MA 105 : Calculus
Division 3, Lecture 28

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Recap of the previous lecture

- Domain additivity for double integrals over rectangles
- Integrable functions
- Algebraic and order properties
- Fubini's theorem on a rectangle. Examples
- Remark about a theorem of Fichtenholtz and Lichtenstein
- Double Riemann sums
- Double integrals over bounded sets
- Volume of a solid bounded by two surfaces defined over a planar domain
- Notion of sets of (two-dimensional) content zero. Examples
- Necessary condition for the integrability of a bounded function over a bounded subset
- Elementary regions. Examples
- Fubini Theorem over Elementary Regions.

Elementary Regions

Let $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ such that $\phi_1 \leq \phi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}.$$

Alternatively, let $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ be continuous functions on $[c, d]$ such that $\psi_1 \leq \psi_2$, and let

$$D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}.$$

In both cases, D is called an **elementary region** in $\mathbb{R}^2$. In the former case, it is said to be of **type** I, and in the latter case, it is said to be of **type** II.

We note that the boundary of an elementary domain in $\mathbb{R}^2$ is of (two-dimensional) content zero. Hence a continuous function defined on such a domain is integrable over it.

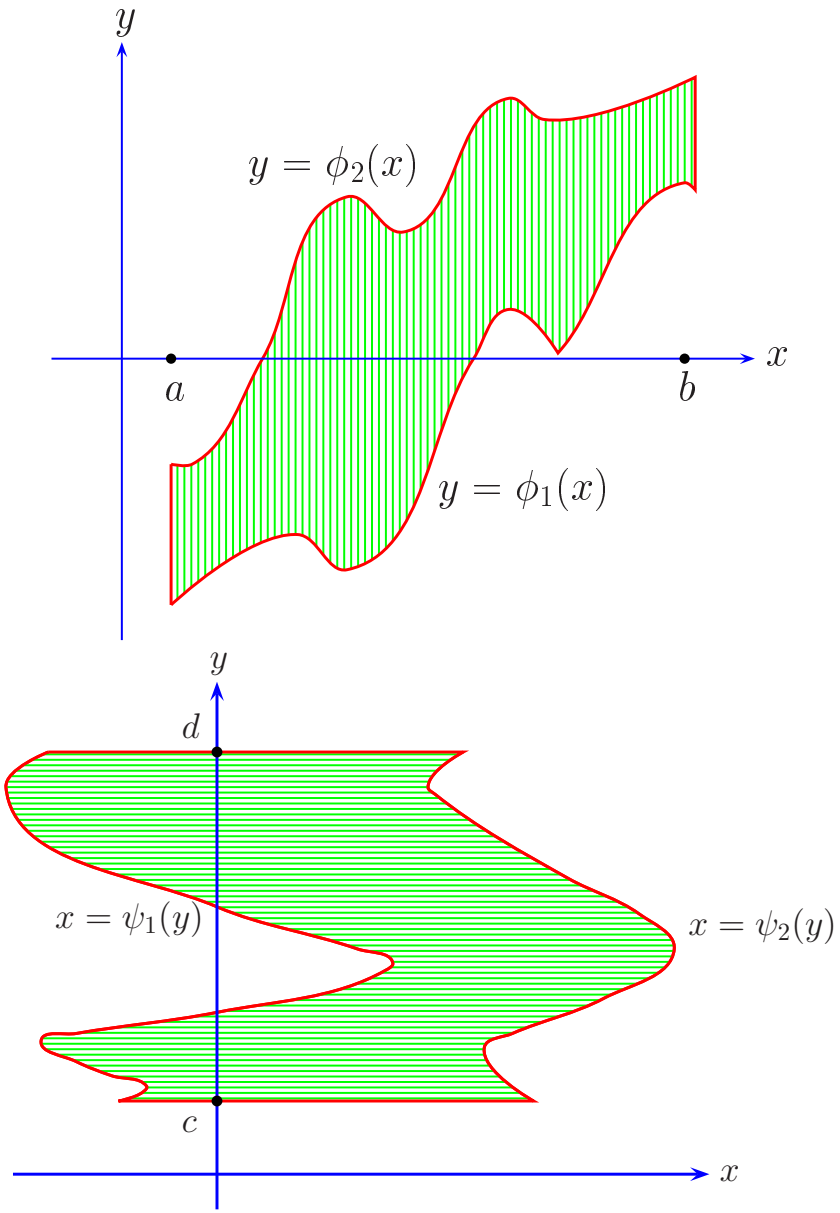


Figure: Elementary regions with boundaries in red colour

Clearly, a rectangle is an elementary region in $\mathbb{R}^2$ of type I as well as of type II.

