MA 105 : Calculus
Division 3, Lecture 27

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Recap of the previous lecture

- Constrained extrema
- Lagrange multiplier theorem. Examples
- The case of three variables
- Multiple constraints. Examples
- Double integrals on a rectangle
- Integrability of a bounded function on a rectangle.
- Examples
- Riemann condition

Double Integral on a Rectangle

Recall that if $R := [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$ with $a < b$, $c < d$. and $f : R \to \mathbb{R}$ be a bounded function, then for a partition P of R, we defined the **lower double sum** $L(P, f)$ and the **upper double sum** $U(P, f)$ of f with respect to P. Moreover, we defined the **lower double integral** $L(f)$ of f and the **upper double integral** $U(f)$ of f by

$$\begin{aligned} L(f) &:= \sup\{L(P, f) : P \text{ is a partition of } R\}, \\ U(f) &:= \inf\{U(P, f) : P \text{ is a partition of } R\}. \end{aligned}$$

Definition

A bounded function $f : R \to \mathbb{R}$ is said to be **(double) integrable** on R if $L(f) = U(f)$.

We have also seen examples of bounded functions that are integrable and also those that are not integrable.

Theorem (Riemann condition)

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$, there is a partition P_ϵ of $[a, b] \times [c, d]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

The proof is very similar to the proof of the result about the Riemann condition in the one variable case.

The Riemann condition can be used to prove many useful results regarding double integration.

(Domain Additivity)

Let $R := [a, b] \times [c, d]$, and let $f : R \to \mathbb{R}$ be a bounded function. Let $s \in (a, b)$, $t \in (c, d)$. Then f is integrable on R if and only if f is integrable on the four subrectangles $[a, s] \times [c, t]$, $[a, s] \times [t, d]$, $[s, b] \times [c, t]$ and $[s, b] \times [t, d]$.
In this case, the integral of f on R is the sum of the integrals of f on the four subrectangles.

Some conventions

Let $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$. In view of domain additivity, we make the following conventions:

- $$\iint_{[a,b]\times[c,d]} f := 0 \quad \text{if } a = b \text{ or } c = d$$
- $$\iint_{[b,a]\times[c,d]} f := -\iint_{[a,b]\times[c,d]} f$$
- $$\iint_{[a,b]\times[d,c]} f := -\iint_{[a,b]\times[c,d]} f$$
- $$\iint_{[b,a]\times[d,c]} f := \iint_{[a,b]\times[c,d]} f.$$

Integrable functions

Let $R := [a, b]\times[c, d]$, and let $f : R \to \mathbb{R}$.

(i) If f is monotonic in each of the two variables, then f is integrable on R.

(ii) If f is bounded on R, and has at most a finite number of discontinuities in R, then f is integrable on R.

Examples:

(i) Let $f(x, y) := [x] + [y]$ for $(x, y) \in R$. Since f is increasing in each variable, f is integrable.

(ii) Let $a, c > 0$ and $r, s \geq 0$. Define $f(x, y) = x^r y^s$ for $(x, y) \in R$. Since f is continuous on R, it is integrable.

(iii) Let $f(0, 0) := 0$ and $f(x, y) := xy/(x^2 + y^2)$ if $(x, y) \in [-1, 1]\times[-1, 1]$ and $(x, y) \neq (0, 0)$. Since f is bounded on R, and it is discontinuous only at $(0, 0)$, f is integrable.

Algebraic and Order Properties

Let $R := [a,b]\times[c,d]$. If $f, g : R \to \mathbb{R}$ are integrable, then

(i) $f+g$ is integrable, and $\int\int_R (f+g) = \int\int_R f + \int\int_R g$,

(ii) αf is integrable, and $\int\int_R \alpha f = \alpha \int\int_R f$ for all $\alpha \in \mathbb{R}$,

(iii) $f \cdot g$ is integrable,

(iv) If there is $\delta > 0$ such that $|f(x,y)| \geq \delta$ for all $(x,y) \in R$ (so that $1/f$ is bounded), then $1/f$ is integrable,

(v) If $f \leq g$, then $\int\int_R f \leq \int\int_R g$,

(vi) $|f|$ is integrable, and $|\int\int_R f| \leq \int\int_R |f|$.

