MA 105 : Calculus
Division 3, Lecture 02

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Recap of the previous lecture

- Generalities about the course
- Text and References
- Basic notations: $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$.
- The set $\mathbb{R}$ of real numbers as a set containing $\mathbb{Q}$ and satisfying the
  - Algebraic Properties
  - Order Properties
  - Completeness Property
- Upper bound, lower bound, supremum,and infimum
- Consequences of the Completeness Property:
  - Existence of infimum of a nonempty set bounded below;
  - Archimedean Property;
  - Existence and uniqueness of positive nth roots of positive real numbers;
  - Existence of a rational as well as an irrational between any two real numbers $a < b$.

Intervals

Given any $a, b \in \mathbb{R}$, we define

- $(a, b) := \{x \in \mathbb{R} : a < x < b\}$ [open interval]
- $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$ [closed interval]
- Semi-open intervals $(a, b]$ and $[a, b)$ are defined similarly.
- It is also useful to consider symbols ∞ and $-\infty$ and define the infinite intervals

$$(a, \infty) := \{x \in \mathbb{R} : x > a\}, \quad [a, \infty) := \{x \in \mathbb{R} : x \geq a\},$$
$$(-\infty, a) := \{x \in \mathbb{R} : x < a\}, \quad (-\infty, a] := \{x \in \mathbb{R} : x \leq a\}.$$

  Also, one writes $\mathbb{R} = (-\infty, \infty)$ and refers to this as an infinite interval, or sometimes, a doubly infinite interval.
- In general, a subset I of $\mathbb{R}$ is an interval if

$$x, y \in I, \ x < y \Longrightarrow [x, y] \subseteq I.$$

- One can show that every interval in $\mathbb{R}$ is open, closed, semi-open, or infinite interval.

Absolute Value

- For $x \in \mathbb{R}$, the absolute value or the modulus of x is

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

- Basic Properties: For any $x, y \in \mathbb{R}$,
  - $|x + y| \leq |x| + |y|$ [Triangle Inequality]
  - $||x| - |y|| \leq |x - y|$.
- Optional Exercises: (i) Show that for any $a, b \in \mathbb{R}$ with $a \geq 0$, $b \geq 0$, and $n \in \mathbb{N}$,

$$\left| \sqrt[n]{a} - \sqrt[n]{b} \right| \leq \sqrt[n]{|a - b|}.$$

(ii) For any $n \in \mathbb{N}$ and any nonnegative real numbers $a_1, \ldots, a_n$, prove the AM-GM inequality:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Functions

- Given sets D, E, a function $f : D \to E$ assigns to each $x \in D$, a unique element of E, denoted $f(x)$. We refer to D as the domain of f and E as the co-domain of f. The set $\{f(x) : x \in D\}$ of all values taken by the function is called the range of f. [For a formal definition, see [GL-1].]
- A function $f : D \to E$ is said to be:
  - one-one (or injective) if for any $x_1, x_2 \in D$,
  $$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$
  - onto (or surjective) if its range is E.
- If $f : D \to E$ is bijective, then it has an inverse $g : E \to D$ with the property that the composites $g \circ f$ and $f \circ g$ are identity functions, i.e.,

$$g(f(x)) = x \text{ for all } x \in D \quad \text{and} \quad f(g(y)) = y \text{ for all } y \in E.$$

Examples and Types of Functions

- Example/Exercise: Consider $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by

  $$f_1(x) = 2x+1, \quad f_2(x) = x^2, \quad f_3(x) = x^3, \quad \text{and} \quad f_4(x) = \frac{1}{x}.$$

  Are these one-one/onto/bijective?
- Functions that mainly arise in Calculus are usually of following types.
  - Polynomial functions
  - Rational functions
  - Algebraic functions
  - Transcendental functions (this includes logarithmic, exponential and trigonometric functions).
- Besides these, we can construct functions by pieceing together known functions (such as those belonging to the above classes). A good example is the absolute value function $x \longmapsto |x|$.

Sequences

Definition

Let X be any set. A sequence in X is a function from the set $\mathbb{N}$ of natural numbers to the set X.

The value of this function at $n \in \mathbb{N}$ is denoted by $a_n \in X$, and a_n is called the nth **term** of the sequence.

We shall use the notation (a_n) to denote a sequence.

Note: $\{a_n : n \in \mathbb{N}\}$ is the set of all terms of the sequence (a_n). Thus if $X := \mathbb{R}$ and $a_n := (-1)^n$ for $n \in \mathbb{N}$, then the sequence (a_n) is given by $-1, 1, -1, 1, \ldots$, but $\{a_n : n \in \mathbb{N}\} = \{-1, 1\}$.

Initially, we let $X := \mathbb{R}$, that is, we consider sequences in $\mathbb{R}$. Later, we shall consider sequences in $\mathbb{R}^2$ and in $\mathbb{R}^3$.