Also, if $a > 0$, then the disk $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq a^2\}$ is an elementary region in $\mathbb{R}^2$, since

$$D = \left\{(x, y) \in \mathbb{R}^2 : -a \leq x \leq a \text{ and } -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}\right\}.$$

Essential feature of an elementary region D in $\mathbb{R}^2$ of type I:
For every $x \in \mathbb{R}$, the vertical cross-section of D at x is a closed and bounded interval.

Essential feature of an elementary region D in $\mathbb{R}^2$ of type II:
For every $y \in \mathbb{R}$, the horizontal cross-section of D at y is a closed and bounded interval.

There do exist bounded subsets of $\mathbb{R}^2$ that are not elementary regions. For example, a star-shaped subset of $\mathbb{R}^2$ or an annulus is not an elementary region, since for some $x \in \mathbb{R}$, the vertical cross-section of D at x is not an interval, and for some $y \in \mathbb{R}$, the horizontal cross-section of D at y is not an interval.

Proposition (Fubini Theorem over Elementary Regions)

Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be continuous.

(i) If $D := \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } \phi_1(x) \leq y \leq \phi_2(x)\}$, where $\phi_1, \phi_2 : [a, b] \to \mathbb{R}$ are continuous, then the iterated integral $\int_a^b \big(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y)dy \big) dx$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

(ii) If $D := \{(x, y) \in \mathbb{R}^2 : c \leq y \leq d \text{ and } \psi_1(y) \leq x \leq \psi_2(y)\}$, where $\psi_1, \psi_2 : [c, d] \to \mathbb{R}$ are continuous, then the iterated integral $\int_c^d \big(\int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \big) dy$ exists and equals the double integral $\iint_D f(x, y)d(x, y)$.

Proof: (i) Let $c := \inf\{\phi_1(x) : x \in [a, b]\}$ and $d := \sup\{\phi_2(x) : x \in [a, b]\}$. Then $D \subset R := [a, b] \times [c, d]$, and the extended function $f^* : R \to \mathbb{R}$ is integrable. Now use the Fubini theorem for f^*.
(ii) A similar argument works here.

Examples:

(i) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1/2 \text{ and } 0 \le y \le x^2\}$ and $f(x, y) := x + y$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$I := \int\int_D (x + y) d(x, y) = \int_0^{1/2} \left(\int_0^{x^2} (x + y) dy \right) dx,$$

which is equal to

$$\int_0^{1/2} \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x^2} dx = \int_0^{1/2} \left(x^3 + \frac{x^4}{2} \right) dx = \frac{3}{160}.$$

Alternatively, since
$D = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le 1/4 \text{ and } \sqrt{y} \le x \le 1/2\}$,

$$I = \int_0^{1/4} \left(\int_{\sqrt{y}}^{1/2} (x+y) dx \right) dy = \int_0^{1/4} \left[\frac{x^2}{2} + xy \right]_{x=\sqrt{y}}^{x=1/2} dy = \frac{3}{160}.$$

(ii) Let $D := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2x\}$ and $f(x, y) := e^{x^2}$ for $(x, y) \in D$. Then f is continuous on the elementary region D. By the Fubini theorem,

$$\iint_D f = \int_0^1 \left(\int_0^{2x} e^{x^2} \, dy \right) dx = \int_0^1 2x \, e^{x^2} \, dx = e - 1.$$

Also, since $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 2 \text{ and } y/2 \leq x \leq 1\}$,

$$\iint_D f = \int_0^2 \left(\int_{y/2}^1 e^{x^2} \, dx \right) dy.$$

However, the integral $\int_{y/2}^1 e^{x^2} \, dx$ cannot be evaluated in terms of known functions.

