,dt \quad \text{for } x \in (0, \infty).$$

Definition

The **arctangent function** $\arctan : \mathbb{R} \to \mathbb{R}$ is defined by

$$\arctan x := \int_0^x \frac{1}{1+t^2}\,dt \quad \text{for } x \in \mathbb{R}.$$

Theorem

$\arctan : \mathbb{R} \to \mathbb{R}$ *is a bounded function. In fact,*

$$-2 < \arctan x < 2 \quad \textit{for all } x \in \mathbb{R}.$$

Definition of π and other trigonometric functions

Definition

$$\pi = 2\sup\{\arctan x : x \in (0,\infty)\}.$$

Then $\pi \in \mathbb{R}$ with $2 \le \pi \le 4$ and $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ is bijective. Thus, the inverse function $\tan : (-\pi/2, \pi/2) \to \mathbb{R}$ is well-defined, and is bijective as well. Now define

$$\sin x := \frac{\tan x}{\sqrt{1+\tan^2 x}} \quad \text{and} \quad \cos x := \frac{1}{\sqrt{1+\tan^2 x}}$$

for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Next, define

$$\sin(\pi/2) := 1 \quad \text{and} \quad \cos(\pi/2) := 0.$$

Finally, extend sin and cos to $\mathbb{R}$ by requiring

$$\sin(x+\pi) := -\sin x \quad \text{and} \quad \cos(x+\pi) := -\cos x \quad \text{for } x \in \mathbb{R}.$$

and extend tan to $\mathbb{R} \setminus \left\{(2n+1)\frac{\pi}{2} : n \in \mathbb{Z}\right\}$ by $\tan x := \dfrac{\sin x}{\cos x}$.

Applications of Riemann Integration

Area under a Curve

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose $f \geq 0$ on $[a, b]$, and let

$$R_f := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.$$

We say that R_f has an **area** if f is Riemann integrable, and then we define

$$\text{Area}\,(R_f) := \int_a^b f(x)dx.$$

If $f : [a, b] \to \mathbb{R}$ is any function, then $f = f^+ - f^-$, where

$$f^+ := \frac{|f| + f}{2} \quad \text{and} \quad f^- := \frac{|f| - f}{2}.$$

Note that $f^+ \geq 0$ and $f^- \geq 0$.

Positive and Negative Parts of a Function

In fact, for $x \in [a, b]$,

$$f^+(x) = \max\{f(x), 0\} \quad \text{and} \quad f^-(x) = -\min\{f(x), 0\}.$$

The functions f^+ and f^- are known as the **positive part** of f and the **negative** part of f, respectively. Clearly, f is bounded if and only if f^+ and f^- are both bounded. Also, f is integrable if and only if f^+ and f^- are both integrable, and then

$$\begin{aligned} \int_a^b f(x)dx &= \int_a^b f^+(x)dx - \int_a^b f^-(x)dx \\ &= \text{Area}\,(R_{f^+}) - \text{Area}\,(R_{f^-}), \end{aligned}$$

which can be called the 'signed area' delineated by the curve $y = f(x)$, $x \in [a, b]$.

