MA 105 : Calculus
Division 3, Lecture 18

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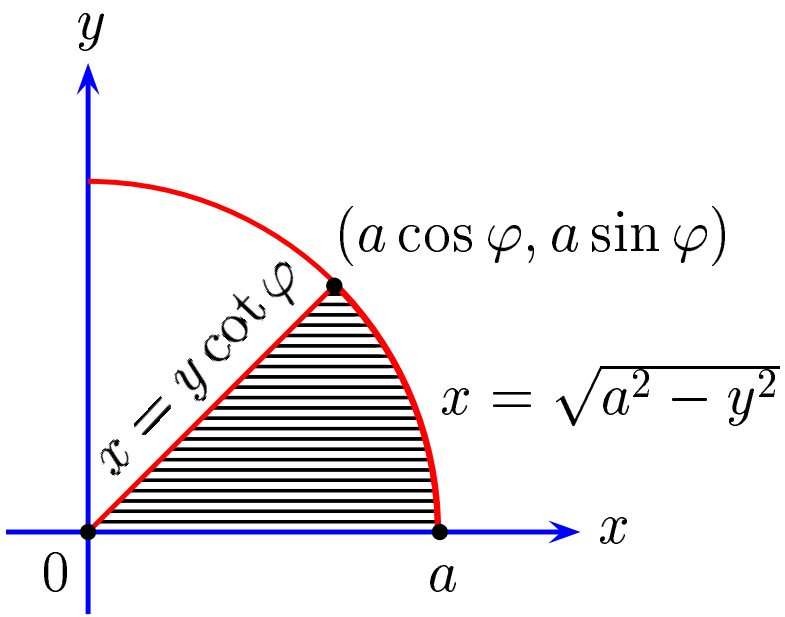
Recap of the previous lecture

- Applications of integration
  - Area of planar regions between two curves. Examples
  - Review of Polar coordinates
  - Area of the sector of a disk
  - Curves given by Polar Equations

Area of a sector of a disk

Let $0 \le \varphi \le \pi/2$, and let R denote the sector of a disc of radius a, marked by the points $(0,0), (a,0)$ and $(a\cos\varphi, a\sin\varphi)$, that is, the region bounded by the curves $x = (\cot\varphi)y$ and $x = \sqrt{a^2 - y^2}$ for $y \in [0, a\sin\varphi]$, and by the x-axis.



$$\text{Then Area}\,(R) \;=\; \int_0^{a\sin\varphi} \left(\sqrt{a^2 - y^2} - (\cot\varphi)y\right) dy = \frac{a^2\varphi}{2}.$$

By symmetry, this result holds for $\varphi \in (\pi/2, \pi]$ as well.

Curves given by Polar Equations

Let R denote the region bounded by the curve $r = p(\theta)$ and the rays $\theta = \alpha$, $\theta = \beta$, where $-\pi \le \alpha < \beta \le \pi$. Thus

$$R := \{(r\cos\theta, r\sin\theta) : \alpha \le \theta \le \beta \text{ and } 0 \le r \le p(\theta)\}.$$

Suppose $p : [\alpha, \beta] \to \mathbb{R}$ is integrable. (If $\alpha = -\pi$ and $\beta = \pi$, then we suppose $p(-\pi) = p(\pi)$.)

- Partition $[\alpha, \beta]$ into $\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta$.
- Pick sample points $\gamma_i \in [\theta_{i-1}, \theta_i]$ for $i = 1, \ldots, n$.
- Area between the rays $\theta = \theta_{i-1}$ and $\theta = \theta_i$ is approximated by the area of a sector of a disc of radius $r_i := p(\gamma_i)$, that is, by $\dfrac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.
- The sum of areas of these sectors is a Riemann sum, namely $\displaystyle\sum_{i=1}^{n} \frac{p(\gamma_i)^2\,(\theta_i - \theta_{i-1})}{2}$.

We define

$$\text{Area}\,(R) := \frac{1}{2}\int_{\alpha}^{\beta} p(\theta)^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta.$$

Examples: (i) Let $a > 0$. Area of the disc enclosed by the circle $r = a$ is equal to $1/2\int_{-\pi}^{\pi} a^2\,d\theta = \pi a^2$.