This example shows that an iterated integral may not always be useful in evaluating a double integral.

Constant Function on a Bounded Subset of $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Define

$$1_D : D \to \mathbb{R} \quad \text{by} \quad 1_D(x, y) := 1 \text{ for all } (x, y) \in D.$$

If the boundary ∂D of D is of (two-dimensional) content zero, then the continuous function 1_D is integrable on D as we saw. The converse also holds, that is, if the function 1_D is integrable on D, then ∂D is of (two-dimensional) content zero.

Examples: Let $R := [a, b] \times [c, d]$. If $D := R$, then we have seen that 1_D is integrable and its double integral is equal to $(b - a)(d - c)$. But if $D := \{(x, y) \in R : x, y \in \mathbb{Q}\}$, then the function $1_D^* : R \to \mathbb{R}$, obtained by extending the function 1_D as usual, is the bivariate Dirichlet function on R. We have seen that 1_D^* is not integrable on R, that is, 1_D is not integrable over D. Note: In this case $\partial D = [a, b] \times [c, d]$, and so ∂D is not of (two-dimensional) content zero.

Area of a Bounded Subset of $\mathbb{R}^2$

Let $D \subset \mathbb{R}^2$ be bounded. We say that D has an **area** if the function 1_D is integrable over D, and then we define

$$\text{Area}\,(D) := \iint_D 1_D(x,y)\,d(x,y).$$

Thus a bounded subset D of $\mathbb{R}^2$ has an area $\iff$ ∂D is of content zero.

Important special case: Consider an elementary domain $D := \{(x,y) \in \mathbb{R}^2 : a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\}$, where $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ are continuous. Then ∂D is of content zero, and by the Fubini theorem,

$$\text{Area}\,(D) = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 1\,dy \right) dx = \int_a^b \big(\phi_2(x) - \phi_1(x)\big)\,dx,$$

which was our definition of the **area between the curves** $y = \phi_1(x)$ and $y = \phi_2(x)$, $x \in [a,b]$.

The following basic inequality is important.

Let D be a bounded subset of $\mathbb{R}^2$ which has an area. If $f : D \to \mathbb{R}$ is an integrable function, and if $|f| \leq \alpha$ on D, then

$$\left| \iint_D f(x,y)d(x,y) \right| \leq \iint_D |f(x,y)|d(x,y) \leq \alpha \,\text{Area}\,(D).$$

The following result is useful in evaluating a double integral.

Theorem (Domain Additivity)

Let D be a bounded subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a bounded function. Suppose $D = D_1 \cup D_2$, where $D_1 \cap D_2$ is of (two-dimensional) content zero. If f is integrable over D_1 and over D_2, then f is integrable over D and

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$

Change of Variables in a Double Integral

The calculation of the double integral $\iint_D f(x,y)d(x,y)$ can often be simplified by reducing it to another double integral $\iint_E g(u,v)d(u,v)$, where the pair (u,v) of new variables is related to the the pair (x,y) of the given variables by a suitable transformation.

In doing so, we look for a function g which is 'simpler' than the given function f such that the domain E of g is also 'simpler' than the given domain D of f.

Let $x^\circ, y^\circ, a_1, b_1, a_2, b_2 \in \mathbb{R}$, and consider the **affine transformation** $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\Psi := (\psi_1, \psi_2)$, where

$$x = \psi_1(u,v) := x^\circ + a_1 u + b_1 v \quad \text{and} \quad y = \psi_2(u,v) := y^\circ + a_2 u + b_2 v.$$

Note that the above equations have a unique solution, that is, the transformation Ψ is one-one $\iff$ $a_1 b_2 - a_2 b_1 \neq 0$.