Area between Curves

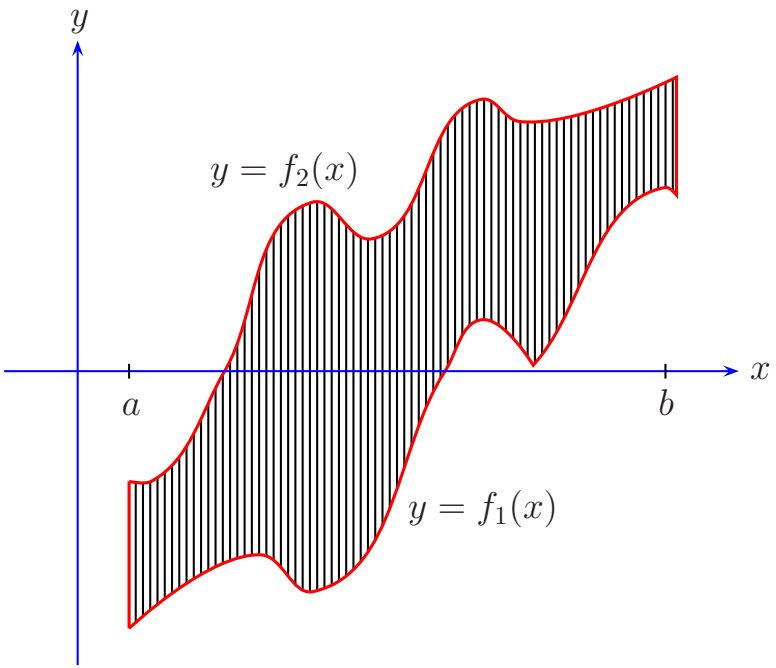
Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be integrable functions such that $f_1 \leq f_2$. Let $R := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\}$ be the **region between the curves** $y = f_1(x)$ and $y = f_2(x)$. Define

$$\text{Area}\,(R) := \text{Area}\,(R_{f_2 - f_1}) = \int_a^b \big(f_2(x) - f_1(x)\big)\,dx.$$

Let $g_1, g_2 : [c, d] \to \mathbb{R}$ be integrable functions such that $g_1 \leq g_2$. Let $R := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}$ be the **region between the curves** $x = g_1(y)$ and $x = g_2(y)$. Define