(ii) Let $a > 0$, and let R denote the region enclosed by the cardioid $r = a(1 + \cos\theta)$. Then

$$\text{Area}\,(R) = \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2\,d\theta = \frac{3a^2\pi}{2}.$$

(iii) Let R denote the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$, where $\theta \in [0, \pi]$. Now $3\sin\theta = 1 + \sin\theta \iff \theta \in \{\pi/6, 5\pi/6\}$, and $1 + \sin\theta \le 3\sin\theta$ if $\theta \in [\pi/6, 5\pi/6]$. Hence

$$\text{Area}\,(R) = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left((3\sin\theta)^2 - (1+\sin\theta)^2\right) d\theta = \pi.$$

Volume of a solid

Let D be a bounded subset of $\mathbb{R}^3$. A cross-section of D obtained by cutting D by a plane in $\mathbb{R}^3$ is called a **slice** of D.

Let $a < b$, and suppose D lies between the planes $x = a$ and $x = b$, which are perpendicular to the x-axis. For $s \in [a, b]$, consider the slice of D by the plane $x = s$, namely $\{(x, y, z) \in D : x = s\}$, and suppose it has an 'area' $A(s)$.

To find the volume of D, we proceed as follows.

- Partition $[a, b]$ into $a = x_0 < x_1 < \cdots < x_n = b$.
- Pick sample points $s_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$.
- Volume between the planes $x = x_{i-1}$ and $x = x_i$ is approximated by the volume of a rectangular slab of width $x_i - x_{i-1}$ and base area $A(s_i)$, that is, by $A(s_i)(x_i - x_{i-1})$.
- The sum of volumes of these slabs is $\displaystyle\sum_{i=1}^{n} A(s_i)(x_i - x_{i-1})$.

Slice Method: We define the **volume** of D by

$$\mathrm{Vol}\,(D) := \int_a^b A(x)\,dx,$$

provided the 'area function' $A : [a, b] \to \mathbb{R}$ is integrable.

Examples:

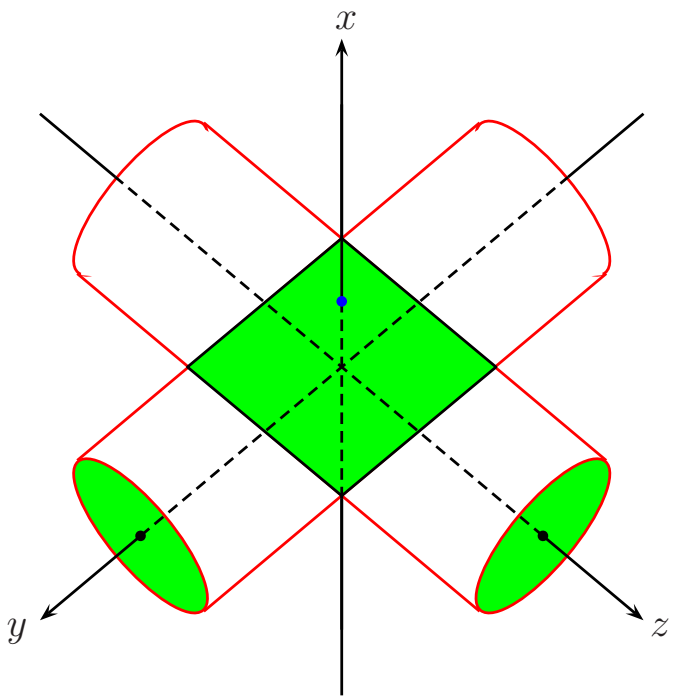
(i) If D is a cylinder with cross-sectional area A and height h, then $\mathrm{Vol}\,(D) = Ah$. (The 'area function' is the constant A.)