In this case, the square region $E := [0,1]\times[0,1]$ is transformed to the parallelogram $D := \Psi(E)$ with vertices $\Psi(0,0) = (x^\circ, y^\circ)$, $\Psi(1,0) = (x^\circ + a_1, y^\circ + a_2)$, $\Psi(0,1) = (x^\circ + b_1, y^\circ + b_2)$ and $\Psi(1,1) = (x^\circ + a_1 + b_1, y^\circ + a_2 + b_2)$. Note that Area $(E) = 1$, and it gets scaled to the area $|a_1 b_2 - a_2 b_1|$ of $D = \Psi(E)$.

In general, let $E \subset \mathbb{R}^2$, and consider $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$. Let $Q_0 := (u_0, v_0)$ be an interior point of E. If $x := \phi_1(u, v)$ and $y := \phi_2(u, v)$ for $(u, v) \in E$, and the functions x and y have partial derivatives at (u_0, v_0), then the **Jacobian** of the transformation Φ at (u_0, v_0) is defined by

$$J(\Phi)(u_0, v_0) := \frac{\partial(x, y)}{\partial(u, v)}(u_0, v_0) := \det \begin{bmatrix} \frac{\partial x}{\partial u}(Q_0) & \frac{\partial x}{\partial v}(Q_0) \\ \frac{\partial y}{\partial u}(Q_0) & \frac{\partial y}{\partial v}(Q_0) \end{bmatrix}.$$

In particular, $J(\Psi)(u, v) = a_1 b_2 - a_2 b_1$, $(u, v) \in \mathbb{R}^2$, for the affine transformation Ψ.

Now suppose E is a square region about Q_0, and the transformation $\Phi := (\phi_1, \phi_2) : E \to \mathbb{R}^2$ is such that the functions ϕ_1 and ϕ_2 are differentiable at Q_0.

Let $P_0 := \Phi(u_0, v_0)$. For $(u, v) \in \mathbb{R}^2$, let

$$\psi_1(u,v) := \phi_1(u_0,v_0) + \frac{\partial \phi_1}{\partial u}\Big|_{(u_0,v_0)}(u-u_0) + \frac{\partial \phi_1}{\partial v}\Big|_{(u_0,v_0)}(v-v_0),$$

$$\psi_2(u,v) := \phi_2(u_0,v_0) + \frac{\partial \phi_2}{\partial u}\Big|_{(u_0,v_0)}(u-u_0) + \frac{\partial \phi_2}{\partial v}\Big|_{(u_0,v_0)}(v-v_0).$$

By the definition of differentiability of ϕ_1 and ϕ_2 at (u_0, v_0),

$$\phi_1(u,v) = \psi_1(u,v)+\epsilon_1(u,v) \quad \text{and} \quad \phi_2(u,v) = \psi_2(u,v)+\epsilon_2(u,v),$$

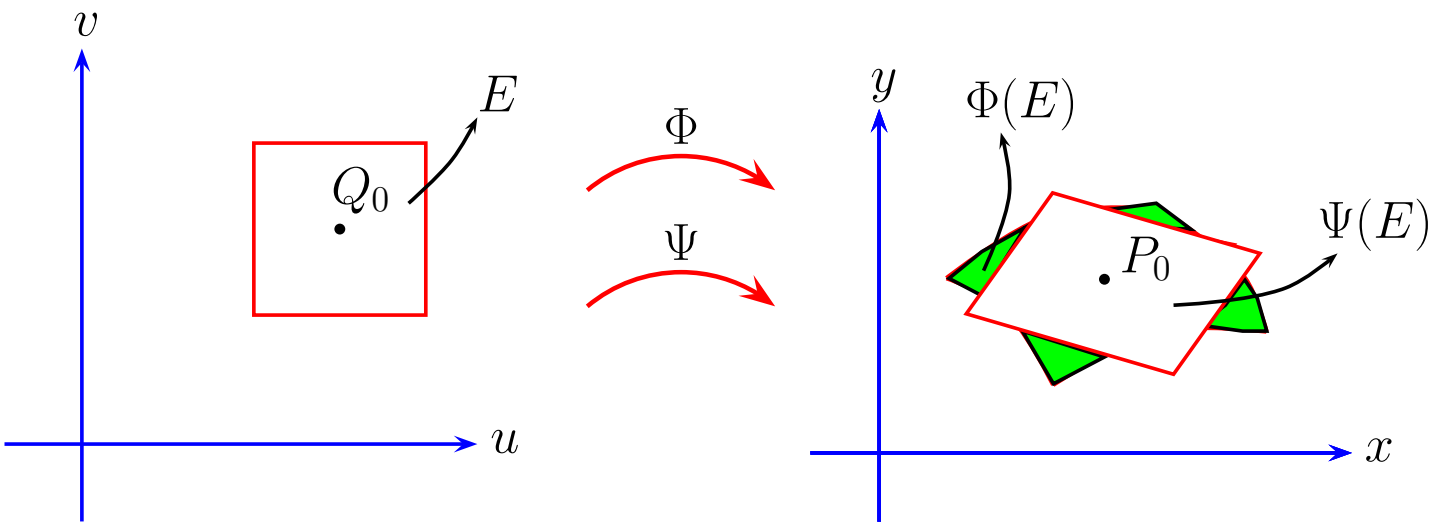
for all $(u, v) \in E$, where $\epsilon_1(u, v)/\|(u - u_0, v - v_0)\| \to 0$ and $\epsilon_2(u, v)/\|(u - u_0, v - v_0)\| \to 0$ as $(u, v) \to (u_0, v_0)$.

