MA 105 : Calculus
Division 3, Lecture 21

Prof. Sudhir R. Ghorpade
IIT Bombay

Recap of the previous lecture

- Euclidean spaces
- Notion of addition and scalar multiplication of vectors
- Dot product. (Cauchy-) Schwarz inequality and Triangle inequality
- Notions of orthogonal vectors, angle between nonzero vectors, neighbourhood of a vector
- Sequence in $\mathbb{R}^2$ and its limit
- Functions on subsets of $\mathbb{R}^2$. Examples
- Continuity of real-valued functions defined on subsets of $\mathbb{R}^2$
- Basic properties of continuous functions

Limits of Functions of Two Variables

Definition

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$. Suppose $(x_0, y_0) \in \mathbb{R}^2$ is such that $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subset D$ for some $r > 0$ [or suppose, more generally, that (x_0, y_0) is a limit point of D, which means that every neighborhood of (x_0, y_0) contains a point of D other than (x_0, y_0)]. We say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists if there is $\ell \in \mathbb{R}$ such that $f(x_n, y_n) \to \ell$ for every sequence $\big((x_n, y_n)\big)$ in D with $(x_n, y_n) \neq (x_0, y_0)$ for all $n \in \mathbb{N}$ and $(x_n, y_n) \to (x_0, y_0)$.

Examples: (i) Let

$$f(x, y) := \begin{cases} x + y & \text{if } (x, y) \in \mathbb{R}^2,\ x \neq y, \\ 1 & \text{if } (x, y) \in \mathbb{R}^2,\ x = y. \end{cases}$$

If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = 1$, and
if $(x_n, y_n) := (1/n, -1/n)$, then $f(x_n, y_n) = 0$ for all $n \in \mathbb{N}$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

(ii) Suppose $f(x,y) := xy/(x^2+y^2)$ for $(x,y) \neq (0,0)$.
If $(x_n, y_n) := (1/n, 1/n)$, then $f(x_n, y_n) = 1/2$, and
if $(x_n, y_n) := (1/n, -1/n)$, then $f(x_n, y_n) = -1/2$ for $n \in \mathbb{N}$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Aliter: Let $(x_n, y_n) := (1/n, (-1)^n/n)$, then for $n \in \mathbb{N}$,
$f(x_n, y_n) = (-1)^n/2$, and so the sequence $(f(x_n, y_n))$ is
divergent. Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(iii) Suppose $f(x,y) := x^2y/(x^4+y^2)$ for $(x,y) \neq (0,0)$.
Suppose $(x_n, y_n) \to (0,0)$ and $(x_n, y_n) \neq (0,0)$. If $x_n := 0$ or
$y_n := 0$, then $f(x_n, y_n) = 0$. Also, if $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and
$y_n := \alpha x_n$ for $n \in \mathbb{N}$, then $f(x_n, y_n) = \alpha x_n/(x_n^2 + \alpha^2) \to 0$.

But if $(x_n, y_n) := (1/n, 1/n^2)$ for $n \in \mathbb{N}$, then $f(x_n, y_n) = 1/2$.
Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity and Limit

Let us relate the concepts of continuity and limit for functions of two variables.

Let $D \subset \mathbb{R}^2$. We say that $(x_0, y_0) \in \mathbb{R}^2$ is an **interior point** of D if there is $r > 0$ such that $B((x_0, y_0), r) \subset D$.

Suppose (x_0, y_0) be an interior point of D [or more generally, suppose $(x_0, y_0) \in D$ and (x_0, y_0) is a limit point of D]. Then $f : D \to \mathbb{R}$ is continuous at $(x_0, y_0) \iff \lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists and it is equal to $f(x_0, y_0)$.

Thus for f to be continuous at an interior point (x_0, y_0) of D,

(i) $f(x_0, y_0)$ has to be defined,

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ must exist, and

(iii) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Limit theorems for functions of two real variables

Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell_f$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = \ell_g$.
The limit theorems for sequences in $\mathbb{R}$ show the following:

(i) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \pm g)(x,y) = \ell_f \pm \ell_g$ resp.

(ii) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \cdot g)(x,y) = \ell_f \cdot \ell_g$.

(iii) If $\ell_g \neq 0$, then $\lim\limits_{(x,y)\to(x_0,y_0)} \left(\frac{f}{g}\right)(x,y) = \frac{\ell_f}{\ell_g}$.

