MA 105 : Calculus
Division 3, Lecture 19

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Recap of the previous lecture

- Curves given by polar equations. Area of related planar regions
- Volume of a solid. Slice Method, Examples
- Solids of Revolution
  - Washer Method
  - Shell Method
  - Examples

Remarks on Volume of a Solid of Revolution

Let D denote a bounded solid of revolution.

Remark 1. In the washer method, the slices (which look like washers) are taken perpendicular to the axis of revolution. On the other hand, in the shell method, the slivers (which look like cylindrical shells) are taken parallel to the axis of revolution.

Remark 2. The basic expression in the washer method is $\pi(r_2^2 - r_1^2)$, where r_2 and r_1 are outer and inner radii of the washer. The basic expression in the shell method is $2\pi r(h_2 - h_1)$, where r is the radius of the sliver, while $h_2 - h_1$ is the height of the sliver.

Remark 3. The volume of D found by the washer method and by the shell method must be the same! This result would follow from a general definition of the volume of a solid in $\mathbb{R}^3$. This can serve as a check on your calculations.

Parametrized Curve

A **parametrized curve** or a **path** C in $\mathbb{R}^2$ is given by $(x(t), y(t))$, where $x, y : [\alpha, \beta] \to \mathbb{R}$ are continuous functions.

Here $[\alpha, \beta]$ is called the **parameter interval**.

We wish to define the 'length' of C.

Basic assumption: The (Euclidean) length of a line segment joining points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is equal to

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We shall assume that C is **smooth**, that is, the functions x, y are **continuously differentiable** on $[\alpha, \beta]$. This means that x, y are differentiable on $[\alpha, \beta]$, and their derivatives x', y' are continuous on $[\alpha, \beta]$.

Arc Length of a Smooth Curve

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 1, \ldots, n$, and draw the line segments joining P_0 to P_1, P_1 to P_2, …, P_{n-1} to P_n.
- The sum of the lengths of these line segments is

$$\sum_{i=1}^{n} \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2}$$

$$= \sum_{i=1}^{n} \sqrt{\big(x'(s_i)\big)^2 + \big(y'(u_i)\big)^2}\,(t_i - t_{i-1}),$$

  for some $s_i,\ u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ by the MVT.

- We define the **arc length** of C by

$$\ell(C) := \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Special Cases

Special cases:

(i) Let a curve C be given by $y = f(x)$, $x \in [a, b]$.
Here $\alpha := a$, $\beta := b$, $x(t) := t$ and $y(t) := f(t)$ for $t \in [a, b]$.
Suppose f is continuously differentiable on $[a, b]$. Then

$$\ell(C) := \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

(ii) Let a curve C be given by $x = g(y)$, $y \in [c, d]$.
Here $\alpha := c$, $\beta := d$, $x(t) := g(t)$ and $y(t) := t$ for $t \in [c, d]$.
Suppose g is continuously differentiable on $[c, d]$. Then

$$\ell(C) := \int_c^d \sqrt{g'(y)^2 + 1}\, dy.$$

Arc Length in Polar coordinates

Let C be given by a polar equation $r = p(\theta)$, $\theta \in [\alpha, \beta]$. As a parametrized curve, C is given by $(x(\theta), y(\theta))$, where

$$x(\theta) := p(\theta)\cos\theta \quad \text{and} \quad y(\theta) := p(\theta)\sin\theta, \quad \theta \in [\alpha, \beta].$$

Suppose the function p is continuously differentiable on $[\alpha, \beta]$.

For $\theta \in [\alpha, \beta]$, we note that $\sqrt{x'(\theta)^2 + y'(\theta)^2}$ is equal to

$$\begin{aligned} & \sqrt{\left(p'(\theta)\cos\theta - p(\theta)\sin\theta\right)^2 + \left(p'(\theta)\sin\theta + p(\theta)\cos\theta\right)^2} \\ = \quad & \sqrt{p(\theta)^2 + p'(\theta)^2}. \end{aligned}$$

Hence

$$\ell(C) := \int_\alpha^\beta \sqrt{p(\theta)^2 + p'(\theta)^2}\, d\theta.$$

Examples

(i) Let C be given by $y = x^2$, $x \in [0, 1]$. Then

$$\begin{aligned} \ell(C) &= \int_0^1 \sqrt{1 + (2x)^2}\, dx = \frac{1}{2} \int_0^2 \sqrt{1 + u^2}\, du \\ &= \frac{1}{2}\sqrt{5} + \frac{1}{4} \ln\left(2 + \sqrt{5}\right). \end{aligned}$$

(Use Integration by Parts. Also, if $f(u) := \ln(u + \sqrt{1 + u^2})$ for $u \in \mathbb{R}$, then note that $f'(u) = 1/\sqrt{1 + u^2}$ for $u \in \mathbb{R}$, and so $\int_0^x \sqrt{1 + u^2}\, du = \frac{1}{2}\left(x\sqrt{1 + x^2} + \ln\left(x + \sqrt{1 + x^2}\right)\right)$ for $x \in \mathbb{R}$.)