Proof: (v) For any partition P of R, $U(P,f) \leq U(P,g)$, and so $\int\int_R f = U(f) \leq U(g) = \int\int_R g$.

(vi) $U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f)$ for each P.
Also, $-|f| \leq f \leq |f| \implies -\int\int_R |f| \leq \int\int_R f \leq \int\int_R |f|$.

Evaluation of a Double Integral

Suppose a function is integrable on a rectangle R. How can we find its double integral?

To evaluate the Riemann integral of an integrable function f on an interval $[a, b]$, we used a powerful result known as the Fundamental Theorem of Calculus (Part II). If we find a function F on $[a, b]$ whose derivative is equal to f on $[a, b]$, then

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a).$$

Later in this course, we shall see some versions of the Fundamental Theorem of Calculus for functions of two/three variables, known as the Green/Stokes theorem.

For the present, we consider an easy and widely used method for evaluating double integrals in which the problem is reduced to a repeated evaluation of Riemann integrals.

Theorem (Fubini Theorem on a Rectangle)

Let $R := [a, b] \times [c, d]$, and let $f : R \to \mathbb{R}$ be integrable. Let I denote the double integral of f on R.

(i) If for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)dy$ exists, then the **iterated integral** $\int_a^b \left(\int_c^d f(x, y)dy \right) dx$ exists and is equal to I.

(ii) If for each fixed $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)dx$ exists, then the **iterated integral** $\int_c^d \left(\int_a^b f(x, y)dx \right) dy$ exists and is equal to I.

(iii) If the hypotheses in both (i) and (ii) above hold, and in particular, if f is continuous on R, then

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx = I = \int_c^d \left(\int_a^b f(x, y)dx \right) dy.$$

The Fubini theorem can be proved by using the Riemann condition for a double integral and for a Riemann integral.

Special case: Let $\phi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ be Riemann integrable. Define $f : R \to \mathbb{R}$ by $f(x, y) := \phi(x)\psi(y)$, $(x, y) \in R$. Then f is integrable on R, and its double integral is equal to

$$\int_a^b \left(\int_c^d \phi(x)\psi(y)dy \right) dx = \left(\int_a^b \phi(x)dx \right) \left(\int_c^d \psi(y)dy \right).$$

In particular, if $r, s \in \mathbb{R}$ with $r \geq 0$ and $s \geq 0$, then

$$\iint_{[a,b]\times[c,d]} x^r y^s d(x, y) = \left(\frac{b^{r+1} - a^{r+1}}{r+1} \right) \left(\frac{d^{s+1} - c^{s+1}}{s+1} \right),$$

provided $0 < a < b$ and $0 < c < d$.

Geometrically, the Fubini theorem says that if f is a nonnegative integrable function on $R := [a,b]\times[c,d]$, then the volume of the solid D under the surface $z = f(x,y)$ and above the rectangle R can be found by the slice method. For $x \in [a,b]$, we find the area $A(x) := \int_c^d f(x,y)dy$, of the cross-section of D perpendicular to the x-axis, or alternatively, for $y \in [c,d]$, we find the area $B(y) := \int_a^b f(x,y)dx$, of the cross-section of D perpendicular to the y-axis. (See Figure.)

$$\text{Vol}\,(D) = \int_a^b A(x)dx = \int_c^d B(y)dy.$$

Examples:

(i) Let $R := [0,1]\times[0,1]$, and $f(x,y) := (x+y)^2$, $(x,y) \in R$. Then f is continuous on R. The double integral of f on R is

$$\begin{aligned}\int_0^1\left(\int_0^1 (x+y)^2dx\right)dy &= \frac{1}{3}\int_0^1 (x+y)^3\Big|_0^1 dy \\ &= \frac{1}{3}\int_0^1 \left((1+y)^3 - y^3\right)dy = \frac{7}{6}.\end{aligned}$$