(ii) Let $a > 0$, and let D denote the solid enclosed by the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. Then D lies between the planes $x = -a$ and $x = a$. For $s \in [-a, a]$, the slice $\{(x, y, z) \in D : x = s\}$ is the square

$$\left\{(s, y, z) \in \mathbb{R}^3 : |y| \le \sqrt{a^2 - s^2} \text{ and } |z| \le \sqrt{a^2 - s^2}\right\},$$

and so $A(s) = \left(2\sqrt{a^2 - s^2}\right)^2 = 4\,(a^2 - s^2)$. Hence

$$\mathrm{Vol}\,(D) = \int_{-a}^{a} A(s)ds = 4\int_{-a}^{a} \left(a^2 - s^2\right) ds = \frac{16a^3}{3}.$$



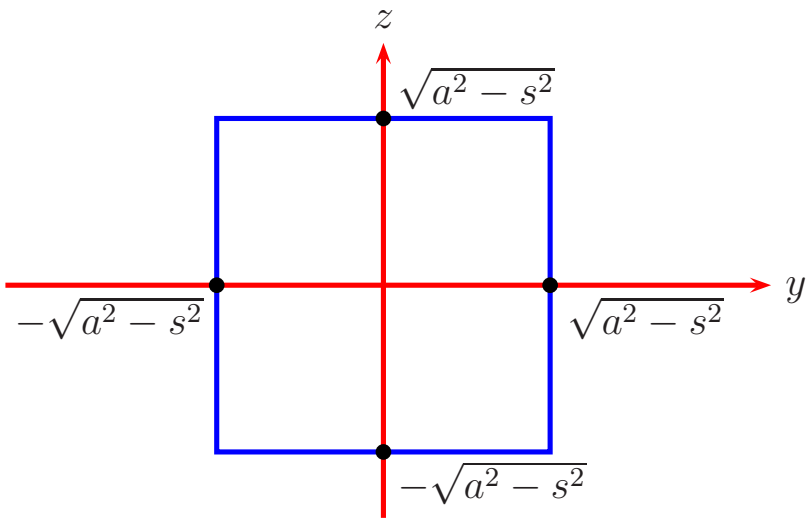


Figure: Solid enclosed by two cylinders and a slice resulting in a square region

Solids of Revolution

If a subset D of $\mathbb{R}^3$ is generated by revolving a planar region about an axis, then D is known as a **solid of revolution**.

Examples:

- Let $a > 0$. The spherical ball

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2\}$$

is obtained by revolving the semidisc

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2 \text{ and } y \geq 0\}$$

about the x-axis.

- Let $b > 0$. The cylindrical solid

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 \leq b^2 \text{ and } 0 \leq y \leq h\}$$

is obtained by revolving the rectangle $[0, b] \times [0, h]$ about the y-axis.

Volume of a Solid of Revolution: Washer Method

- Let D be the solid obtained by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$, and the lines $x = a$ and $x = b$, about the x-axis, where $0 \leq f_1 \leq f_2$.
- The slice of D at $x \in [a, b]$ looks like a circular washer, that is, a disc of radius $f_2(x)$ from which a smaller disc of radius $f_1(x)$ has been removed, and so the area of the slice is $A(x) := \pi\,(f_2(x)^2 - f_1(x)^2)$.
- Suppose f_1 and f_2 are integrable on $[a, b]$. Then the area function A is integrable on $[a, b]$, and by the slice method,

$$\text{Vol}\,(D) = \int_a^b A(x)dx = \pi \int_a^b \left(f_2(x)^2 - f_1(x)^2\right) dx.$$

This special case of the slice method is called the washer method.

If the inner radius of a washer is equal to 0, then the washer is in fact a disk. If this is the case for every $x \in [a, b]$, then the washer method is also known as the disk method.