(ii) Let C be given by $x = (2y^6 + 1)/8y^2$, $y \in [1, 2]$. Then

$$\int_1^2 \left(1 + \left(y^3 - \frac{1}{4y^3}\right)^2\right)^{1/2} dy = \int_1^2 \left(y^3 + \frac{1}{4y^3}\right) dy = \frac{123}{32}.$$

(iii) Let $a > 0$ and $\varphi \in [0, \pi]$. Let C denote the arc of a circle of radius a given by $x(\theta) := a\cos\theta$, $y(\theta) := a\sin\theta$ for $\theta \in [0, \varphi]$. Then C is given by the polar equation $r = p(\theta)$, where $p(\theta) = a$ for $\theta \in [0, \phi]$, and so

$$\ell(C) = \int_0^{\varphi} \sqrt{a^2 + 0^2}\, d\theta = a\,\varphi.$$

Hence the length of a circle of radius a is $\int_{-\pi}^{\pi} a\, d\theta = 2\pi a$.

(iv) Let C be given by $r = 1 + \cos\theta$ for $\theta \in [0, \pi]$. Then

$$\begin{aligned} \ell(C) &= \int_0^{\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= \int_0^{\pi} \sqrt{2(1+\cos\theta)}\, d\theta = 2\int_0^{\pi} \cos\frac{\theta}{2}\, d\theta = 4. \end{aligned}$$

(Note: $\cos(\theta/2) \geq 0$ for $\theta \in [0, \pi]$.)

Curves in $\mathbb{R}^3$

Suppose C is a smooth parametrized curve in $\mathbb{R}^3$ given by $(x(t), y(t), z(t))$, where $x,\ y,\ z : [\alpha, \beta] \to \mathbb{R}$ are continuously differentiable functions on $[\alpha, \beta]$.

In analogy with the definition of the arc length of a curve in $\mathbb{R}^2$, we define the **arc length** of C by

$$\ell(C) := \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Example

Let C denote a **helix** in $\mathbb{R}^3$ given by $x(t) := a\cos t,\ \ y(t) := a\sin t,\ \ z(t) := b\,t,\ \ t \in [\alpha, \beta]$, where $a,\ b, \in \mathbb{R},\ a > 0$ and $b \neq 0$. Then

$$\ell(C) = \int_\alpha^\beta \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2}\, dt = (\beta - \alpha)\sqrt{a^2 + b^2}.$$

Surface of Revolution

A **surface of revolution** is generated when a curve C in $\mathbb{R}^2$ is revolved about a line L in $\mathbb{R}^2$.

First suppose the curve C is a slanted line segment P_1P_2 of length λ_2, and C does not cross L. Let d_1 and d_2 denote the distances of P_1 and P_2 from L with $d_1 \leq d_2$. Then the surface of revolution is a frustum F of a right circular cone with base radii d_1 and d_2, and slant height λ_2. We find its surface area.

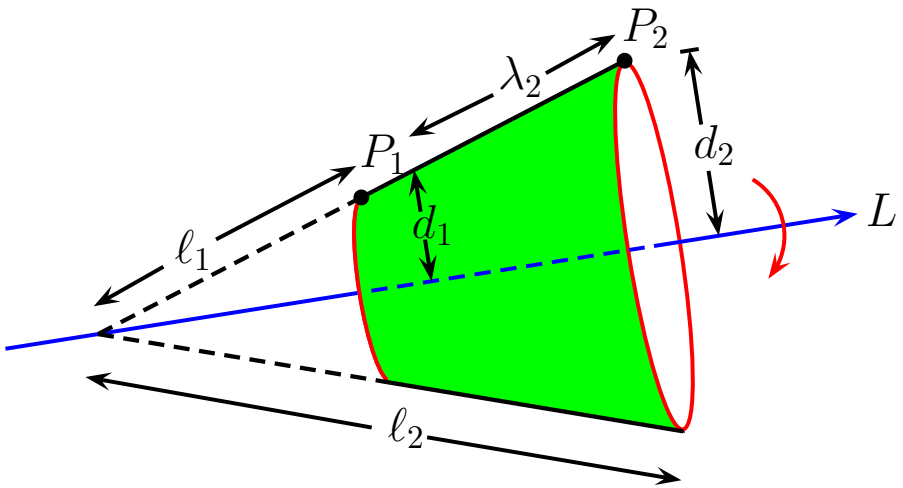


Figure: Frustum of a right circular cone

Consider a cone with base radius d and slant height ℓ. If we slit open this cone, we obtain a sector of a disk of radius ℓ.

