MA 105 : Calculus
Division 3, Lecture 20

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Euclidean Spaces

Let $m \in \mathbb{N}$, and consider the m dimensional **Euclidean space**

$$\mathbb{R}^m := \{(x_1, \ldots, x_m) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, m\}.$$

If $m = 1$, then an element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar** and if $m > 1$, then an element of $\mathbb{R}^m$ is called a **vector**.
We shall consider sequences in $\mathbb{R}^m$, and functions from subsets of $\mathbb{R}^m$ to $\mathbb{R}$, and we shall study their properties.
We shall set up the basic structure in $\mathbb{R}^m$, and then restrict ourselves to $m := 2, 3$.

For $\boldsymbol{x} := (x_1, \ldots, x_m)$, $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$, let

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &:= (x_1 + y_1, \ldots, x_m + y_m) \in \mathbb{R}^m, \\ a\,\boldsymbol{x} &:= (a\,x_1, \ldots, a\,x_m) \in \mathbb{R}^m, \end{aligned}$$

called the **sum** of $\boldsymbol{x}$ and $\boldsymbol{y}$, and the **scalar multiple** of a and $\boldsymbol{x}$.

Norm of an element of $\mathbb{R}^m$

Let $\boldsymbol{x} := (x_1, \ldots, x_m) \in \mathbb{R}^m$. We define the **norm** of $\boldsymbol{x}$ by

$$\|\boldsymbol{x}\| := (x_1^2 + \cdots + x_m^2)^{1/2}.$$

For $m = 1$, the norm of $x \in \mathbb{R}$ is its absolute value $|x|$. Note:

$$\max\{|x_1|, \ldots, |x_m|\} \leq \|\boldsymbol{x}\| \leq |x_1| + \cdots + |x_m|.$$

Let $\mathbf{0} := (0, \ldots, 0)$. Then $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \mathbf{0}$.

For $\boldsymbol{x} := (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\boldsymbol{y} := (y_1, \ldots, y_m) \in \mathbb{R}^m$, let

$$\boldsymbol{x} \cdot \boldsymbol{y} \;:=\; x_1 y_1 + \cdots + x_m y_m \in \mathbb{R},$$

called the **dot product** or the **scalar product** of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Theorem (Cauchy-Schwarz Inequality and Triangle Inequality)

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|$ and $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

Proof: If $\|\boldsymbol{x}\| = 0$ or $\|\boldsymbol{y}\| = 0$, then $\boldsymbol{x} = \mathbf{0}$ or $\boldsymbol{y} = \mathbf{0}$, so done.

Suppose $\|\boldsymbol{x}\| \neq 0$ and $\|\boldsymbol{y}\| \neq 0$. Then

$$\frac{|x_j|}{\|\boldsymbol{x}\|}\frac{|y_j|}{\|\boldsymbol{y}\|} \leq \frac{1}{2}\left(\frac{|x_j|^2}{\|\boldsymbol{x}\|^2} + \frac{|y_j|^2}{\|\boldsymbol{y}\|^2}\right) \quad \text{for } j = 1, \ldots, m,$$

since $ab \leq (a^2 + b^2)/2$ for all $a, b \in \mathbb{R}$. Hence

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \sum_{j=1}^{m} |x_j|\,|y_j| \leq \frac{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}{2}(1+1) = \|\boldsymbol{x}\|\|\boldsymbol{y}\|.$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$. As a consequence,

$$\begin{aligned} \|\boldsymbol{x}+\boldsymbol{y}\|^2 &= \sum_{j=1}^{m}(x_j+y_j)^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\,\boldsymbol{x}\cdot\boldsymbol{y} \\ &\leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 + 2\,\|\boldsymbol{x}\|\|\boldsymbol{y}\| = (\|\boldsymbol{x}\| + \|\boldsymbol{y}\|)^2. \end{aligned}$$

Thus $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$. □

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$. We say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are **orthogonal** or **perpendicular** (to each other) if

$$\boldsymbol{x} \cdot \boldsymbol{y} = 0.$$

If $\boldsymbol{x} \neq \boldsymbol{0}$ and $\boldsymbol{y} \neq \boldsymbol{0}$, then the **angle** between them is the unique $\theta \in [0, \pi]$ such that $\cos\theta = \boldsymbol{x} \cdot \boldsymbol{y} / \|\boldsymbol{x}\| \, \|\boldsymbol{y}\|$. (Cosine Rule)

Let $m \in \mathbb{N}$ and $\boldsymbol{x}_0 \in \mathbb{R}^m$. For $\boldsymbol{x} \in \mathbb{R}^m$, $\|\boldsymbol{x} - \boldsymbol{x}_0\|$ is called the **distance** of $\boldsymbol{x}$ from $\boldsymbol{x}_0$. For $r > 0$, the subset

$$B(\boldsymbol{x}_0, r) := \{\boldsymbol{x} \in \mathbb{R}^m : \|\boldsymbol{x} - \boldsymbol{x}_0\| < r\}$$

of $\mathbb{R}^m$ is called the **neighbourhood** of $\boldsymbol{x}_0$ of radius r.