$$\text{Area}\,(R) := \int_c^d \big(g_2(y) - g_1(y)\big)\,dy.$$

If two curves cross each other a finite number of times, then we must find areas of several regions between them separately, and add them up.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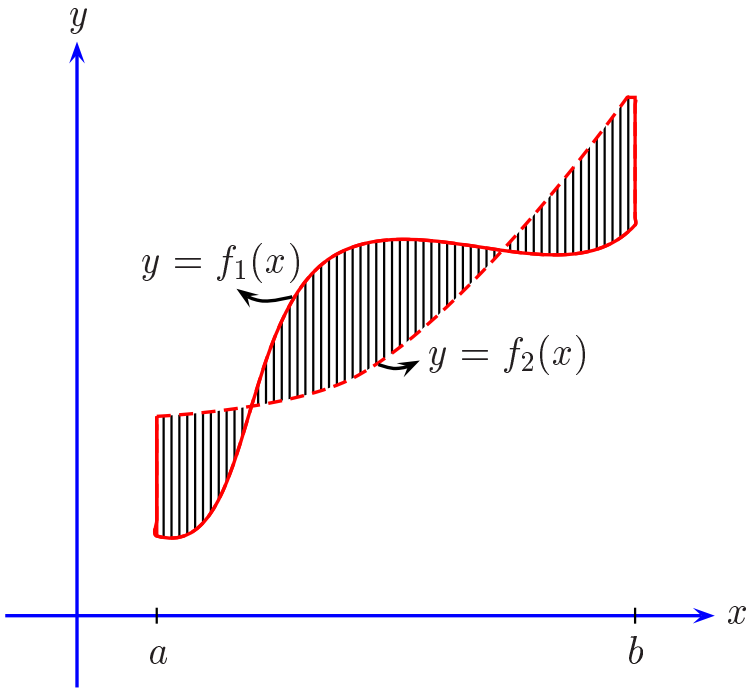
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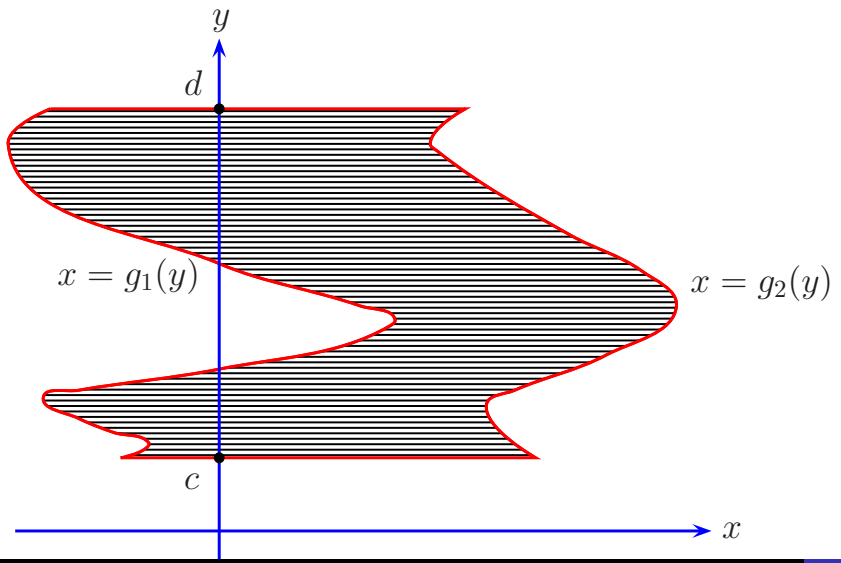
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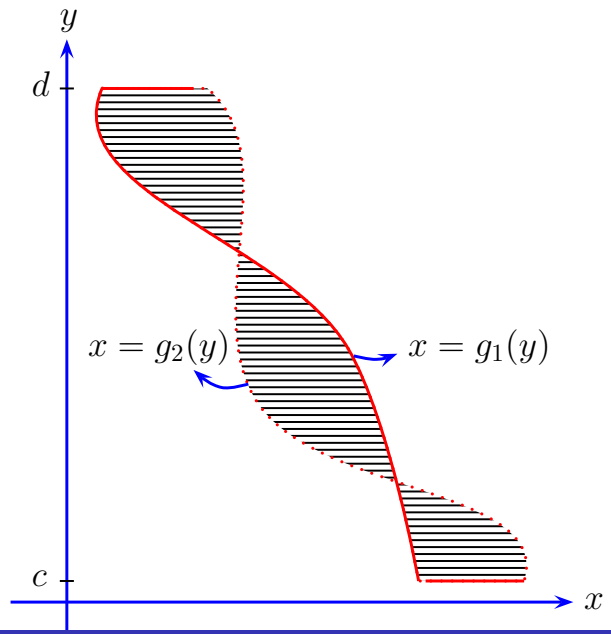
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Examples

(i) Let R denote the region enclosed by the loop of the curve $y^2 = x(1-x)^2$, that is, the region bounded by the curves $y = -\sqrt{x}(1-x)$ and $y = \sqrt{x}(1-x)$.
Now $\sqrt{x}(1-x) = -\sqrt{x}(1-x) \iff x = 0$ or 1, and $\sqrt{x}(1-x) \geq -\sqrt{x}(1-x)$ for $x \in [0,1]$. Hence

$$\text{Area}\,(R) = \int_0^1 \left(\sqrt{x}(1-x) - (-\sqrt{x}(1-x))\right) dx = \frac{8}{15}.$$

(ii) Let R denote the region bounded by the curves $x = -2y^2$ and $x = 1 - 3y^2$.
Now $-2y^2 = 1 - 3y^2 \iff y = \pm 1$, and $-2y^2 \leq 1 - 3y^2$ if $y \in [-1,1]$. Hence

$$\text{Area}\,(R) = \int_{-1}^1 \left(1 - 3y^2 - (-2y^2)\right) dy = \int_{-1}^1 (1 - y^2)\, dy = \frac{4}{3}.$$

Polar coordinates

Review:

The function $\cos^{-1} : [-1, 1] \to [0, \pi]$ is one-one and onto.