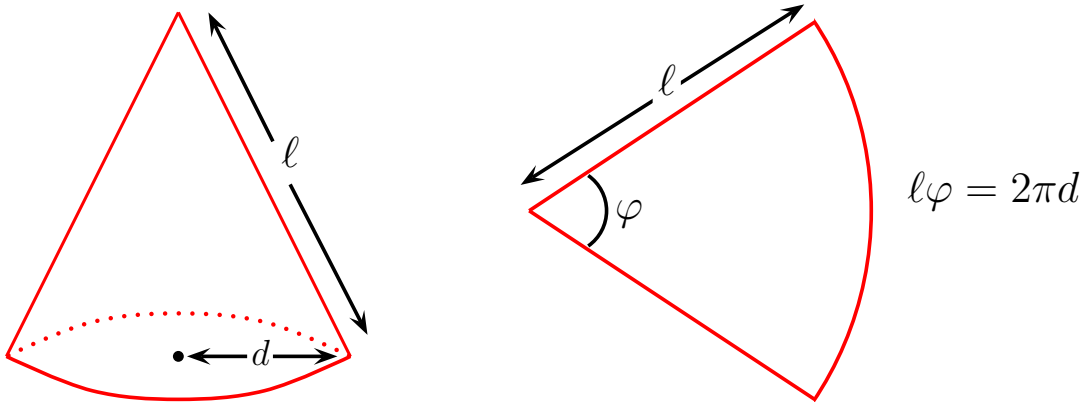


Figure: Right circular cone and sector of a disk

Since $\ell\varphi = 2\pi d$, the surface area of the cone is equal to

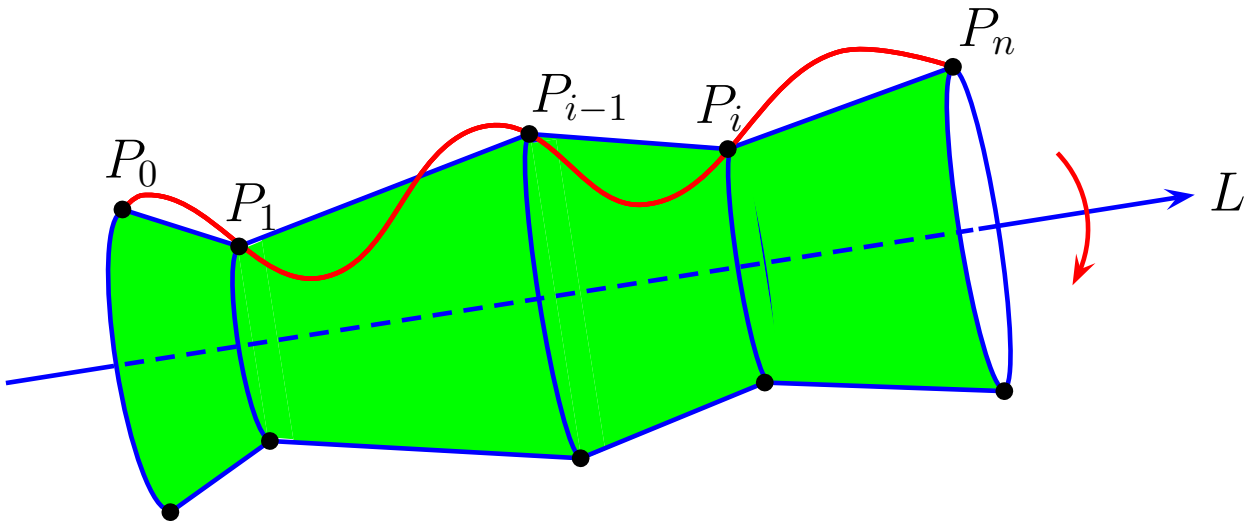
$$\frac{1}{2}\ell^2\varphi = \frac{1}{2}\ell^2\frac{2\pi d}{\ell} = \pi\ell d.$$

Hence the surface area of the frustrum F of the cone is $\pi\ell_2 d_2 - \pi\ell_1 d_1 = \pi(d_1 + d_2)(\ell_2 - \ell_1) = \pi(d_1 + d_2)\lambda_2$.