(iv) If $f(x,y) \leq g(x,y)$ for all $(x,y) \neq (x_0,y_0)$ near (x_0,y_0), then $\ell_f \leq \ell_g$.

(v) (Sandwich theorem) If $f(x,y) \leq h(x,y) \leq g(x,y)$ for all (x,y) near (x_0,y_0) and $(x,y) \neq (x_0,y_0)$, and if $\ell_f = \ell = \ell_g$, then $\lim\limits_{(x,y)\to(x_0,y_0)} h(x,y) = \ell$.

Equivalent condition for a limit to exist

(Recall the ϵ-δ condition for Continuity)

Let $D \subset \mathbb{R}^2$, $f : D \to \mathbb{R}$ and $(x_0, y_0) \in D$. Then f is continuous at (x_0, y_0) $\iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$(x, y) \in D,\ \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - f(x_0, y_0)| < \epsilon.$$

Similarly,

(ϵ-δ condition for Limit)

Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Suppose $B((x_0, y_0), r) \setminus \{(x_0, y_0)\} \subset D$ for some $r > 0$ [or more generally, suppose, (x_0, y_0) is a limit point of D]. Let $\ell \in \mathbb{R}$. Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = \ell \iff$ the following ϵ-δ condition holds: For every $\epsilon > 0$, there is $\delta > 0$ such that $(x, y) \in D$, $0 < \|(x, y) - (x_0, y_0)\| < \delta \implies |f(x, y) - \ell| < \epsilon$.

Examples:

(i) Let $f(x,y) := x^2y/(x^2+y^2)$ for $(x,y) \neq (0,0)$. Now

$$|f(x,y)| = |x|\frac{|xy|}{x^2+y^2} \leq \frac{|x|}{2} \quad \text{for } (x,y) \in \mathbb{R}^2.$$

Let $\epsilon > 0$. If $\delta := 2\epsilon$, then

$$0 < \|(x,y)\| < \delta \implies |f(x,y)-0| \leq \frac{|x|}{2} \leq \frac{\|(x,y)\|}{2} < \frac{\delta}{2} = \epsilon.$$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Aliter: If we define $f(0,0) := 0$, then f is continuous at $(0,0)$ as we have seen before, and so $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to 0.

(ii) Let $f(x,y) := \dfrac{x^2+y^2}{1-\sqrt{x^2+y^2+1}}$ for $(x,y) \neq (0,0)$.

'Rationalizing' the fraction, we obtain

$f(x,y) = -1 - \sqrt{x^2+y^2+1}$ for $(x,y) \neq (0,0)$, and so
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = -1 - \sqrt{1} = -2$. (Check: $\delta := \epsilon$ works.)

(iii) Let $f(x,y) := y \sin \dfrac{1}{x^2+y^2}$ for $(x,y) \neq (0,0)$.

Since $|f(x,y) - 0| \leq |y| \leq \|(x,y)\|$ for $(x,y) \neq (0,0)$,
$\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. (Clearly, $\delta := \epsilon$ works.)

Thumb rule: Suppose $f(x,y) = p(x,y)/q(x,y)$ for $(x,y) \neq (0,0)$, where $p(x,y)$ and $q(x,y)$ are homogeneous polynomials of degrees m and n respectively, and $p(0,0) = 0 = q(0,0)$. If $m > n$, then $f(x,y) \to 0$ as $(x,y) \to (0,0)$, and if $m \leq n$, then this limit does not exist.

Partial Differentiation

Recall: Let $D \subset \mathbb{R}$, and let c be an interior point of D. A function $f : D \to \mathbb{R}$ is said to have a **derivative** at c if $\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists, and then it is denoted by $f'(c)$.

Definition

Let $D \subset \mathbb{R}^2$, and let (x_0, y_0) be an interior point of D.
A function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** x at (x_0, y_0) if

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

exists, and then it is denoted by $f_x(x_0, y_0)$ or by $\dfrac{\partial f}{\partial x}(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the horizontal line segment $\{(x, y_0) : x \in (x_0 - r, x_0 + r)\} \subset D$.

Geometric interpretation

Let C denote the curve obtained by intersecting the graph of f by the horizontal plane $y = y_0$. The partial derivative of f with respect to x at (x_0, y_0) equals the slope of the tangent to the curve C at (x_0, y_0). It can be interpreted as the rate of change in f along the x-axis at (x_0, y_0).

Computationally, $f_x(x_0, y_0)$ is obtained by differentiating f with respect to x at x_0, treating y as the constant y_0.