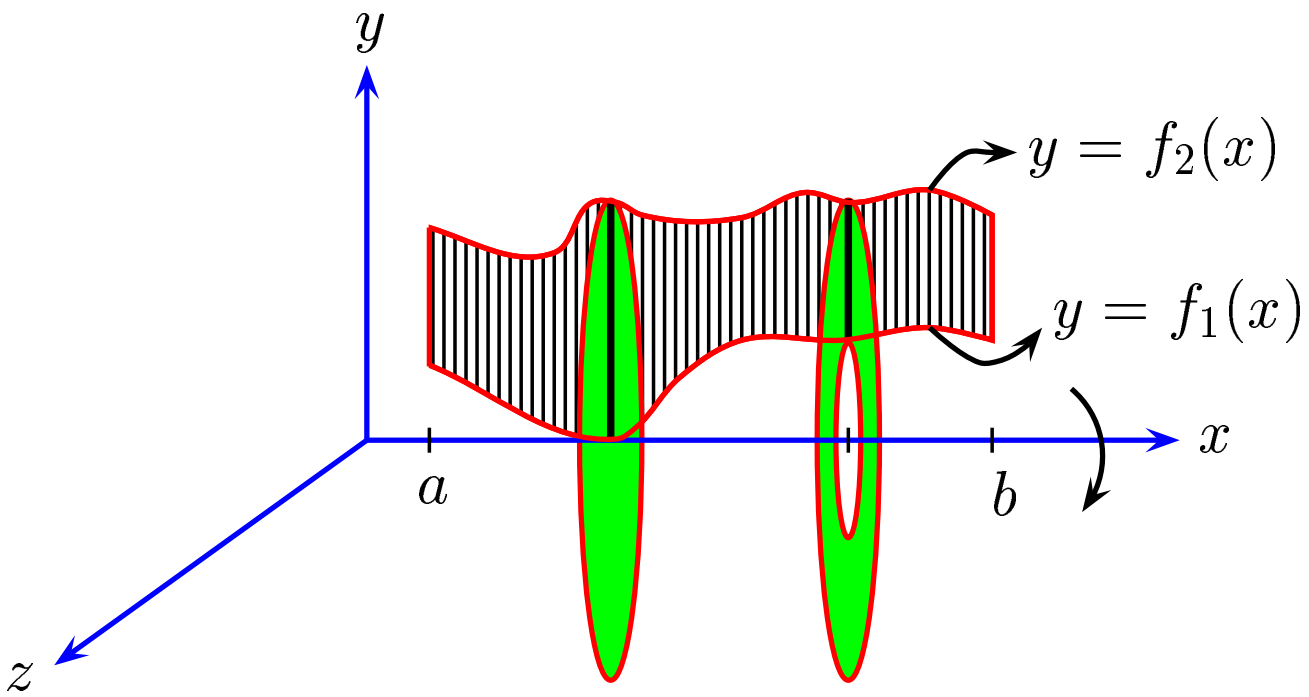


Figure: Illustrations of a disk and of a washer

Examples:

(i) Let D denote the solid obtained by rotating the region between the curves $y = x$ and $y = x^2$ about the x-axis.

- Let $f_1(x) := x^2$ and $f_2(x) := x$ for $x \in [0, 1]$. The curves $y = f_1(x)$ and $y = f_2(x)$ intersect at $x = 0$ and $x = 1$, and $0 \leq f_1 \leq f_2$ on $[0, 1]$.
- By the washer method,

$$\text{Vol}\,(D) = \pi \int_0^1 \left((x)^2 - (x^2)^2\right) dx = \pi \int_0^1 (x^2 - x^4) dx = \frac{2\pi}{15}.$$

(ii) Let D denote the spherical ball with centre at $(0,0,0)$, and radius $a>0$. Then D is obtained by revolving the semidisc $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le a^2 \text{ and } y\ge 0\}$ about the x-axis.

- Let $f_1(x):=0$ and $f_2(x):=\sqrt{a^2-x^2}$ for $x\in[-a,a]$. The curves $y=f_1(x)$ and $y=f_2(x)$ intersect at $x=-a$ and $x=a$, and $0\le f_1\le f_2$ on $[-a,a]$.
- By the disc method,

$$\text{Vol}\,(D)=\pi\int_{-a}^{a}\left(\left(\sqrt{a^2-x^2}\right)^2-0^2\right)dx=\pi\int_{-a}^{a}(a^2-x^2)dx=\frac{4\pi a^3}{3}.$$

Let D be the solid obtained by revolving the region between the curves $x = g_1(y)$ and $x = g_2(y)$, $c \leq y \leq d$, about the y-axis, where $0 \leq g_1 \leq g_2$. Then, as before,

$$\text{Vol}\,(D) = \pi \int_c^d \left(g_2(y)^2 - g_1(y)^2\right) dy.$$

Example:

Let D denote the solid obtained by revolving the region in the first quadrant between the parabolas $y = x^2$ and $y = 2 - x^2$ about the y-axis. Now $\sqrt{y} = \sqrt{2-y} \iff y = 1$. By the disk method,

$$\text{Vol}\,(D) = \pi \int_0^1 (\sqrt{y})^2 dy + \pi \int_1^2 \left(\sqrt{2-y}\right)^2 dy = \pi\left(\frac{1}{2} + \frac{1}{2}\right) = \pi.$$

Volume of a Solid of Revolution: Shell Method

Let D be a bounded subset of $\mathbb{R}^3$. A cross-section of D obtained by piercing a cylinder through D is called a **sliver** of D.

- Let $f_1, f_2 : [a, b] \to \mathbb{R}$ be such that $f_1 \leq f_2$, and suppose $0 \leq a < b$. Let D denote the solid generated by revolving the region between the curves $y = f_1(x)$ and $y = f_2(x)$, and the lines $x = a$ and $x = b$ about the y-axis.
- For $s \in [a, b]$, the sliver $\{(x, y, z) \in D : x^2 + z^2 = s^2\}$ of D by the cylinder $x^2 + z^2 = s^2$ is a right circular cylinder having height $f_2(s) - f_1(s)$ and radius s, and so its surface area is $2\pi s\big(f_2(s) - f_1(s)\big)$. (To be justified later)
- Suppose f_1 and f_2 are integrable on $[a, b]$. Since D is 'made up' of these cylindrical slivers (or shells), we define

$$\text{Vol}\,(D) := 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)\,dx,$$

given by the shell method.

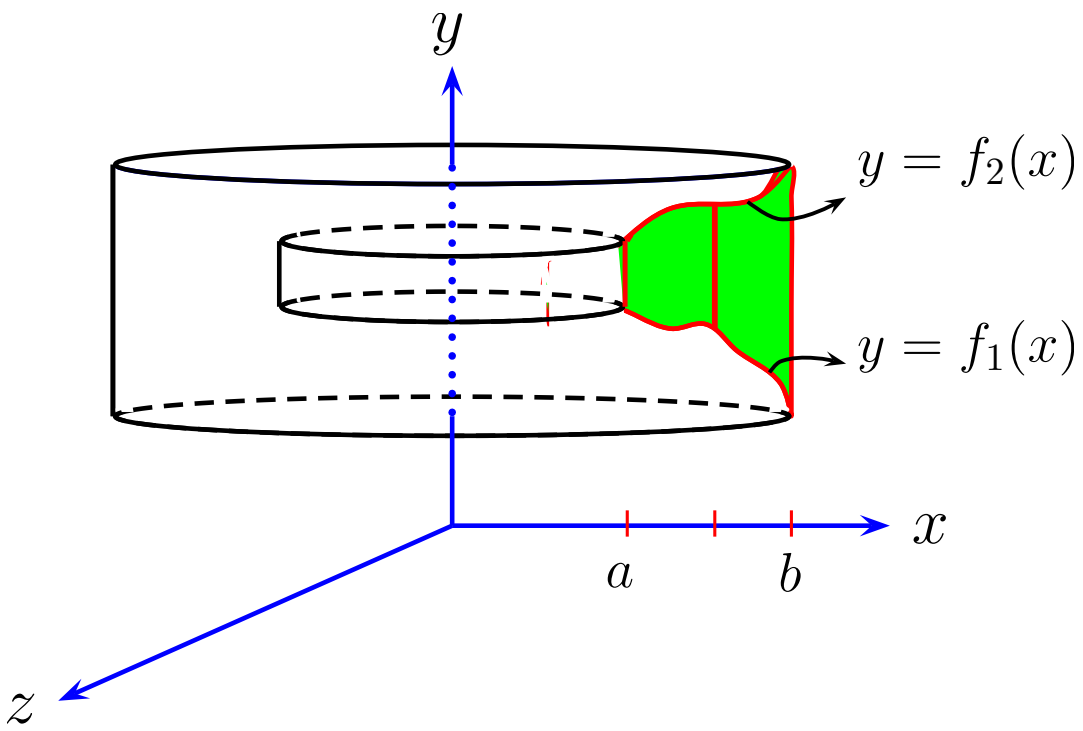


Figure: Illustration of the Shell Method

$$\text{Vol}\,(D) := 2\pi \int_a^b x\big(f_2(x) - f_1(x)\big)\,dx.$$

Shell Method (continued)