If $m := 1$, then $B(\boldsymbol{x}_0, r)$ is the interval $(x_0 - r, x_0 + r)$ in $\mathbb{R}$.

If $m := 2$ and $\boldsymbol{x}_0 := (x_0, y_0)$, then $B(\boldsymbol{x}_0, r)$ is the disk $\{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < r^2\}$ in $\mathbb{R}^2$.

If $m := 3$ and $\boldsymbol{x}_0 := (x_0, y_0, z_0)$, then $B(\boldsymbol{x}_0, r)$ is the ball $\{(x, y, z) \in \mathbb{R}^3 : (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$ in $\mathbb{R}^3$.

Sequence in $\mathbb{R}^2$ and its Limit

A **sequence** in $\mathbb{R}^2$ is a function from $\mathbb{N}$ to $\mathbb{R}^2$. We denote the nth term of such a sequence by (x_n, y_n), and the sequence itself by $\big((x_n, y_n)\big)$.

Definition

A sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ is **convergent** in $\mathbb{R}^2$ if both (x_n) and (y_n) are convergent sequences in $\mathbb{R}$. In this case, we write

$$(x_n, y_n) \to (x_0, y_0), \text{ where } x_n \to x_0 \text{ and } y_n \to y_0.$$

For $n \in \mathbb{N}$,

$$\max\{|x_n - x_0|, |y_n - y_0|\} \leq \|(x_n, y_n) - (x_0, y_0)\| \leq |x_n - x_0| + |y_n - y_0|.$$

Hence $(x_n, y_n) \to (x_0, y_0) \iff \|(x_n, y_n) - (x_0, y_0)\| \to 0.$

Functions on subsets of $\mathbb{R}^2$

Let D be a subset of $\mathbb{R}^2$, and let f be a real-valued function defined on D. The subset

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 : (x, y) \in D\}$$

is called the **graph** of f. It is the surface $z = f(x, y)$ in $\mathbb{R}^3$.

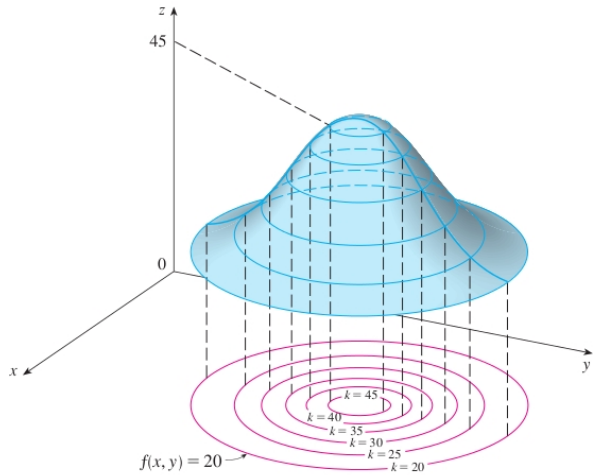
Let $c \in \mathbb{R}$. Then the set
$\{(x, y, c) \in \mathbb{R}^3 : (x, y) \in D$ and $f(x, y) = c\}$ is called a **contour line** of f. It is the intersection of the graph of f by the horizontal plane $z = c$ in $\mathbb{R}^3$.

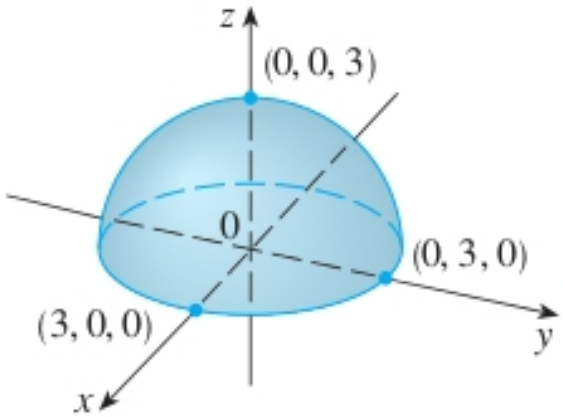
Also, $\{(x, y) \in D : f(x, y) = c\} \subset \mathbb{R}^2$ is called a **level curve** of f. It is the projection of the corresponding contour line on the plane $z = 0$ in $\mathbb{R}^3$.