Similarly, a function $f : D \to \mathbb{R}$ is said to have a **partial derivative with respect to** y at (x_0, y_0) if

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

exists, and then it is denoted by $f_y(x_0, y_0)$ or by $\dfrac{\partial f}{\partial y}(x_0, y_0)$.

This definition also makes sense if there is $r > 0$ such that the vertical line segment $\{(x_0, y) : y \in (y_0 - r, y_0 + r)\} \subset D$.

Let $a < b$ and $c < d$. Consider

$$D := \{(x, c) \in \mathbb{R}^2 : a \leq x \leq b\} \cup \{(a, y) \in \mathbb{R}^2 : c \leq y \leq d\},$$

and let $f : D \to \mathbb{R}$. Although there is no interior point of D, we can still define $f_x^+(a, c)$, $f_x^-(b, c)$, $f_y^+(a, c)$ and $f_y^-(a, d)$ as the appropriate left hand and the right hand partial derivatives.

Partial derivatives of sums, products, quotients and compositions of functions of two variables can be found in exactly the same manner as the derivatives of a function of one variable.

Definition

If the partial derivatives of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ of f exist at (x_0, y_0), then

$$(\nabla f)(x_0, y_0) := (f_x(x_0, y_0), f_y(x_0, y_0)) \in \mathbb{R}^2$$

is called the **gradient** of f at (x_0, y_0).

Partial Differentiation: Examples

(i) Let $f(x,y) := x^2 + y^2$ for $(x,y) \in \mathbb{R}^2$. Partial derivatives of f exist at all points of $\mathbb{R}^2$. In fact, for $(x_0, y_0) \in \mathbb{R}^2$,

$$f_x(x_0, y_0) = 2x_0 \quad \text{and} \quad f_y(x_0, y_0) = 2y_0.$$

(ii) Let $f(x,y) := \sqrt{x^2 + y^2}$ for $(x,y) \in \mathbb{R}^2$.

Let $(x_0, y_0) \neq (0,0)$. Then

$$f_x(x_0, y_0) = \frac{x_0}{\sqrt{x_0^2 + y_0^2}} \quad \text{and} \quad f_y(x_0, y_0) = \frac{y_0}{\sqrt{x_0^2 + y_0^2}}.$$

But f does not have either partial derivative at $(0,0)$ since $\lim_{h \to 0} |h|/h$ does not exist. Note: f is continuous at $(0,0)$.

(iii) Let $f(x,y) := xy/(x^2 + y^2)$ if $(x,y) \neq (0,0)$, and $f(0,0) = 0$. It is easy to see that $f_x(0,0) = 0 = f_y(0,0)$. We have already seen that f is not continuous at $(0,0)$.

Higher Order Partial Derivatives

Let $D \subset \mathbb{R}^2$, and $f : D \to \mathbb{R}$. Suppose $f_x(x_0, y_0)$ exists for every $(x_0, y_0) \in D$. If the function $f_x : D \to \mathbb{R}$ has a partial derivative with respect to x at (x_0, y_0), then it is denoted by $f_{xx}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial x^2}(x_0, y_0)$.

Also, if $f_x : D \to \mathbb{R}$ has a partial derivative with respect to y at (x_0, y_0), then it is denoted by $f_{xy}(x_0, y_0)$ or by $\dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

Similarly, we define $f_{yy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$, or $\dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$.

In general, the **mixed partial derivatives** $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ may not be equal.

Mixed Partials

Example: Let $f(x,y) := x\,y\dfrac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$, and $f(0,0) := 0$. Then $f_x(0,y_0) = -y_0$ for $y_0 \in \mathbb{R}$, and $f_y(x_0,0) = x_0$ for $x_0 \in \mathbb{R}$. Hence $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

Theorem (Mixed Partials Theorem)

Let $D \subset \mathbb{R}^2$, and let (x_0,y_0) be an interior point of D. Then there is $r > 0$ such that
$S := \{(x,y) \in \mathbb{R}^2 : |x-x_0| < r \text{ and } |y-y_0| < r\} \subset D$.
Consider $f : S \to \mathbb{R}$, and suppose f_x and f_y exist on S. If one of the mixed partials f_{xy} or f_{yx} exists on S, and it is continuous at (x_0,y_0), then the other mixed partial exists at (x_0,y_0), and $f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0)$.

We omit a proof of this result since it is rather involved.