Now suppose C is parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$.

- Partition $[\alpha, \beta]$ into $\alpha = t_0 < t_1 < \cdots < t_n = \beta$.
- Let $P_i := \big(x(t_i), y(t_i)\big)$ for $i = 0, 1, \ldots, n$, and draw the line segments P_0P_1, P_1P_2, …, $P_{n-1}P_n$.

Let $d_0, d_1, d_2, \ldots, d_n$ be the distances of $P_0, P_1, P_2, \ldots, P_n$ from the line L. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the lengths of the line segments $P_0P_1, P_1P_2, \ldots, P_{n-1}P_n$. Suppose they don't cross L.



Fix $i \in \{1, \ldots, n\}$. When the line segment $P_{i-1}P_i$ is revolved about the line L, it generates a frustum F_i (of a right circular cone) whose surface area is $\pi(d_{i-1} + d_i)\lambda_i$.

Let $\rho(t)$ denote the distance of the point $(x(t), y(t))$ on the curve C from the line L. Then $d_i = \rho(t_i)$ for $i = 0, 1, \ldots, n$.

Thus the sum of the surface areas of the frustrums $F_1, \ldots, F_n$ is

$$\pi \sum_{i=1}^{n} \big(\rho(t_{i-1}) + \rho(t_i)\big)\lambda_i,$$

If the functions x' and y' are continuously differentiable on $[\alpha, \beta]$, then the length λ_i of the line segment $P_{i-1}P_i$ is given by

$$\begin{aligned} \lambda_i &= \sqrt{\big(x(t_i) - x(t_{i-1})\big)^2 + \big(y(t_i) - y(t_{i-1})\big)^2} \\ &= \sqrt{x'(s_i)^2 + y'(u_i)^2}\,(t_i - t_{i-1}) \end{aligned}$$

for some $s_i,\ u_i \in (t_{i-1}, t_i)$ for $i = 1, \ldots, n$ (by the MVT).

Area of Surface of Revolution

Let C be a smooth curve parametrized by $(x(t), y(t))$, $t \in [\alpha, \beta]$. Suppose the curve C does not cross the line L given by $ax + by + c = 0$. We define the **area of the surface** S generated by revolving C about the line L by

$$\text{Area}\,(S) := 2\pi \int_{\alpha}^{\beta} \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

where $\rho(t)$ is the distance of $(x(t), y(t))$ from the line L,

that is, $\rho(t) := |ax(t) + by(t) + c|/\sqrt{a^2 + b^2}$ for $t \in [a, b]$.

Note: Since the curve C does not cross the line L, the curve C lies entirely on one of the sides of the line L, that is,
either $ax(t) + by(t) + c \geq 0$ for all $t \in [\alpha, \beta]$,
or $ax(t) + by(t) + c \leq 0$ for all $t \in [\alpha, \beta]$.

Special Cases:

(i) Let the line L be the x-axis, and let the curve C be given by $y = f(x)$ for $x \in [a, b]$, where f is continuously differentiable. If $f \geq 0$ on $[a, b]$ or $f \leq 0$ on $[a, b]$, then

$$\text{Area}(S) = 2\pi \int_a^b |f(x)| \sqrt{1 + f'(x)^2}\, dx.$$

(ii) Let the line L be the y-axis, and let the curve C be given by $x = g(y)$ for $y \in [c, d]$, where g is continuously differentiable. If $g \geq 0$ on $[c, d]$ or $g \leq 0$ on $[c, d]$, then

$$\text{Area}(S) = 2\pi \int_c^d |g(y)| \sqrt{1 + g'(y)^2}\, dy.$$

(iii) Let the line L be given by $\theta = \gamma$, where $\gamma \in (-\pi, \pi]$, and let the curve C be given by $r = p(\theta)$ for $\theta \in [\alpha, \beta]$, where p is continuously differentiable on $[\alpha, \beta]$. Suppose C does not cross L. Now the curve C is also given by $(p(\theta)\cos\theta, p(\theta)\sin\theta)$ for $\theta \in [\alpha, \beta]$.

Also, $\rho(\theta) = p(\theta)|\sin(\theta - \gamma)|$ for $\theta \in [\alpha, \beta]$.