Contour lines and level curves help us visualize the behaviour of a function f on its domain in $\mathbb{R}^2$. Such a visualization is not possible for functions defined on subsets of $\mathbb{R}^3$.

Contour lines and level curves

The picture below on the left gives a general illustration of contour lines and level curves.





Examples: (i) Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$, and let

$$f(x, y) := \sqrt{9 - x^2 - y^2} \quad \text{for } (x, y) \in D.$$

The graph of f is the upper hemisphere of radius 3 centered at the origin. This is depicted by the picture on the right above. It is easy to visualize what the contour lines and the level curves are in thie case.

(ii) Let $D := \mathbb{R}^2$, and let $f(x, y) := x^2 + y^2$ for $(x, y) \in D$. The graph of f is a paraboloid.

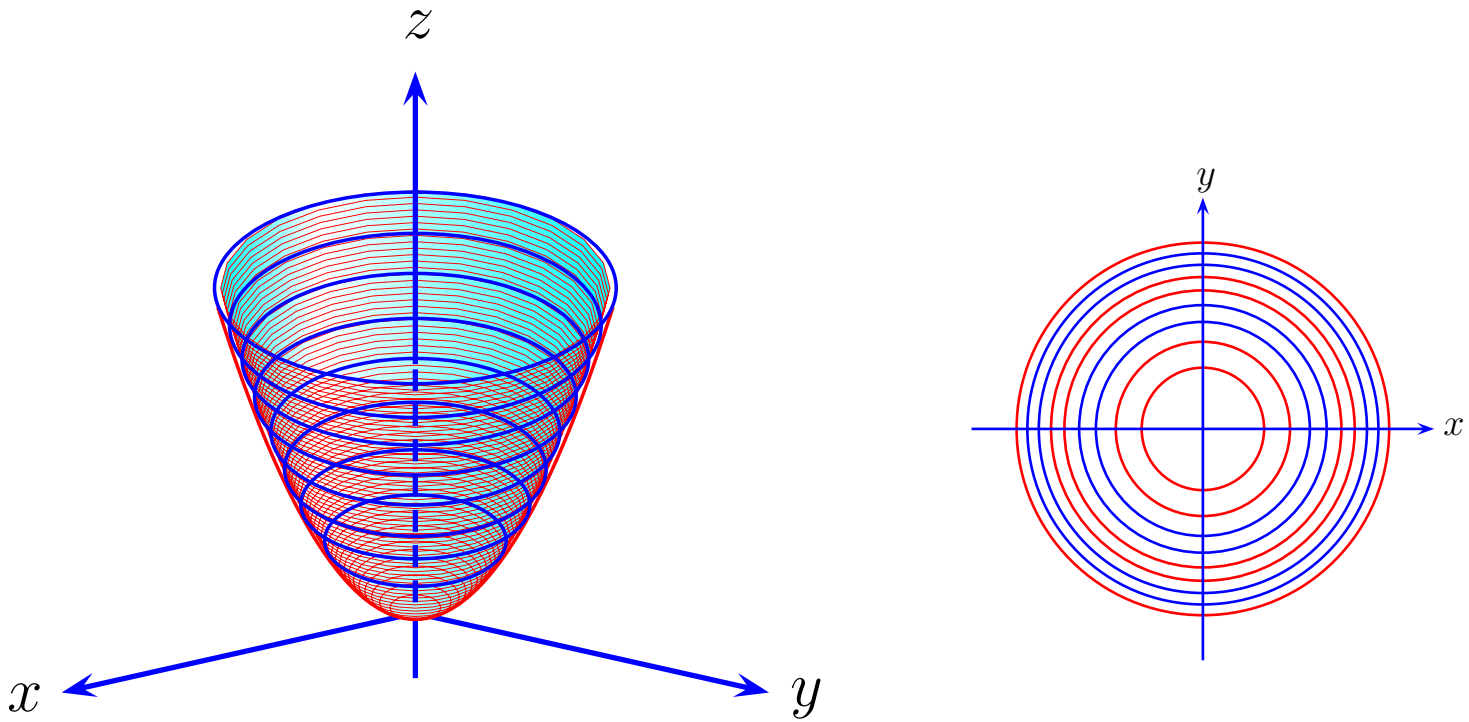


Figure: Contour lines and the corresponging level curves

(iii) Let $D := \{(x, y) \in \mathbb{R}^2 : x \neq y\}$, and let $f(x, y) := (x + y)/(x - y)$ for $(x, y) \in D$. Then the contour line corresponding to $c = 2$ is $\{(x, x/3, 2) : x \in \mathbb{R} \setminus \{0\}\}$. Level curves are lines through the origin in $\mathbb{R}^2$.