(ii) Let $R := [0,1]\times[0,1]$, $f(0,0) := 0$, and for $(x,y) \neq (0,0)$, let $f(x,y) := xy(x^2-y^2)/(x^2+y^2)^3$. For $x \in [0,1]$, $x \neq 0$,

$$A(x) := \int_0^1 f(x,y)dy = \int_0^1 \frac{xy(x^2-y^2)}{(x^2+y^2)^3}dy = \frac{x}{2(1+x^2)^2}.$$

(Substitute $x^2+y^2 = u$.) Also, $A(0) = \int_0^1 0dy = 0$. Hence

$$\int_0^1 \left(\int_0^1 f(x,y)dy \right) dx = \int_0^1 A(x)dx = \int_0^1 \frac{x}{2(1+x^2)^2}dx = \frac{1}{8}.$$

By interchanging x and y, $\displaystyle\int_0^1 \left(\int_0^1 f(x,y)dx \right) dy = -\frac{1}{8}$.

Thus the two iterated integrals exist, but they are not equal.

Note that since $f(1/n, 1/2n) = 24n^2/125$ for all $n \in \mathbb{N}$, the function f is not bounded on R, and so it is not integrable on R. Thus Fubini's theorem is not applicable.

Remark

Around 1910, Fichtenholz and Lichtenstein proved the following result.

Theorem

Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. If for each fixed $x \in [a,b]$, the Riemann integral $\int_c^d f(x,y)dy$ exists, and if for each fixed $y \in [c,d]$, the Riemann integral $\int_a^b f(x,y)dx$ exists, then the **iterated integrals** $\int_a^b \big(\int_c^d f(x,y)dy \big) dx$ and $\int_c^d \big(\int_a^b f(x,y)dx \big) dy$ exist and are equal.

The above result is stronger than the Fubini theorem since it does not assume that the function $f : [a,b] \times [c,d] \to \mathbb{R}$ is integrable; it only assumes that f is bounded on $[a,b]\times[c,d]$.

Double Riemann Sum

If we are not able to evaluate the double integral exactly, we attempt to find its approximations.

Given a bounded function $f : R \to \mathbb{R}$, and a partition $P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$, of $R := [a, b] \times [c, d]$, a double sum of the form

$$S(P, f) := \sum_{i=1}^{n} \sum_{j=1}^{k} f(s_i, t_j)(x_i - x_{i-1})(y_j - y_{j-1}),$$

where $(s_i, t_j) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, is called a double Riemann sum for f corresponding to P.

Note: $L(P, f) \leq S(P, f) \leq U(P, f)$ for any $s_1, \ldots, s_n \in [a, b]$ and $t_1, \ldots, t_k \in [c, d]$.

We define the **mesh** of a partition
$P := \{(x_i, y_j) : i = 0, 1, \ldots, n \text{ and } j = 0, 1, \ldots, k\}$ by
$\mu(P) := \max\{x_1 - x_0, \ldots, x_n - x_{n-1},\ y_1 - y_0, \ldots, y_k - y_{k-1}\}$.

Theorem: Let f be integrable on R, and let $\epsilon > 0$. Then there is $\delta > 0$ such that $U(P, f) - L(P, f) < \epsilon$ for every partition P satisfying $\mu(P) < \delta$.

Corollary: If f is integrable on R and (P_n) is a sequence of partitions of R with $\mu(P_n) \to 0$, then $U(P_n, f) - L(P_n, f) \to 0$. Further, if $S(P_n, f)$ is a double Riemann sum corresponding to P_n and f, then $S(P_n, f) \to \iint_R f$.

Proof: Given $\epsilon > 0$, find $\delta > 0$ as in the theorem. Then there is $n_0 \in \mathbb{N}$ such that $\mu(P_n) < \delta$ for all $n \geq n_0$, and so $U(P_n, f) - L(P_n, f) < \epsilon$. Thus $U(P_n, f) - L(P_n, f) \to 0$.
Since $L(P_n, f) \leq S(P_n, f) \leq U(P_n, f)$, and
$L(P_n, f) \leq L(f) = \iint_R f = U(f) \leq U(P_n, f)$, we see that
$\left|S(P_n, f) - \iint_R f\right| \leq U(P_n, f) - L(P_n, f) \to 0$. □

Caution:

If we define

$$\mu(P) := \max\{(x_i - x_{i-1})(y_j - y_{j-1}) : 1 \le i \le n;\ 1 \le j \le k\},$$

then we may not obtain $S(P_n, f) \to \iint_R f$ as $\mu(P_n) \to 0$ for every integrable function f on R. An example of a function of this kind is a bit involved.