Examples

Examples:

1. $a_n := n$ for $n \in \mathbb{N}$: $1, 2, 3, 4, \ldots$
2. $a_n := 1/n$ for $n \in \mathbb{N}$: $1,\ 1/2,\ 1/3,\ 1/4, \ldots$
3. $a_n := n^2$ for $n \in \mathbb{N}$: $1, 4, 9, 16, \ldots$
4. $a_n := \sqrt{2}$ for $n \in \mathbb{N}$: $\sqrt{2}, \sqrt{2}, \ldots$ This is an example of a constant sequence.
5. $a_n := 2^n$ for $n \in \mathbb{N}$: $2, 4, 8, 16, \ldots$
6. $a_n := (-1)^n$ for $n \in \mathbb{N}$: $-1, 1, -1, 1, \ldots$
7. $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3$: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ This sequence is known as the Fibonacci sequence.
8. $a_n := 1/2 + \cdots + 1/2^n$ for $n \in \mathbb{N}$. Check: $a_n = 1 - (1/2^n)$.

Bounded sequences

A sequence (a_n) of real numbers is said to be **bounded above** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded above, that is, if there is a real number α such that $a_n \leq \alpha$ for every $n \in \mathbb{N}$.

A sequence (a_n) of real numbers is said to be **bounded below** if the set $\{a_n : n \in \mathbb{N}\}$ is bounded below, that is, if there is a real number β such that $\beta \leq a_n$ for every $n \in \mathbb{N}$.

A sequence (a_n) of real numbers is said to be **bounded** if it is bounded above as well as bounded below, that is, if there are real numbers α, β such that $\beta \leq a_n \leq \alpha$ for every $n \in \mathbb{N}$.

If a sequence is not bounded, it is said to be **unbounded**.

Let us check which of the sequences mentioned earlier are bounded above and/or bounded below.

Examples of bounded and unbounded sequences:

- $a_n := n$ for $n \in \mathbb{N}$: $\beta = 1$.
- $a_n := 1/n$ for $n \in \mathbb{N}$: $\beta = 0$, $\alpha = 1$.
- $a_n := n^2$ for $n \in \mathbb{N}$: $\beta = 1$.
- $a_n := \sqrt{2}$ for $n \in \mathbb{N}$: $\beta = \sqrt{2} = \alpha$.
- $a_n := 2^n$ for $n \in \mathbb{N}$: $\beta = 2$.
- $a_n := (-1)^n$ for $n \in \mathbb{N}$: $\beta = -1$, $\alpha = 1$.
- $a_1 := 1, a_2 := 1$ and $a_n := a_{n-1} + a_{n-2}$ for $n \geq 3$: $\beta = 1$.
- $a_n := 1/2 + \cdots + 1/2^n$ for $n \in \mathbb{N}$: $\beta = 1/2$, $\alpha = 1$.

Note: The sequence given by $a_n := n$ for $n \in \mathbb{N}$, is not bounded above by the Archimedean property of $\mathbb{R}$.

Toward convergence of a sequence

Let (a_n) be a sequence in $\mathbb{R}$, and let a be a real number.

- Convergence of a sequence (a_n) to a real number a should mean that the term a_n is as close to a as we like for all sufficiently large n.
- How to make this precise? We require for every fixed positive real number ϵ, the term a_n is within ϵ distance of a for all sufficiently large n. In other words, we require

$$a_n \in (a - \epsilon,\ a + \epsilon) \quad \text{or equivalently,} \quad |a_n - a| < \epsilon$$

  to be satisfied for all sufficiently large n.
- The phrase "for all sufficiently large n" can be made precise by saying that there exists some $n_0 \in \mathbb{N}$ such that the above condition holds for all n from n_0 onward, that is, it holds for all $n \geq n_0$.

Definition of convergence of a sequence

Definition

Let (a_n) be a sequence of real numbers. We say that (a_n) is convergent if there is $a \in \mathbb{R}$ such that the following condition holds. For every $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq n_0$.

This is known as the ϵ-n_0 definition of convergence of a sequence.

In this case, we say that (a_n) **converges** to a, or that a is a **limit** of (a_n), and we write

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a \ \ (\text{as } n \to \infty).$$

If a sequence does not converge, we say that the sequence **diverges** or it is **divergent**.

The convergence of a sequence is unaltered if a finite number of its terms are replaced by some other terms.

Examples:

(i) Let $a \in \mathbb{R}$ and $a_n := a$ for all $n \in \mathbb{N}$. Then $a_n \to a$. We can let $n_0 := 1$.

(ii) $a_n := 1/n$ for all $n \in \mathbb{N}$. Then $a_n \to 0$.

Let $\epsilon > 0$ be given. We want to find $n_0 \in \mathbb{N}$ such that $|(1/n) - 0| < \epsilon$ for all $n \geq n_0$.

Choose any $n_0 \in \mathbb{N}$ which is greater than $1/\epsilon$. This is possible because of the Archimedean property of $\mathbb{R}$. For example, we can let $n_0 := [1/\epsilon] + 1$.

(iii) $a_n := 2/(n^2+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\left|\frac{2}{n^2+1} - 0\right| = \frac{2}{n^2+1} < \frac{2}{n^2} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > \sqrt{2}/\sqrt{\epsilon}$. For example, let $n_0 := [\sqrt{2}/\sqrt{\epsilon}] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.

(iv) $a_n := 5/(3n+1)$ for $n \in \mathbb{N}$. Then $a_n \to 0$. Now

$$\frac{5}{3n+1} < \frac{5}{3n} \quad \text{for all } n \in \mathbb{N}.$$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > 5/3\epsilon$. For example, let $n_0 := [5/3\epsilon] + 1$. Then $|a_n - 0| < \epsilon$ for all $n \geq n_0$.