Thus the affine transformation $\Psi := (\psi_1, \psi_2) : \mathbb{R}^2 \to \mathbb{R}^2$ maps Q_0 to P_0, and approximates the transformation Φ around Q_0.

Also, by the definition of $\Psi = (\psi_1, \psi_2)$, we obtain

$$J(\Psi) = \left(\frac{\partial \psi_1}{\partial u}\frac{\partial \psi_2}{\partial v} - \frac{\partial \psi_1}{\partial v}\frac{\partial \psi_2}{\partial u}\right) = \left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u}\right)(Q_0) = J(\Phi)(Q_0).$$

Hence $\text{Area}\,(\Psi(E)) = |J(\Psi)|\text{Area}\,(E) = |J(\Phi)(Q_0)|\text{Area}\,(E)$.



To state our next result, we need the following definition.
A subset Ω of $\mathbb{R}^2$ is called **open** if every point of Ω is an interior point of Ω.

Theorem (Change of Variables Formula)

Let D be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose Ω is an open subset of $\mathbb{R}^2$ and $\Phi : \Omega \to \mathbb{R}^2$ is a one-one transformation such that $\Phi := (\phi_1, \phi_2)$, where ϕ_1 and ϕ_2 have continuous partial derivatives in Ω and $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$. Let $E \subset \Omega$ be such that $\Phi(E) = D$. Then E is a closed and bounded subset of Ω, and E has an area. Moreover, $f \circ \Phi : E \to \mathbb{R}$ is continuous, and

$$\iint_D f(x, y)d(x, y) = \iint_E (f \circ \Phi)(u, v)|J(\Phi)(u, v)|d(u, v).$$

Note: The hypothesis $J(\Phi)(u, v) \neq 0$ for all $(u, v) \in \Omega$ can be weakened by assuming only that $J(\Phi)(u, v) \geq 0$ for all $(u, v) \in \Omega$ or $J(\Phi)(u, v) \leq 0$ for all $(u, v) \in \Omega$, and $J(\Phi)(u, v) = 0$ only for (u, v) in a subset of Ω of (two-dimensional) content zero.

Compare the above result on Change of Variables for double integrals with the result on Integration by Substitution for Riemann integrals proved in Lecture 12:

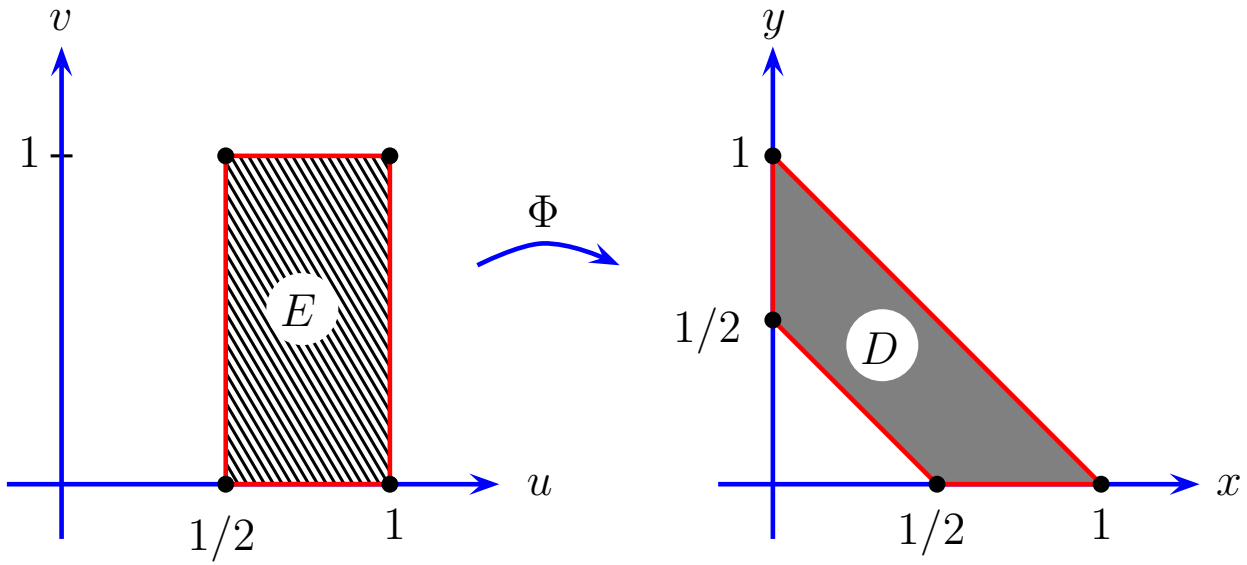
(Integration by substitution)

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $\phi : [\alpha, \beta] \to \mathbb{R}$ be a continuously differentiable function such that $\phi([\alpha, \beta]) = [a, b]$. If $\phi'(t) \neq 0$ for all $t \in (\alpha, \beta)$, then

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))|\phi'(t)|dt.$$

Example

Let $D := \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0 \text{ and } 1 \leq 2(x + y) \leq 2\}$, and define $f : D \to \mathbb{R}$ by $f(x, y) := y/(x + y)$ for $(x, y) \in D$.



Let $u := x+y,\ v := y/(x+y)$, that is, $x := u(1-v),\ y := uv$.

We let $\Omega := \{(u, v) \in \mathbb{R}^2 : u > 0\}$, and define $\Phi : \Omega \to \mathbb{R}^2$ by $\Phi(u, v) = \big(u(1 - v), uv\big)$. Then the map Φ is one-one from Ω onto $\{(x, y) \in \mathbb{R}^2 : x + y > 0\}$. If $\Phi = (\phi_1, \phi_2)$, then the partial derivatives of ϕ_1 and ϕ_2 exist and are continuous.

Also,

$$J(\Phi)(u,v) = \det \begin{bmatrix} 1-v & -u \\ v & u \end{bmatrix} = u \neq 0 \quad \text{for all } (u,v) \in \Omega.$$

Further, if $E := [1/2, 1] \times [0, 1]$, then $\Phi(E) = D$ and $(f \circ \Phi)(u, v) = v$ for all $(u, v) \in E$. Since f is continuous on D, the Change of Variables formula gives

$$\begin{aligned} \iint_D f(x,y)\,d(x,y) &= \iint_E f(u(1-v), uv)|u|\,d(u,v) \\ &= \iint_E uv\,d(u,v) \\ &= \left(\int_{1/2}^{1} u\,du\right)\left(\int_0^1 v\,dv\right) \\ &= \frac{3}{16}. \end{aligned}$$

Polar Transformation Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\Phi(r, \theta) := (r\cos\theta,\ r\sin\theta)$. If $\Phi := (\phi_1, \phi_2)$, then ϕ_1 and ϕ_2 have continuous partial derivatives in $\mathbb{R}^2$ and

$$J(\Phi)(r,\theta) = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r \quad \text{for all } (r,\theta) \in \mathbb{R}^2.$$

Thus the Jacobian of Φ is nonzero on $\{(r,\theta) \in \mathbb{R}^2 : r \neq 0\}$.
Also, the function $\Phi : \{(r,\theta) \in \mathbb{R}^2 : r > 0 \text{ and } -\pi < \theta \leq \pi\}$ $\to \{(x,y) \in \mathbb{R}^2 : (x,y) \neq (0,0)\}$ is one-one and onto.
In this case, the following result holds.