The preceding corollary allows us to find limits of certain sequences. For example, let

$$s_n := \frac{1}{n^4} \sum_{i=1}^{n} \sum_{j=1}^{n} (i + j)^2 \quad \text{for } n \in \mathbb{N}.$$

Then $s_n = \dfrac{1}{n^2} \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{i}{n} + \frac{j}{n}\right)^2 = S(\mathrm{P}_n, f)$, where $f(x, y) := (x + y)^2$ for $(x, y) \in [0, 1] \times [0, 1]$, and $\mu(\mathrm{P}_n) = 1/n$ for $n \in \mathbb{N}$. Hence $s_n \to \int_0^1 \int_0^1 (x + y)^2 d(x, y) = 7/6$.

Double Integral over a Bounded Set

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Consider a rectangle $R := [a, b] \times [c, d]$ such that $D \subset R$, and define $f^* : R \to \mathbb{R}$ by

$$f^*(x, y) := \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is **integrable over** D if f^* is integrable on R, and in this case, the **double integral of** f **over** D is defined to be the double integral of f^* on R, that is,

$$\iint_D f(x, y) d(x, y) := \iint_R f^*(x, y) d(x, y).$$

By the domain additivity of double integrals on rectangles, the integrability of f over D and the value of its double integral are independent of the choice of a rectangle R containing D and the corresponding extension f^* of f to R.

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be integrable over D. We may also denote the double integral of f over D by $\iint_D f$.

If f is nonnegative, then the **volume** of the solid under the surface given by $z = f(x, y)$ and above the region D is defined to be the double integral of f over D. Thus

$$\text{Vol}\,(E_f) := \iint_D f(x, y)d(x, y),$$

where $E_f := \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } 0 \leq z \leq f(x, y)\}$.

Next, if $g : D \to \mathbb{R}$ is integrable, and $f \leq g$ on D, then

$$\iint_D \big(g(x, y) - f(x, y)\big)d(x, y)$$

is defined to be the **volume between the surfaces** given by $z = f(x, y)$ and $z = g(x, y)$.

The double integral over a bounded subset of $\mathbb{R}^2$ has algebraic and order properties analogous to those of the double integral on a rectangle.

In order to seek conditions under which a bounded function f defined on a bounded subset D of $\mathbb{R}^2$ is integrable over D, we introduce a new concept.

A bounded subset E of $\mathbb{R}^2$ is said to be of (two-dimensional) **content zero** if for every $\epsilon > 0$, there are finitely many rectangles whose union contains E and the sum of whose areas is less than ϵ.

A set of (two-dimensional) content zero is a 'thin' subset of $\mathbb{R}^2$.

Examples:
(i) Every finite subset of $\mathbb{R}^2$ is of content zero.

(ii) The infinite subset $\{(1/n, 1/k) : n, k \in \mathbb{N}\}$ of $\mathbb{R}^2$ is of content zero.

(iii) The subset $\{(x, y) \in [0,1]\times[0,1] : x, y \in \mathbb{Q}\}$ of $\mathbb{R}^2$ is not of content zero. (Consider any ϵ less than 1.)

(iv) Let $\varphi : [a, b] \to \mathbb{R}$ be an integrable function. Then its graph $E := \{(x, \varphi(x)) : x \in [a, b]\}$ is of content zero. To see this, let $\epsilon > 0$. By the Riemann condition, there is a partition $P := \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(P, \varphi) - L(P, \varphi) < \epsilon$. Then $E \subset \bigcup_{i=1}^{n} R_i$, where $R_i := [x_{i-1}, x_i]\times[m_i(\varphi), M_i(\varphi)]$ and

$$\text{Area}\,(R_1) + \cdots + \text{Area}\,(R_n) = U(P, \varphi) - L(P, \varphi) < \epsilon.$$

Similarly, if $\psi : [c, d] \to \mathbb{R}$ is an integrable function, then the set $E := \{(\psi(y), y) : y \in [c, d]\}$ is of content zero.