Let $g_1, g_2 : [c, d] \to \mathbb{R}$ be integrable functions such that $g_1 \leq g_2$, and suppose $0 \leq c < d$. Let D denote the solid obtained by revolving the region between the curves $x = g_1(y)$ and $x = g_2(y)$ and between the lines $y = c$ and $y = d$ about the x-axis.

For $t \in [c, d]$, the sliver $\{(x, y, z) \in D : y^2 + z^2 = t^2\}$ of D by the cylinder $y^2 + z^2 = t^2$ is a right circular cylinder having height $g_2(t) - g_1(t)$ and radius t, and so its surface area is $2\pi t\big(g_2(t) - g_1(t)\big)$.

Since D is 'made up' of these cylindrical slivers (or shells), we define

$$\text{Vol}\,(D) := 2\pi \int_c^d y\big(g_2(y) - g_1(y)\big)\,dy.$$

Examples

(i) Let D denote the solid obtained by revolving the region bounded by the curves $y = 2x^2 - x^3$ and $y = 0$ about the y-axis. Now $2x^2 - x^3 = 0 \iff x = 0$ or $x = 2$. By the shell method,

$$\text{Vol}\,(D) = 2\pi \int_0^2 x(2x^2 - x^3 - 0)dx = \frac{56\pi}{5}.$$

(ii) Let D denote the solid obtained by revolving the rectangle $[0, b] \times [0, h]$ about the y-axis. By the shell method,

$$\text{Vol}\,(D) = 2\pi \int_0^b x\,(h - 0)\,dx = \pi h b^2.$$

(iii) Let E denote the solid obtained by revolving the rectangle $[0, b] \times [0, h]$ about the x-axis. By the shell method,

$$\text{Vol}\,(E) = 2\pi \int_0^h y\,(b - 0)\,dy = \pi b h^2.$$

(iv) Let D denote the solid in $\mathbb{R}^3$ obtained by revolving the region in the first quadrant bounded by the curve $y = x^3$ and the line $y = 4x$ about the line $x = -1$.