Let $P := (x, y) \neq (0, 0)$. There are unique r, $\theta \in \mathbb{R}$ such that

$$r > 0,\ \theta \in (-\pi, \pi],\ x = r\cos\theta \text{ and } y = r\sin\theta.$$

In fact, $r := \sqrt{x^2 + y^2}$ and

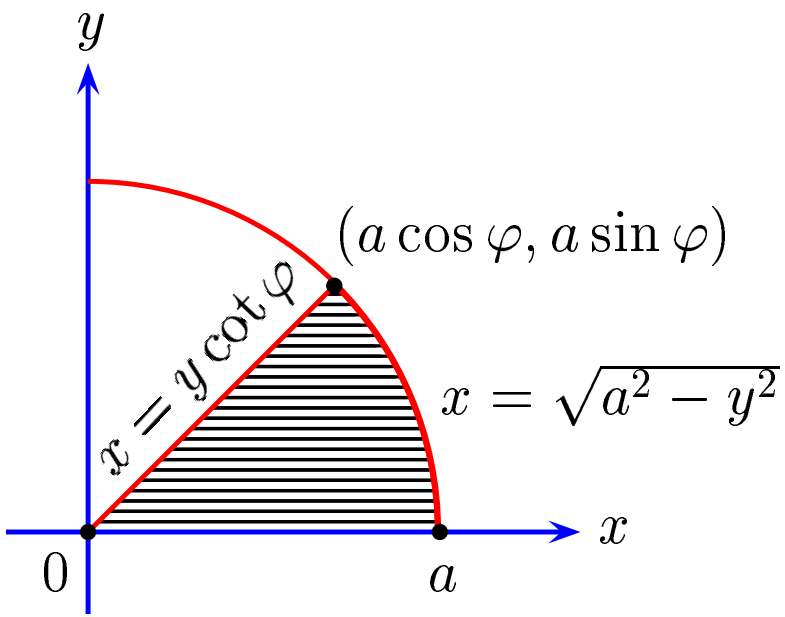
$$\theta := \begin{cases} \cos^{-1}(x/r) & \text{if } y \geq 0, \\ -\cos^{-1}(x/r) & \text{if } y < 0. \end{cases}$$

(If $y < 0$, then $|x/r| < 1$, and $-\cos^{-1}(x/r) \in (-\pi, 0)$.)

The pair (r, θ) is defined as the **polar coordinates** of P.

Area of a sector of a disk

Let $0 \le \varphi \le \pi/2$, and let R denote the sector of a disc of radius a, marked by the points $(0,0), (a,0)$ and $(a\cos\varphi, a\sin\varphi)$, that is, the region bounded by the curves $x = (\cot\varphi)y$ and $x = \sqrt{a^2 - y^2}$ for $y \in [0, a\sin\varphi]$, and by the x-axis.



$$\text{Then Area}\,(R) \;=\; \int_0^{a\sin\varphi} \left(\sqrt{a^2 - y^2} - (\cot\varphi)y\right) dy = \frac{a^2\varphi}{2}.$$

By symmetry, this result holds for $\varphi \in (\pi/2, \pi]$ as well.

Curves given by Polar Equations

Let R denote the region bounded by the curve $r = p(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$, where $-\pi \le \alpha < \beta \le \pi$. Thus

$$R := \{(r\cos\theta, r\sin\theta) : \alpha \le \theta \le \beta \text{ and } 0 \le r \le p(\theta)\}.$$

Suppose $p : [\alpha, \beta] \to \mathbb{R}$ is integrable. (If $\alpha = -\pi$ and $\beta = \pi$, then we suppose $p(-\pi) = p(\pi)$.)

- Partition $[\alpha, \beta]$ into $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$.
- Pick sample points $\gamma_i \in [\theta_{i-1}, \theta_i]$ for $i = 1, \ldots, n$.
- Area between the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$ is approximated by the area of a sector of a disc of radius $r_i := p(\gamma_i)$, that is, by $\dfrac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.
- The sum of areas of these sectors is a Riemann sum, namely $\displaystyle\sum_{i=1}^{n} \frac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.