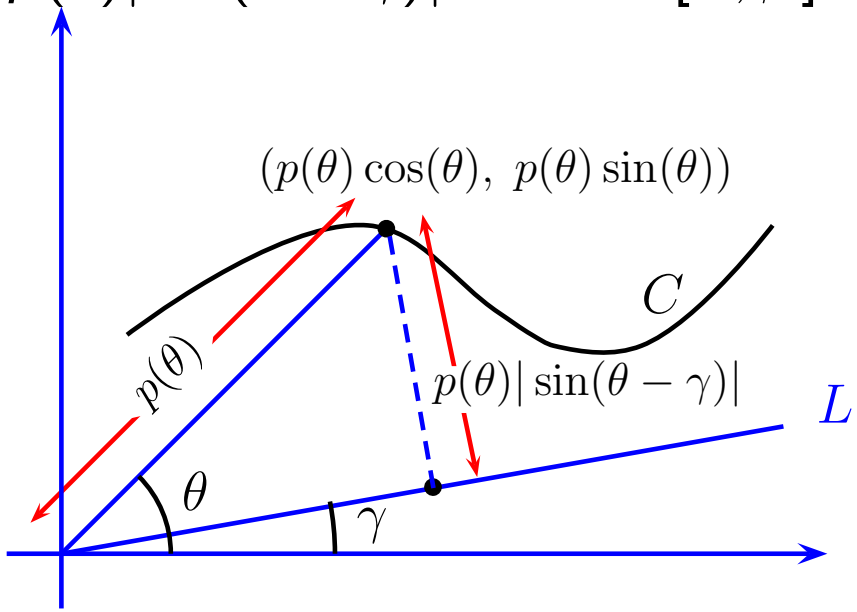


Figure: Distance of a point on a polar curve from a ray.

Thus $\text{Area}(S) = 2\pi \int_\alpha^\beta p(\theta)|\sin(\theta-\gamma)|\sqrt{p(\theta)^2 + p'(\theta)^2}\,d\theta.$

Examples:

(i) [Tut Sheet 5, # 5] Let S denote the surface generated by revolving the curve $y = (x^3/3) + (1/4x)$, $x \in [1, 3]$, about the line $y = -1$. Then

$$\begin{aligned}
\text{Area}(S) &= 2\pi \int_1^3 (y+1)\sqrt{1+(y')^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right) \sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right) \left(x^2 + \frac{1}{4x^2}\right) dx \\
&= 1823\pi/18.
\end{aligned}$$

(iii) Let $0 < b < a$ and let C denote the circle given by $(a + b\cos t, b\sin t)$, $t \in [-\pi, \pi]$. Let S denote the surface generated by revolving the curve C about the y-axis. Then $a + b\cos t > 0$ for all $t \in [-\pi, \pi]$, and so

$$\begin{aligned}\text{Area}\,(S) &= 2\pi\int_{-\pi}^{\pi}(a+b\cos t)\sqrt{(-b\sin t)^2+(b\cos t)^2}\,dt\\ &= 2\pi b\int_{-\pi}^{\pi}(a+b\cos t)dt\\ &= 4\pi^2 ab.\end{aligned}$$

Note: S is in fact the surface of a torus in $\mathbb{R}^3$.

(iii) Let $a>0$, and S denote the surface generated by revolving the semicircle $p(\theta)=a$, $\theta\in[0,\pi]$, about the x-axis. Then

$$\text{Area}(S)=2\pi\int_0^{\pi}a\sin\theta\sqrt{a^2+0^2}\,d\theta=4\pi a^2.$$

Note: S is in fact the sphere of radius a in $\mathbb{R}^3$.

Remaining Tut Problem

In the last tutorial, all problems of Tut Sheet 5, except # 4, 5, 7, were covered. Of those that remain, (7) was worked out in the previous lecture and (5) in this lecture. We will now work out:

(4) Find the arc length of the each of the curves described below.

(i) the cycloid $x = t - \sin t,\ y = 1 - \cos t,\ 0 \le t \le 2\pi$

(ii) $y = \int_0^x \sqrt{\cos 2t}\, dt,\ 0 \le x \le \pi/4.$

Solution: The arc length in (i) is given by

$$\int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)}dt = \int_0^{2\pi} 2|\sin(t/2)|dt = 8.$$

and using the FTC, the arc length in (ii) is given by

$$\int_0^{\pi/4} \sqrt{1+y'^2}dx = \int_0^{\pi/4} \sqrt{1+\cos(2x)}dx = \sqrt{2}\int_0^{\pi/4} |\cos(x)|dx = 1.$$

Thus Tutorial Sheet 5 is now completely done!