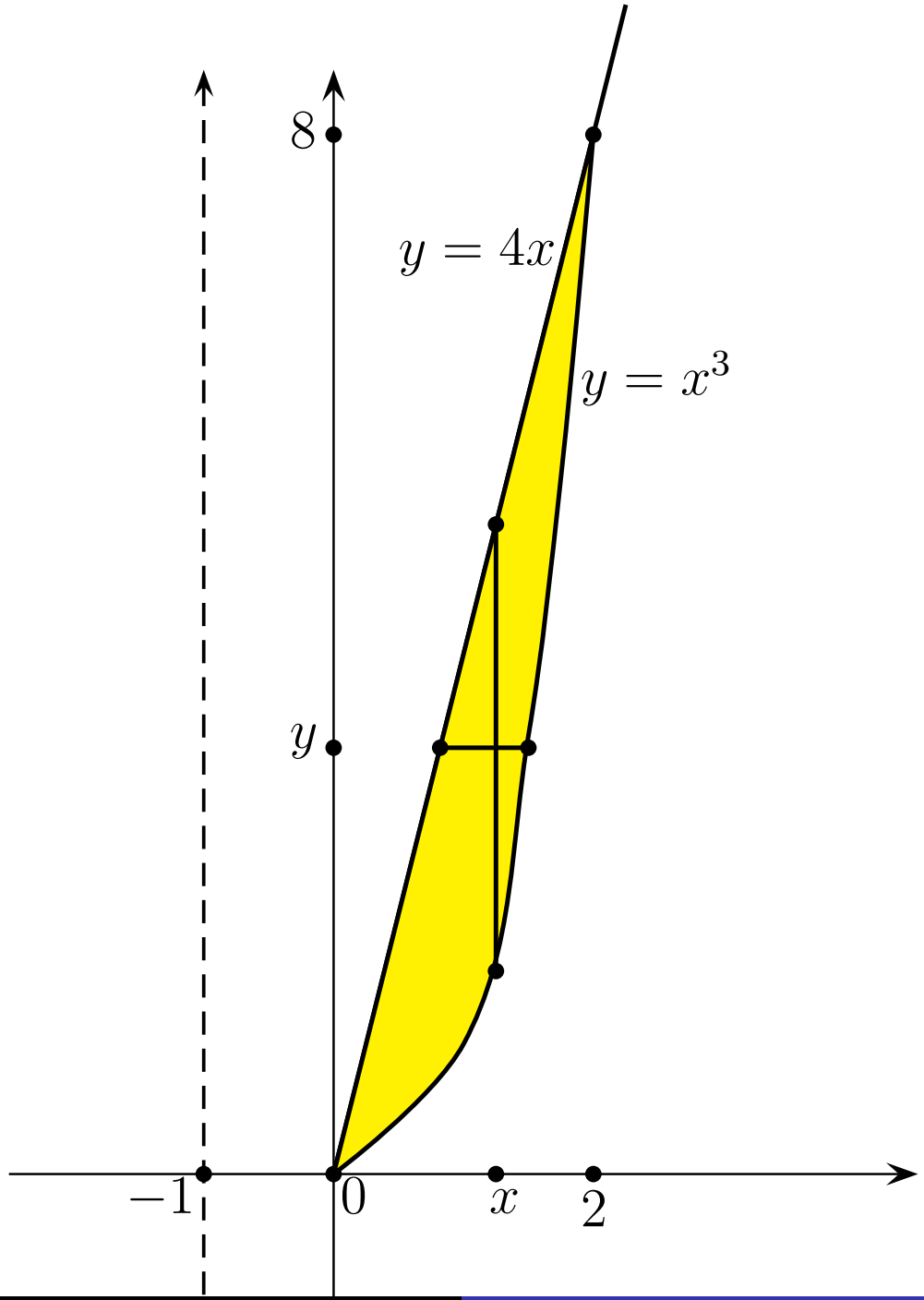
Now in the first quadrant, $x^3 = 4x \iff x = 0$ or $x = 2$.

- By the washer method,

$$\begin{aligned}\mathrm{Vol}\,(D) &= \pi \int_0^8 \Big(\big(y^{1/3} + 1\big)^2 - \big((y/4) + 1\big)^2 \Big)\, dy \\ &= \pi \int_0^8 \big(y^{2/3} + 2y^{1/3} - (y^2/16) - (y/2)\big)\, dy = \frac{248\pi}{15}.\end{aligned}$$

- By the shell method,

$$\begin{aligned}\mathrm{Vol}\,(D) &= 2\pi \int_0^2 (x+1)\big(4x - x^3\big)\, dx \\ &= 2\pi \int_0^2 \big(4x + 4x^2 - x^3 - x^4\big)\, dx = \frac{248\pi}{15}.\end{aligned}$$


8
$y = 4x$
$y = x^3$
y
-1
0
x
2

Remarks on Volume of a Solid of Revolution

Let D denote a bounded solid of revolution.

Remark 1. In the washer method, the slices (which look like washers) are taken perpendicular to the axis of revolution. On the other hand, in the shell method, the slivers (which look like cylindrical shells) are taken parallel to the axis of revolution.

Remark 2. The basic expression in the washer method is $\pi(r_2^2 - r_1^2)$, where r_2 and r_1 are outer and inner radii of the washer. The basic expression in the shell method is $2\pi r(h_2 - h_1)$, where r is the radius of the sliver, while $h_2 - h_1$ is the height of the sliver.

Remark 3. The volume of D found by the washer method and by the shell method must be the same! This result would follow from a general definition of the volume of a solid in $\mathbb{R}^3$. This can serve as a check on your calculations.