(v) Let functions x, $y : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), and let their derivatives be bounded on (a, b). Then the set $E := \{(x(t), y(t)) \in \mathbb{R}^2 : t \in [a, b]\}$ is of two-dimensional content zero. We prove this as follows.

For $n \in \mathbb{N}$, consider the partition of $[a, b]$ into n equal parts, and let s_i be the mid-point of the ith subinterval for $i = 1, \ldots, n$. Let $t \in [a, b]$. Then there is $i \in \{1, \ldots, n\}$ such that $|t - s_i| \leq (b - a)/2n$. By the Mean Value Theorem, there are c_x and c_y between t and s_i such that

$$x(t)-x(s_i) = x'(c_x)(t-s_i) \quad \text{and} \quad y(t)-y(s_i) = x'(c_y)(t-s_i).$$

Let $\alpha \in \mathbb{R}$ with $|x'|, |y'| \leq \alpha$ on (a, b), and $\beta := b - a$. Then $|x(t) - x(s_i)|, |y(t) - y(s_i)| \leq \alpha\beta/2n$, and so $(x(t), y(t))$ lies in a square S_i centered at $(x(s_i), y(s_i))$ and of area $\alpha^2\beta^2/n^2$.

Thus E is contained in the union of the n squares $S_1, \ldots, S_n$, the sum of whose areas is equal to $\alpha^2\beta^2/n$.

Given $\epsilon > 0$, choose $n \in \mathbb{N}$ such that $n > \alpha^2(b - a)^2/\epsilon$. Then the set E is contained in the union of these n squares, the sum of whose areas is less that ϵ. Hence E is of two-dimensional content zero.

Theorem

Let D be a bounded subset of $\mathbb{R}^2$, and $f : D \to \mathbb{R}$ be a bounded function. If the boundary ∂D of D is of (two-dimensional) content zero and if the set of discontinuities of f in D is also of (two-dimensional) content zero, then f is integrable over D.

We omit the proof.

The above theorem says that the two conditions
(i) 'the boundary ∂D of D is of content zero' and
(ii) 'the set of discontinuities of f in D is of content zero'
together are sufficient to guarantee the integrability of f on D.
While neither condition can be dropped from the above theorem, the integrability of f does not necessarily imply either condition.

(Find examples to illustrate the above remarks.)

Elementary Regions

Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ such that $\phi_1 \leq \phi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}.$$

Alternatively, let $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ be continuous functions on $[c, d]$ such that $\psi_1 \leq \psi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}.$$

In both cases, D is called an **elementary region** in $\mathbb{R}^2$. In the former case, it is said to be of **type** I, and in the latter case, it is said to be of **type** II.

We note that the boundary of an elementary domain in $\mathbb{R}^2$ is of (two-dimensional) content zero. Hence a continuous function defined on such a domain is integrable over it.