Proposition

Let D be a closed and bounded subset of $\mathbb{R}^2$ which has an area, and let $f : D \to \mathbb{R}$ be continuous. Suppose the set $E := \{(r,\theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi, (r\cos\theta, r\sin\theta) \in D\}$, has an area. Then the function $(r,\theta) \longmapsto f(r\cos\theta, r\sin\theta)$ is continuous on E and
$\iint_D f(x,y)d(x,y) = \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r,\theta)$.

Examples

Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Define

$$E := \{(r, \theta) \in \mathbb{R}^2 : r \geq 0, -\pi \leq \theta \leq \pi \text{ and } (r\cos\theta, r\sin\theta) \in D\}.$$

Then $E := [0, 1] \times [-\pi, \pi]$.

(i) Let $f(x, y) := \sqrt{1 - x^2 - y^2}$ for $(x, y) \in D$. Then

$$\begin{aligned}
\iint_D f(x, y)d(x, y) &= \iint_E f(r\cos\theta, r\sin\theta)\, r\, d(r, \theta) \\
&= \int_{-\pi}^{\pi} \left(\int_0^1 \sqrt{1 - r^2}\, r\, dr \right) d\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} \left(\int_0^1 \sqrt{s}\, ds \right) d\theta = \frac{2\pi}{3}.
\end{aligned}$$

(ii) Let $f(x, y) := e^{x^2+y^2}$ for $(x, y) \in D$. Then

$$\iint_D f(x, y)d(x, y) = \int_{-\pi}^{\pi} \left(\int_0^1 e^{r^2} r\ dr \right) d\theta = \int_{-\pi}^{\pi} \left(\frac{e - 1}{2} \right) d\theta = \pi(e-1).$$

Triple Integrals

The theory of double integrals of functions of two variables readily extends to triple integrals of functions of three variables. No new ideas are needed. We shall, therefore, mention only a few important points.

Consider a **cuboid** $K := [a, b] \times [c, d] \times [p, q]$, where $a < b$, $c < d$, $p < q$. We consider a partition P of K into several subcuboids.

For a bounded function $f : K \to \mathbb{R}$, we define the corresponding lower triple sum $L(P, f)$, the upper triple sum $U(P, f)$, the lower triple integral $L(f)$, and the upper triple integral $U(f)$. As before, f is **(triple) integrable** on K, if $L(f) = U(f)$. In this case, its **triple integral** is denoted by

$$\iiint_K f(x, y, z) d(x, y, z) \quad \text{or} \quad \iiint_K f.$$

If $f := 1$ on K, then f is integrable on K, and its triple integral is equal to the volume of K, whereas if f is the trivariate Dirichlet function on K, then f is not integrable on K.

Further, f is integrable on K if and only if it satisfies the Riemann condition. If f is monotonic in each of the three variables, or if f is continuous on K, then f is integrable.

Fubini's theorem on cuboids

Let f be integrable on K, and let I denote its triple integral. Suppose for each fixed $x \in [a,b]$, the double integral $\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)$ exists. Then the **iterated integral** $\int_a^b \big(\iint_{[c,d]\times[p,q]} f(x,y,z)d(y,z)\big)dx$ exists and equals I. Further, if for each fixed $(x,y) \in [a,b]\times[c,d]$, the Riemann integral $\int_p^q f(x,y,z)dz$ exists, then the **iterated integral** $\int_a^b \big[\int_c^d \big(\int_p^q f(x,y,z)dz\big)dy\big]dx$ exists and equals I.

There are other versions of this result with the roles of x, y, z interchanged. They are used in evaluating triple integrals.