Continuity of a function defined on a subset of $\mathbb{R}^2$

Definition

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. We say that f is **continuous** at (x_0, y_0) if

$$(x_n, y_n) \to (x_0, y_0) \implies f(x_n, y_n) \to f(x_0, y_0)$$

for every sequence $\big((x_n, y_n)\big)$ in D.

We say that f is **continuous on** D if f is continuous at each point of D.

We say that f is **discontinuous** on D if it is not continuous at some point of D.

Examples: A constant function is clearly continuous. Also, it is clear from the definition that $f, g : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) := x$ and $g(x, y) := y$ for $(x, y) \in \mathbb{R}^2$ are continuous.

Basic properties of continuous functions

- Let $D \subset \mathbb{R}^2$, and $f, g : D \to \mathbb{R}$. Let $(x_0, y_0) \in D$, and let f and g be continuous at (x_0, y_0). Then $f + g$ and $f \cdot g$ are continuous at (x_0, y_0). Further, f/g is continuous at (x_0, y_0) if $g((x_0, y_0)) \neq 0$. (Limit theorems for sequences.)
- Let $D \subset \mathbb{R}^2, E \subset \mathbb{R}$. Suppose $f : D \to \mathbb{R}$ and $g : E \to \mathbb{R}$ are functions such that $f(D) \subset E$, f is continuous at $(x_0, y_0) \in D$, and g is continuous at $f(x_0, y_0) \in E$. Then the **composite function** $g \circ f : D \to \mathbb{R}$ is continuous at (x_0, y_0). (The proof is easy.)

  Note: We can use this together with simple examples such as those in the preceding slide to arrive at a large class of examples of continuous functions.

Examples:

(i) Let p be a **polynomial function in two variables**, that is, let $k = 0$ or $k \in \mathbb{N}$ and for $(x, y) \in \mathbb{R}^2$, let

$$p(x, y) := \sum_{i \geq 0, j \geq 0, i+j \leq k} a_{i,j} x^i y^j, \quad \text{where } a_{i,j} \in \mathbb{R}.$$

Then p is a continuous function on $\mathbb{R}^2$.

(ii) A **rational function in two variables**, that is, a function given by $r(x, y) := p(x, y)/q(x, y)$, where p and q are polynomial functions in two variables, is continuous at each $(x_0, y_0) \in \mathbb{R}^2$ at which $q((x_0, y_0)) \neq 0$.

(iii) Let $p(x, y)$ be a polynomial function in two variables, and define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) := e^{p(x,y)}$ for $(x, y) \in \mathbb{R}^2$. Then f is continuous on $\mathbb{R}^2$ since the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(z) := e^z$ for $z \in \mathbb{R}$, is continuous on $\mathbb{R}$.

Equivalent condition for continuity of a function

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at $(x_0, y_0) \iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there exists $\delta > 0$ such that $(x, y) \in D$ and

$$\|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \epsilon.$$

Examples:

Let $f : \mathbb{R}^2 \to \mathbb{R}$. Suppose $f(0, 0) := 0$.

(i) Suppose $f(x, y) := xy/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$.
Let $(x_n, y_n) := (1/n, 1/n)$, $n \in \mathbb{N}$. Then $(x_n, y_n) \to (0, 0)$, but $f(x_n, y_n) \to 1/2 \neq f(0, 0)$. Hence f is not continuous at $(0, 0)$.

(ii) Suppose $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y) - f(0,0)| = |x| \frac{|xy|}{x^2+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

If $(x_n, y_n) \to (0,0)$, then $f(x_n, y_n) \to 0 = f(0,0)$. Hence f is continuous at 0.

(iii) Suppose $f(x,y) := x^2y/(x^4+y^2)$ for $(x,y) \neq (0,0)$.
If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = n/(1+n^2) \to 0$.
Can we say f is continuous at $(0,0)$?
Let $(x_n, y_n) := (1/n, 1/n^2)$, then $(x_n, y_n) \to (0,0)$, but $f(x_n, y_n) \to 1/2 \neq f(0,0)$. Hence f is not continuous at $(0,0)$.

(iv) Suppose $f(x,y) := x^3y/(x^4+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y) - f(0,0)| = |x| \frac{|x^2y|}{x^4+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

As in (ii) above, f is continuous at $(0,0)$.