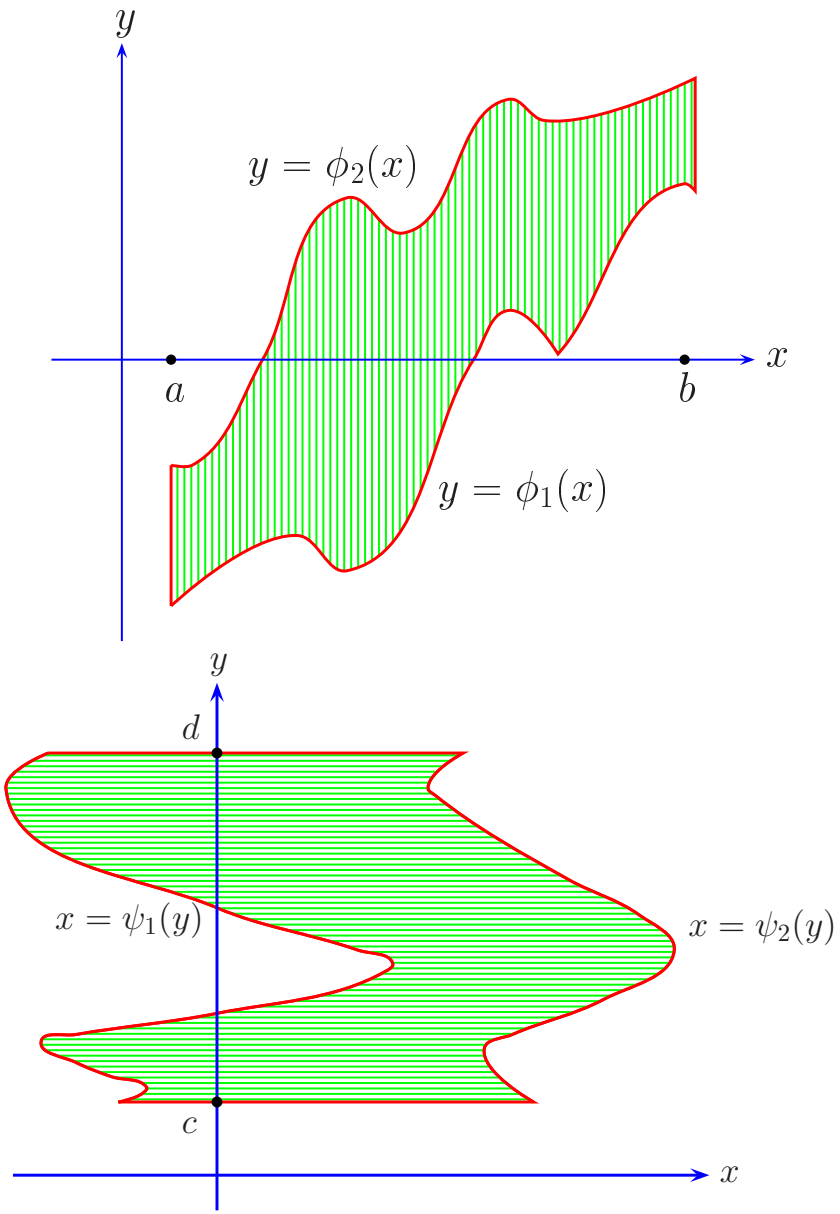

Figure: Elementary regions with boundaries in red colour

Clearly, a rectangle is an elementary region in $\mathbb{R}^2$ of type I as well as of type II.
Also, if $a > 0$, then the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$ is an elementary region in $\mathbb{R}^2$, since

$$D = \left\{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a \text{ and } -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}\right\}.$$

Essential feature of an elementary region D in $\mathbb{R}^2$ of type I:
For every $x \in \mathbb{R}$, the vertical cross-section of D at x is a closed and bounded interval.
Essential feature of an elementary region D in $\mathbb{R}^2$ of type II:
For every $y \in \mathbb{R}$, the horizontal cross-section of D at y is a closed and bounded interval.

There do exist bounded subsets of $\mathbb{R}^2$ that are not elementary regions. For example, a star-shaped subset of $\mathbb{R}^2$ or an annulus is not an elementary region, since for some $x \in \mathbb{R}$, the vertical cross-section of D at x is not an interval, and for some $y \in \mathbb{R}$, the horizontal cross-section of D at y is not an interval.

Proposition (Fubini Theorem over Elementary Regions)

Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous, then the iterated integral $\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \right) dx$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

(ii) If $D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$, where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are continuous, then the iterated integral $\int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \right) dy$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

Proof: (i) Let $c := \inf\{\phi_1(x) : x \in [a, b]\}$ and $d := \sup\{\phi_2(x) : x \in [a, b]\}$. Then $D \subset R := [a, b] \times [c, d]$, and the extended function $f^* : R \to \mathbb{R}$ is integrable. Now use the Fubini theorem for f^*. (ii) A similar argument works here.